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I.C.T.P., P.O. BOX 586, 34100 TRIESTE, ITALY, CABLE: CENTRATOM TRIESTE

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Survey on quasi-projective moduli of polarized manifolds

E. Viehweg
Fachbereich 6, Mathematik
Universität - GHS - Essen
Universitätsstrasse 3
D-45117 Essen 1
Germany

MAIN BUILDING STRADA COSTIERA, 11 TEL. 22401 TELEFAX 224163 TELEX 460392 ADRIATICO GUEST HOUSE VIA GRIGNANO, 9 TEL. 224241 TELEFAX 224531 TELEX 460449
MICROPROCESSOR LAB. VIA BEIRUT, 31 TEL. 224471 TELEFAX 224600 TELEX 460392 GALILEO GUEST HOUSE VIA BEIRUT, 7 TEL. 22401 TELEFAX 2240310 TELEX 460392

Survey on Quasi-Projective Moduli of Polarized Manifolds

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Eckart Viehweg

Universität - GH - Essen, Fachbereich 6, Mathematik
Universitätsstraße 3, D-45117 Essen 1 , Germany
(MAT900 at vm.hrz.uni-essen.de)

B. Riemann claimed that the conformal structure of a Riemann surface of genus g is determined by $3g-3$ parameters, which he proposed to name "moduli". Following A. Grothendieck and D. Mumford [8] we will consider "algebraic moduli". Let us recall D. Mumford's strengthening of B. Riemann's statement.

Theorem 1 (Mumford [8]) *Let k be an algebraically closed field and, for $g \geq 2$,*

$$\mathfrak{C}_g(k) = \{ \text{ projective curves of genus } g, \text{ defined over } k\} / \cong .$$

Then there exists a quasi-projective coarse moduli variety C_g of dimension $3g-3$. i.e. a variety C_g and a natural bijection $\mathfrak{C}_g(k) \cong C_g(k)$ where $C_g(k)$ denotes the k-valued points of C_g.

Later we will give the exact definition of "natural" and of coarse moduli schemes. Let us just remark at this point that $\mathfrak{C}_g(S)$ denotes the set of isomorphism classes of flat morphisms $f : X \to S$, whose fibers $f^{-1}(s)$ belong to $\mathfrak{C}_g(k)$, and that "natural" implies that for a family $f : X \to S \in \mathfrak{C}_g(S)$ the induced map $S(k) \to C_g(k)$ comes from a morphism of schemes $\phi : S \to C_g$.

In the spirit of B. Riemann's result one should ask for a description of algebraic parameters or, at least, for a description of an ample sheaf on C_g. Using the notations introduced above one finds that for each $\nu \geq 0$ there is some $p > 0$ and an invertible sheaf $\lambda_\nu^{(p)}$ on C_g, with $\phi^* \lambda_\nu^{(p)} = \det(f_* \omega_{X/S}^\nu)^p$.

Addendum 2 (Mumford [8]) *For ν, μ and p sufficiently large, for*

$$\alpha = (2g-2) \cdot \nu - (g-1) \quad \text{and} \quad \beta = (2g-2) \cdot \nu \cdot \mu - (g-1)$$

the sheaf $\lambda_{\nu \cdot \mu}^{(p)\alpha} \otimes \lambda_\nu^{(p)-\beta \cdot \mu}$ is ample.

In the sequel all schemes are supposed to be defined over an algebraically closed field k of characteristic zero.

Trying to generalize Mumford's result to higher dimensions, one first remarks that the genus g of a projective curve X determines the Hilbert polynomial $h(\nu) = \chi(\omega_X^\nu) = (2g-2)\cdot\nu-(g-1)$ of X. Hence, if $h(T) \in \mathbb{Q}[T]$ is a polynomial of degree n, with $h(\mathbb{Z}) \subset \mathbb{Z}$, one should consider the subset

$$\mathfrak{C}_h(k) = \{X;\ X \text{ projective manifold, } \omega_X \text{ ample and } h(\nu) = \chi(\omega_X^\nu) \text{ for all } \nu\}/\cong$$

of

$$\mathfrak{C}(k) = \{X;\ X \text{ projective manifold, } \omega_X \text{ ample }\}/\cong.$$

Theorem 3 ([9], II) *Keeping the above notations, there exists a coarse quasi-projective moduli scheme C_h for $\mathfrak{C}_h(k)$.*

For $\eta \in \mathbb{N}$ there exists some $p \in \mathbb{N}$ and an invertible sheaf $\lambda_\eta^{(p)}$ on C_h such that for all $g: Y \to S \in \mathfrak{C}_h(S)$ and for the induced morphisms $\varphi: S \to C_h$ one has

$$\varphi^*\lambda_\eta^{(p)} = \det(g_*\omega_{Y/S}^\eta)^p.$$

If $\eta >$ Max $\{\eta_0;$ with $h(\eta_0) = 0\} \cup \{1\}$ then the sheaf $\lambda_\eta^{(p)}$ is ample.

Theorem 4 (Gieseker) *If $\deg(h) = 2$, i.e. if one considers surfaces of general type, then one may start in 3 with*

$$\tilde{\mathfrak{C}}(k) = \{X;\ X \text{ projective normal surface with rational double points, with } \omega_X \text{ ample }\}/\cong.$$

D. Gieseker's result is stronger than stated here. His method implies, that one obtains an ample sheaf of the form $\lambda_{\nu\cdot\mu}^{(p)^{h(\nu)}} \otimes \lambda_\nu^{(p)^{-h(\nu\cdot\mu)\cdot\mu}}$.

In our talk we will only consider the moduli problems $\mathfrak{C}_h(k)$, introduced above. We will sketch the methods which allow to proof the theorems 3 and 4 and discuss some partial results concerning moduli of singular reduced schemes. Some of the results have analogues for higher dimensional varieties with arbitrary polarizations, i.e. for pairs $(X, \mathcal{H})$ where $\mathcal{H}$ is an ample invertible sheaf on X (see [8], p.: 97). We define $(X, \mathcal{H}) \equiv (X', \mathcal{H}')$ if there exists an isomorphism $\tau: X \to X'$ such that $\mathcal{H}$ and $\tau^*\mathcal{H}'$ are numerically equivalent, and $(X, \mathcal{H}) \sim (X', \mathcal{H}')$ if there are isomorphisms $\tau: X \to X'$ and $\tau^*\mathcal{H}' \to \mathcal{H}$.

Theorem 5 ([9], III) *Let $h \in \mathbb{Q}[T_1, T_2]$ be a polynomial of degree n in T_1 and let*

$$\mathfrak{M}(k) = \{(X, \mathcal{H});\ X \text{ projective manifold, } \mathcal{H} \text{ ample invertible and } \omega_X \text{ semi-ample }\}/\simeq.$$

Then there exists a coarse quasi-projective moduli scheme M_h for the moduli problem $\mathfrak{M}_h(k)$ of polarized manifolds $(X, \mathcal{H}) \in \mathfrak{M}(k)$ with

$$h(\alpha, \beta) = \chi(\mathcal{H}^\alpha \otimes \omega_X^\beta) \quad \text{for all} \quad \alpha, \beta \in \mathbb{N}.$$

Assume one has chosen natural numbers ϵ, r, r' and γ such that, for all $(X, \mathcal{H}) \in \mathfrak{M}_h(k)$, one has:

i. $\mathcal{H}^\gamma$ *is very ample and without higher cohomology.*

ii. $\epsilon > c_1(\mathcal{H}^\gamma)^n + 1$.

iii. $r = \dim_k(H^0(X, \mathcal{H}^\gamma))$ *and* $r' = \dim_k(H^0(X, \mathcal{H}^\gamma \otimes \omega_X^{\epsilon\cdot\gamma}))$.

Then, for some $p > 0$, *there exists an ample invertible sheaf* $\lambda^{(p)}_{\gamma,\epsilon\cdot\gamma}$ *on* M_h *with: For* $(g : Y \to S, \mathcal{L}) \in \mathfrak{M}_h(S)$ *let* $S \to M_h$ *be the induced morphism. Then*

$$\varphi^*\lambda^{(p)}_{\gamma,\epsilon\cdot\gamma} = \det(g_*(\mathcal{L}^\gamma \otimes \omega_{X/S}^{\epsilon\cdot\gamma}))^r \otimes \det(g_*\mathcal{L}^\gamma)^{-r'}.$$

One may replace the moduli problem $\mathfrak{M}(k)$ in 5 by any submoduli problem, as long as the additional conditions are deformation invariants. If one considers in this way moduli of abelian varieties, K-3 surfaces, Calabi-Yau manifolds or, more generally, if for all manifolds X in $\mathfrak{M}_h(k)$ one knows that $\omega_X^\delta = \mathcal{O}_X$, then the moduli scheme M_h in 5 carries an ample sheaf $\lambda^{(p)}$ with $\varphi^*\lambda^{(p)} = g_*\omega_{X/S}^{\delta\cdot p}$.

Finally, building up on 5 one obtains the existence of a coarse moduli scheme for polarized manifolds up to numerical equivalence.

Theorem 6 ([10]) *Given* $\mathfrak{M}$ *and* h *as in 5, there exists a coarse quasi-projective moduli scheme* P_h *for* $\mathfrak{P}_h = \mathfrak{M}_h/\equiv$.

The construction of P_h is done by using moduli of abelian varieties with a given finite morphism to a fixed quasi-projective scheme. This scheme will be the moduli space M_h from theorem 5. One can give an explicite ample sheaf on P_h, but its definition requires some work.

All the moduli mentioned have been constructed beforehand as analytic or algebraic spaces (Narasimhan-Simha, Tankeev, Artin, Popp, Mumford-Fogarty, Kollár, ..., see [8], Appendix to Chapter 5).

These notes are based on the author's manuscript on "Quasi-Projective Moduli of Polarized Manifolds" which hopefully will appear 1995 as a monograph. The construction of the ample sheaves in 3, 5 and 6, much nicer than those obtained in [9] and slightly better than those of [2], is given there.

1 Moduli schemes and Hilbert Schemes

Starting from $\mathfrak{C}_h(k)$ one defines the functor of canonically polarized manifolds with Hilbert polynomial h by:

$$\mathfrak{C}_h(S) = \{f : Y \to S; f \text{ flat and } f^{-1}(s) \in \mathfrak{C}_h(k), \text{ for all } s \in S\}/\cong,$$

where $\cong$ stands for S-isomorphisms, and $\mathfrak{C}_h(\tau : S' \to S)(f : Y \to S) = (pr_2 : Y \times_S S' \to S')$.

Definition 7 (Mumford, [8]) Given the functor $\mathfrak{C}_h$: (Schemes $/k$) $\to$ (Sets) a scheme C_h will be called a *coarse moduli scheme* for $\mathfrak{C}_h$, if there exists a natural transformation

$$\Theta : \mathfrak{C}_h \longrightarrow \mathrm{Hom}(\ \ , C_h)$$

satisfying:

(a) $\Theta(\operatorname{Spec} k) : \mathfrak{C}_h(k) \to \operatorname{Hom}(\operatorname{Spec} k, C_h) = C_h(k)$ is bijective.

(b) Given a scheme B and a natural transformation $\chi : \mathfrak{C}_h \to \operatorname{Hom}(\ \ , B)$, there is a unique morphism $\psi : M_h \to B$ such that $\chi = \Psi \circ \Theta$, for the natural transformation $\Psi : \operatorname{Hom}(\ \ , M_h) \to \operatorname{Hom}(\ \ , B)$ induced by Ψ.

The construction of moduli schemes starts with A. Grothendieck's Hilbert scheme $\operatorname{Hilb}^l_{h'}$ and with the construction of the *Hilbert scheme* $H \subset \operatorname{Hilb}^l_{h'}$ *of ν-canonically embedded manifolds*:

Boundedness. By Matzusaka's big theorem (see [7]) there exists some $\nu_0 > 0$ such that ω_X^ν is very ample and without higher cohomology, for all $X \in \mathfrak{C}_h(k)$ and for all $\nu \geq \nu_0$.

For $l = h(\nu) - 1$ one obtains an embedding $\iota : X \to \mathbb{P}^l$ and $\mathcal{O}_X(1) = \iota^* \mathcal{O}_{\mathbb{P}^l}(1) = \omega_X^\nu$. In particular $h'(\mu) = \chi(\mathcal{O}_X(\mu))$, for $h'(T) = h(\nu \cdot T)$. Grothendieck constructed in [3] a scheme $\operatorname{Hilb}^l_{h'}$, whose points parametrize subschemes X of $\mathbb{P}^l$ with $h'(\mu) = \chi(\mathcal{O}_X(\mu))$, and a universal family

$$\begin{array}{ccc} X^l_{h'} & \xrightarrow{\ \iota\ } & \mathbb{P}^l \times \operatorname{Hilb}^l_{h'} \\ & {}_f\searrow \quad \swarrow_{pr_2} & \\ & \operatorname{Hilb}^l_{h'} & \end{array}$$

and, step by step, one replaces $\operatorname{Hilb}^l_{h'}$ by the subscheme H whose points $s \in H$ satisfy:

1. $\iota(f^{-1}(s))$ does not lie in a hyperplane.
2. $f^{-1}(s) \in \mathfrak{C}_h(k)$.
3. $\omega^\nu_{f^{-1}(s)} = \mathcal{O}_{f^{-1}(s)}(1)$.

We have to verify, that those conditions really define a subscheme. This is obvious for 1). For 2) one uses that smoothness and connectedness of the fibres are open conditions, and the last one is quite simple to verify. In other terms, $\mathfrak{C}_h$ satsifies:

Local closedness. If $f : Y \to S$ is a flat projective morphism and if $\mathcal{L}$ is an invertible sheaf on Y then there is a locally closed subscheme T of S with:
A morphism $S' \to S$ factors over T if and only if $pr_2 : Y \times_S S' \to S' \in \mathfrak{C}_h(S')$ and if $pr_1^* \mathcal{L} = \omega^\nu_{Y \times_S S'/S'}$.

Let $f : \mathcal{Y} \to H \in \mathfrak{C}_h(H)$ be the family obtained by restricting $X^l_{h'} \to \operatorname{Hilb}^l_{h'}$ to H. The embedding ι induces an isomorphism $\varrho : \mathbb{P}(f_* \omega^\nu_{\mathcal{Y}/H}) \to \mathbb{P}^l \times H$ and for some invertible sheaf $\mathcal{B}$ on H one has $f_* \omega^\nu_{\mathcal{Y}/H} \cong \bigoplus^{h(\nu)} \mathcal{B}$. By the choice of H the universal property of $X^l_{h'} \to \operatorname{Hilb}^l_{h'}$ carries over and, for $g : Y \to S \in \mathfrak{C}_h(S)$ and $\rho : \mathbb{P}(g_* \omega^\nu_{Y/S}) \xrightarrow{\cong} \mathbb{P}^l \times S$, there is a unique morphism $S \to H$, such that $(g : Y \to S, \rho)$ is the pullback of $(f : \mathcal{Y} \to H, \varrho)$. In particular, for $\tau \in \mathbb{P}G = \mathbb{P}Gl(h(\nu), k)$, replacing ϱ by $(\tau \times \mathrm{id}) \circ \varrho : \mathbb{P}(f_* \omega^\nu_{\mathcal{Y}/H}) \to \mathbb{P}^l \times H$, one obtains a morphism $H \to H$, again denoted by τ. The uniqueness implies that $\sigma : \mathbb{P}G \times H \to H$ with $\sigma(\tau, h) = \tau(h)$ is an action of $\mathbb{P}G$ on H.

Definition 8 $\mathbb{P}G$ acts *properly* on H if the induced morphism $\psi = \sigma \times pr_2 : \mathbb{P}G \times H \to H \times H$ is proper.

If one tries to show that ψ is proper by using the "Valuative Criterion for Properness" ([4], II, 4.7), one has to verify that $\mathfrak{C}_h$ satisfies a third condition:

Separatedness If R is a discrete valuation ring with quotient field K and if

$$f_i : X_i \to S = \mathrm{Spec}\,(R) \in \mathfrak{C}_h(S)$$

are two families, then every $\mathrm{Spec}\,(K)$-isomorphism $\delta : (X_1)_K \to (X_2)_K$ extends to an isomorphism $X_1 \cong X_2$ over S.

Again, it is easy to verify the separatedness for $\mathfrak{C}_h$. For some $g : Z \to S$ there are birational morphisms $\varphi_i : Z \to X_i$ with $\delta \circ \varphi_1 = \varphi_2$ and δ induces an isomorphism

$$g_* \omega^{\mu}_{Z/S} = f_{2*} \omega^{\mu}_{X_2/S} \cong f_{1*} \omega^{\mu}_{X_1/S}.$$

However, X_i is isomorphic to $\mathbf{Spec}(\bigoplus_{\mu \geq 0} f_{i*} \omega^{\mu}_{X_1/S})$.

Remark 9 For $\deg h = 2$, i.e. for moduli problems of surfaces, the three properties: Boundedness, Local closedness and Separatedness hold true for the moduli functor $\tilde{\mathfrak{C}}_h$ in 4.

2 Geometric Invariant Theory

Most of the content of this section is due to D. Mumford, [8].

In the last chapter we obtained an action of $\mathbb{P}G = \mathbb{P}Gl(l+1,k)$ or of its finite cover $G = Sl(l+1,k)$ on a scheme H, via $\sigma : G \times H \to H$. We will stick to this example, even if most of the content of this section holds true for all linear reductive groups G.

Definition 10 A scheme M, together with a morphism $\pi : H \to M$ is called a *good quotient of H by G* if

(a) $\pi \circ \sigma = \pi \circ pr_2$

(b) $\mathcal{O}_M = (\pi_* \mathcal{O}_H)^G \subset \pi_* \mathcal{O}_H$

(c) for a G invariant closed subset W of H, the image $\pi(W)$ is closed. Moreover, for two disjoint G-invariant subsets W_1 and W_2, one has $\pi(W_1) \cap \pi(W_2) = \emptyset$.

The sheaf $(\pi_* \mathcal{O}_H)^G$ of G-invariant functions is defined in the following way. For $U \subset M$ a function $f : \pi^{-1}(U) \to \mathbf{A}^1$ is in $(\pi_* \mathcal{O}_H)^G(U)$ if and only if

$$f \circ \sigma = f \circ pr_2 : G \times \pi^{-1}(U) \to \mathbf{A}^1.$$

A good quotient is a *categorial quotient*, i.e. whenever $\epsilon : H \to Z$ is a morphism with

$$\epsilon \circ \sigma = \epsilon \circ pr_2 : G \times H \to Z,$$

then there is a unique morphism $\delta : M \to Z$ with $\epsilon = \delta \circ \pi$.

Since we considered a proper action of G on H, we can add one more property. If G_x denotes the G-orbit of $x \in H$, then $G_x \times \{x\}$ is the image of $G \times \{x\}$ under ψ. Hence the properness of the group action implies that all orbits are closed. The property c) in 10 implies that $\pi^{-1}\pi(x)$ must consist of a single orbit. One finds that a good quotient $\pi : H \to M$ for a proper group action is a geometric quotient:

Definition 11 A good quotient $\pi : H \to M$ is called a *geometric quotient*, if one has in addition:

(d) for every $x \in H$ the fibre $\pi^{-1}(\pi(x))$ is the orbit G_x.

If $H = \operatorname{Spec} A$ is affine, then the existence of quotients is well known. In fact, D. Hilbert has shown that A^G is again an affine k-algebra and $M = \operatorname{Spec} A^G$ is a good quotient. Since we assumed G to act proper, it is even a geometric quotient.

Before being able to present Mumford's criterion for the existence of quotients of quasi-projective schemes under an action of $G = Sl(l+1,k)$, we need the notation of G-linearized invertible sheaves or, for later use, of G-linearized locally free sheaves. If $\mathcal{E}$ is locally free on H we denote by $\mathbf{V}(\mathcal{E}^\vee)$ the induced geometric vector bundle (see [4], II, Ex. 5.18).

Definition 12 A G-linearization of $\mathcal{E}$ is a lifting of the G-action $\sigma : G \times H \to H$ to an action $\sigma' : G \times \mathbf{V}(\mathcal{E}^\vee) \to \mathbf{V}(\mathcal{E}^\vee)$, given by morphisms of vector bundles

$$\sigma'(g, -) : \mathbf{V}(\mathcal{E}^\vee) \longrightarrow \mathbf{V}(\mathcal{E}^\vee).$$

The action σ' induces an isomorphism

$$\phi : \sigma^*\mathcal{E} \xrightarrow{\cong} pr_2\mathcal{E}$$

with certain compatibilities, and usually we will refere to ϕ instead of σ'.

Example 13 If G acts linearly on $\mathbb{P}^M$ and on H and if $\iota : H \to \mathbb{P}^M$ is a G-invariant embedding, then $\iota^*\mathcal{O}_{\mathbb{P}^M}(1)$ has a G-linearization. In fact, $\mathbf{V}(\mathcal{O}_{\mathbb{P}^M}(1))$- zero section is nothing but the natural map $\bigoplus^{M+1} k - (0,\dots,0) \to \mathbb{P}^M$.

Example 14 Let $\delta : G \to Gl(r,k)$ be a representation. Then G acts on $H \times \oplus^r k$ by

$$g(h \times \underline{v}) = g(h) \times \delta(g)(\underline{v}).$$

We will denote the induced G-linearization by

$$\phi_\delta : \sigma^* \bigoplus^r \mathcal{O}_H \longrightarrow pr_2^* \bigoplus^r \mathcal{O}_H.$$

If $\mathcal{L}$ is an invertible sheaf with a G-linearization one can talk about the G-invariant sections $H^0(H,\mathcal{L})^G$. In fact,

$$H^0(H,\mathcal{L}) = \{s : H \longrightarrow \mathbf{V}(\mathcal{L}^{-1}); s \text{ a section }\}$$

and s is G-invariant if $s = s^g$ for

$$s^g : H \xrightarrow{g=\sigma(g,-)} H \xrightarrow{s} \mathbf{V}(\mathcal{L}^{-1}) \xrightarrow{\sigma'(g^{-1},-)} \mathbf{V}(\mathcal{L}^{-1}).$$

Following [8], but changing the notations slightly, we define:

Definition 15 Let $G = Sl(l+1,k)$ be acting on H with finite stabilizers and let $\mathcal{L}$ be a G-linearized sheaf. Then

1. $x \in H$ is called stable with respect to σ and $\mathcal{L}$ if, for some $N > 0$, there is a section $t \in H^0(H, \mathcal{L}^N)^G$ with:
   (a) $H_t = H - V(t)$ is affine, where $V(t)$ denotes the zero locus of t.
   (b) $t(x) \neq 0$
   (c) the induced action of G on H_t is closed.
2. $H(\mathcal{L})^s = \{x \in H; x \text{ stable with respect to } \sigma \text{ and } \mathcal{L}\}$.

If the group G is acting properly, then 1), c) holds true automatically. It is easy to see that $\mathcal{L}|_{H(\mathcal{L})^s}$ is ample. The main property of stable points is:

Theorem 16 (Mumford) *There exists a geometric quotient $\pi : H(\mathcal{L})^s \to M$ and an ample invertible sheaf $\lambda^{(p)}$ on M such that $\mathcal{L}^p = \pi^*\lambda^{(p)}$, for some $p \gg 0$.*

The proof of theorem 16 is quite simple. One constructs for the sets H_t in 15, 1), the geometric quotient M_t using Hilbert's finiteness theorem. It is a categorial quotient and the uniqueness of such allows to glue the M_t to M. It is more difficult to show that the existence of M gives nice functorial properties for the set of stable points:

Theorem 17 (Mumford) *Let $G = Sl(l+1,k)$ act on the scheme H with finite stabilizers and let $\mathcal{L}$ be a G-linearized invertible sheaf.*

1. *If $\iota : H_{\text{red}} \to H$ is the natural morphism then $(H(\mathcal{L})^S)_{\text{red}} = H_{\text{red}}(\iota^*\mathcal{L})^S$.*
2. *If $\iota : H_0 \hookrightarrow H$ is a locally closed G-invariant subscheme then $H_0 \cap H(\mathcal{L})^s \subset H_0(\iota^*\mathcal{L})^s$.*
3. *If in 2) $\mathcal{L}$ is ample on H and if H_0 is projective then $H_0 \cap H(\mathcal{L})^s = H_0(\iota^*\mathcal{L})^s$.*

As in [8] the theorem 17 allows to show that the question whether a given point $x \in H$ belongs to $H(\mathcal{L})^s$, or not, can be decided by studying compactifications of G_x.

Corollary 18 *Assume in 17 that for some $x \in H$ there exists a scheme H', an open embedding $\iota : H \to H'$ and for some $N > 0$ a coherent subsheaf $\mathcal{G}$ of $\iota_*\mathcal{L}^N$ such that:*

(a) The closure $\overline{G_x}$ of G_x in H' is projective.

(b) $\mathcal{G}|_H$ is isomorphic to $\mathcal{L}^N$ and $\mathcal{G}$ is generated by global sections.

(c) *On $\overline{G_x}$ there is an effective Cartier divisor D_x with $(D_x)_{\mathrm{red}} = \overline{G_x} - G_x$ and an inclusion*

$$\mathcal{O}_{\overline{G_x}}(D_x) \longrightarrow (\mathcal{G}|_{\overline{G_x}})/_{\mathrm{torsion}}$$

which is surjective over G_x.

Then $x \in H(\mathcal{L})^s$.

Sketch of proof.
One can find a finite dimensional G-invariant subspace V of $H^0(H, \mathcal{L}^N)$ which generates $\mathcal{G}$ and, replacing N by some multiple, which contains a section t_0 with $V(t_0) = \overline{G_x} - G_x$. Replacing H' by the closure of H in $\mathbb{P}(V)$ one may assume that $\mathcal{G} = \mathcal{O}_{H'}(1)$. By 17 one obtains

$$\overline{G_x}(\mathcal{O}_{\overline{G_x}}(1))^s = \overline{G_x} \cap H'(\mathcal{O}_{H'}(1))^s \subset H'(\mathcal{O}_{H'}(1))^s$$

and

$$H \cap H'(\mathcal{O}_{H'}(1))^s \subset H(\mathcal{O}_H(1))^s = H(\mathcal{L})^s.$$

Hence one may assume that H consists of one orbit G_x and that $H' = \overline{G_x}$. Let t_0 be the global section of $\mathcal{O}_{\overline{G_x}}(1)$ with $V(t_0) = \overline{G_x} - G_x$. Replacing the zero divisor D of t_0 by a finite sum of conjugates one may assume that D is G-invariant. Since the image of a character $Sl(l+1,k) \to k^*$ is finite one may assume that t_0 is G-invariant. By definition of stability one obtains that $x \in \overline{G_x}(\mathcal{O}_{\overline{G_x}}(1))^s$. □

3 Weak positivity and Stability

Definition 19 Let Z be a reduced quasi-projective scheme and let $\mathcal{G}$ be a locally free sheaf on Z. We call $\mathcal{G}$ *weakly positive* (over Z) if for all ample invertible sheaves $\mathcal{H}$ on Z and for all $\alpha > 0$ the sheaf $S^\alpha(\mathcal{G}) \otimes \mathcal{H}$ is ample.

In [9] we had to use a stronger positivity condition. In the forthcoming monograph we get along with "weak positivity", as defined above. However, in order to prove the positivity results needed in the next section, one better starts with the slightly more general definition "weakly positive over a subscheme". Weakly positive sheaves have properties similar to those of ample sheaves. In particular one can show:

Properties 20

1. Direct sums and tensor products of weakly positive sheaves are weakly positive.
2. A locally free sheaf $\mathcal{G}$ is weakly positive (over Z) if and only if the same holds true for $S^\mu(\mathcal{G})$, for some $\mu > 0$.
3. If $\mathcal{G}$ is weakly positive then $\Lambda^\mu \mathcal{G}$ and all locally free quotient sheaves are weakly positive.
4. If $\tau : Z' \to Z$ is a finite surjective morphism and if the trace map induces a splitting of $\mathcal{O}_Z \to \tau_* \mathcal{O}_{Z'}$, then a locally free sheaf $\mathcal{G}$ on Z is weakly positive if and only if $\tau^* \mathcal{G}$ is weakly positive.

5. If Z is proper and $\mathcal{G}$ locally free on Z, then $\mathcal{G}$ is weakly positive over Z if and only if $\mathcal{G}$ is numerically effective, i.e. for all morphisms $\gamma : C \to Z$ from a proper non-singular curve and for all invertible quotients Q of $\gamma^*\mathcal{G}$ one has $\deg(Q) \geq 0$.

6. If Z is proper, $\tau : Z' \to Z$ finite, surjective and if $\mathcal{G}$ is locally free on Z, then $\mathcal{G}$ is weakly positive if and only if $\tau^*\mathcal{G}$ is weakly positive.

To formulate the stability criterion we consider projective schemes $\bar{G}$ and $\bar{H}$, containing G and H as open dense subschemes, and a quasi-projective scheme Z, containing $G \times H$, such that:

1. The morphisms $\sigma : G \times H \to H, pr_1 : G \times H \to G$ and $pr_2 : G \times H \to H$ extend to morphisms $\varphi : Z \to \bar{H}$, $p_1 : Z \to \bar{G}$ and $p_2 : Z \to \bar{H}$.
2. For $U = \varphi^{-1}(H)$ and $V = p_2^{-1}(H)$ the morphisms $\varphi_U = \varphi|_U$ and $p_{2V} = p_{2|_V}$ are projective.
3. $Z = U \cup V$ and $G \times H = U \cap V$.

Such a Z is easily constructed as a subscheme of $G \times H \times H$. For the last property "$G \times H = U \cap V$" one needs that G acts properly.

Theorem 21 *Let $G = Sl(l+1, k)$ act properly on a reduced scheme H. Let $\mathcal{L}$ be an ample invertible sheaf on H, G-linearized by $\phi : \sigma^*\mathcal{L} \to pr_2^*\mathcal{L}$ and let $\delta : G \to Sl(r, k)$ be a representation with finite kernel. Assume for some $\bar{H}, \bar{G}$ and Z, as above, there is a locally free sheaf $\mathcal{F}$ on Z with:*

> *(a) There are isomorphisms $\gamma_U : \varphi_U^* \bigoplus^r \mathcal{L} \to \mathcal{F}_U = \mathcal{F}|_U$ and $\gamma_V : p_{2V}^* \bigoplus^r \mathcal{L} \to \mathcal{F}_V = \mathcal{F}|_V$ such that $\Phi = \gamma_V^{-1}|_{U\cap V} \circ \gamma_U|_{U \cap V}$ is the G-linearization $\phi_\delta \otimes \phi$ (see 14).*
>
> *(b) $\mathcal{F}$ is weakly positive over Z.*

Then $H = H(\mathcal{L})^s$.

Addendum 22 *Assume that in addition there is an ample invertible sheaf λ on H, G-linearized by $\phi' : \sigma^*\lambda \to pr_2^*\lambda$, and an invertible sheaf Λ on Z with :*

> *(c) There are isomorphisms $\gamma'_U : \varphi_U^*\lambda \to \Lambda_U = \Lambda|_U$ and $\gamma'_V : p_{2V}^*\lambda \to \Lambda_V = \Lambda|_V$ such that $\gamma_V'^{-1}|_{U\cap V} \circ \gamma'_U|_{U\cap V}$ is the G linearization ϕ'.*
>
> *(d) For some $\alpha, \iota > 0$ the sheaf $\Lambda^\iota \otimes \det(\mathcal{F})^{-\alpha}$ is weakly positive over Z.*

Then $H = H(\lambda)^s$.

Sketch of the proof.
One has the isomorphism $\gamma_V : \bigoplus^r \mathcal{L}_V \to \mathcal{F}_V$ or, equivalently, a morphism $\epsilon_V : \mathcal{L}_V \to \bigoplus^r \mathcal{F}_V$, which splitts locally. Blowing up Z, if necessary, $\mathcal{L}_V$ extends to an invertible sheaf $\mathcal{M}$ on Z and ϵ_V to a locally splitting inclusion $\epsilon : \mathcal{M} \to \bigoplus^r \mathcal{F}$. If $\underline{s} : \bigoplus^r \mathcal{M} \to \mathcal{F}$ denotes the induced morphism let D be the zero divisor of $\det(\underline{s})$, hence $\mathcal{M}^r = \det(\mathcal{F}) \otimes \mathcal{O}_Z(-D)$. The dual of ϵ induces a surjection

$$S^r \bigoplus^r \bigwedge^{r-1} \mathcal{F} = S^r \bigoplus^r (\mathcal{F}^\vee \otimes \det \mathcal{F}) \longrightarrow \mathcal{M}^{-r} \otimes \det(\mathcal{F})^r = \det(\mathcal{F})^{r-1} \otimes \mathcal{O}_Z(D).$$

By 20, the sheaf on the left hand side is weakly positive and hence the sheaf $\det(\mathcal{F})^{r-1}\otimes\mathcal{O}_Z(D)$, as well. The assumption d) implies that $\Lambda^{r'}\otimes\mathcal{O}_Z(D')$ is weakly positive, for some $r'>0$ and for some multiple D' of D. The main observation is:

Claim 23 $U - D'_{\text{red}} = U\cap V = G\times H$.

Proof. For $x\in H$, the fibre $U_x=\sigma^{-1}(x)$ in $G\times H$ is isomorphic to G and $\overline{U_x}=\varphi^{-1}(x)$ is a compactification of this fibre. Assume, blowing up $\overline{U_x}$, if necessary, that the finite map $G\to \mathbb{P}Gl(l+1,k)$ extends to a proper morphism $\overline{U_x}\to\mathbb{P}$ where $\mathbb{P}=\mathbb{P}(\bigoplus^r k^r)$ is the usual compactification of $\mathbb{P}Gl(l+1,k)$. On $\mathbb{P}$ one has the tautological map $\mathcal{O}_{\mathbb{P}}(-1)\to\bigoplus^r\mathcal{O}^r_{\mathbb{P}}$ and the induced universal bases $\underline{s}':\bigoplus^r\mathcal{O}_{\mathbb{P}}(-1)\to\mathcal{O}_{\mathbb{P}}^{\oplus r}$. The property a) in 26 implies that the pullback of $\underline{s}'$ to $\overline{U_x}$ is the same as $\underline{s}|_{\overline{U_x}}$. Hence the pullback $\overline{U_x}-U_x$ of the zero divisor $\mathbb{P}-\mathbb{P}G$ of $\det(\underline{s}')$ is the same as $D|_{\overline{U_x}}$. □

Let us collect what we know up to now. Blowing up $\bar{H},\bar{G}$ and Z, one finds an extension $\bar{\lambda}$ of λ to $\overline{H}$ and an ample sheaf on Z of the form $p_2^*\bar{\lambda}\otimes\mathcal{O}_Z(A)$, for some divisor A on Z, supported outside of $G\times H$. One may choose $\bar{\lambda}$ and A, such that $\gamma_V'^{-1}:\Lambda_V\to p_{2V}^*\lambda$ extends to an inclusion $\Lambda\to p_2^*\bar{\lambda}$. Hence for some divisor F supported in $Z-V$, the sheaf $\Lambda\otimes\mathcal{O}_Z(A+F)$ is ample. By definition of weak positivity, the same holds true for all $\alpha>0$ and for

$$\Lambda^{1+r'\cdot\alpha}\otimes\mathcal{O}_Z(A+F+\alpha\cdot D').$$

If $x\in H$ is a given point, this sheaf, restricted to $\overline{U_x}$ is of the form $\mathcal{O}_{\overline{U_x}}(\Delta_\alpha)$ where Δ_α is a divisor, supported in $\overline{U_x}-U_x$, with high multiplicities, for $\alpha\gg 0$. The second projection maps $\overline{U_x},U_x$ to $\overline{G_x},G_x$ in H. A careful study of the sheaves involved allows to verify that the assumptions of 18 hold true, for some subsheaf $\mathcal{G}$ of $\iota_*\lambda^{N'}$, for $N'\gg 0$. □

4 Geometric Invariant Theory on Hilbert Schemes

Let us return to the moduli functor $\mathfrak{C}_h$ (or $\tilde{\mathfrak{C}}_h$) studied in section 1 to the Hilbert scheme H and to the universal family $f:\mathcal{Y}\to H\in\mathfrak{C}_h(H)$. Recall that $G=Sl(l+1,k)$ acts properly on H, i.e. the morphism

$$\psi=\sigma\times pr_2:G\times H\to H\times H$$

is proper. The fibres of ψ are isomorphic to the stabilizers of the points and since G is affine, the stabilizers are finite.

Proposition 24 *If $\pi:H\to M$ is a geometric quotient of H by G, then M is a coarse moduli scheme.*

In fact, the property a) in the definition 7 holds true, since $\pi^{-1}\pi(x)=G_x$, and property b) follows since M is a categorial quotient. In order to apply 21 or 22 we need G-linearized sheaves. They exist, since the action σ lifts to an action $\sigma':G\times\mathcal{Y}\to\mathcal{Y}$. One obtains:

Lemma 25 *If for some $\eta>0$ the sheaf $f_*\omega^\eta_{\mathcal{Y}/H}$ is locally free of rank $r(\eta)>0$, then the sheaf $\lambda_\eta=\det(f_*\omega^\eta_{\mathcal{Y}/H})$ is G-linearized. Moreover, for the number ν used to construct H, the sheaf*

$$f_*\omega^\nu_{\mathcal{Y}/H}=\bigoplus^{h(\nu)}\mathcal{B}$$

has a G-linearization $\Phi = \phi_\delta \otimes \phi$, where ψ is a G-linearization of $\mathcal{B}$ and where ψ_δ is induced by the trivial representation.

There is also a natural ample sheaf $\mathcal{A}$ on H:

Lemma 26 *Using the notations from 25, the sheaf*

$$\mathcal{A} = \lambda_{\mu\cdot\nu}^{r(\nu)} \otimes \lambda_{\nu}^{-r(\mu\cdot\nu)\cdot\mu}$$

is ample and G-linearized on H, for $\mu \gg 0$.

$\mathcal{A}$ is the sheaf induced by the Plücker embedding. For $\mu \gg 0$ the multiplication map

$$S^{\mu}(f_*\omega_{Y/H}^{\nu}) \cong \mathcal{B}^{\mu} \otimes S^{\mu}(\bigoplus^{r(\nu)} \mathcal{O}_H) \longrightarrow f_*\omega_{Y/H}^{\nu\cdot\mu}$$

is surjective and the induced map

$$\bigwedge^{r(\nu\cdot\mu)} S^{\mu}(\bigoplus^{r(\nu)} \mathcal{O}_H) \longrightarrow \lambda_{\nu\cdot\mu} \otimes \mathcal{B}^{-\mu\cdot r(\mu\cdot\nu)}$$

induces an embedding of H in some projective space. $\mathcal{A}$ is nothing but the $r(\nu)$-th power of the right hand side.

For curves and surfaces, D. Mumford and D. Gieseker were able to show that $H = H(\mathcal{A})^s$. For $\deg h \geq 3$, we have to look for other ample sheaves on H. This is done by the following theorem, which builds up on results and methods, due to T. Fujita, Y. Kawamata, J. Kollár and the author (see [2]).

Theorem 27 *For $g : Y \to S \in \mathfrak{C}_h(S)$ (or for surfaces, in $\widetilde{\mathfrak{C}}_h(S)$) one has, for $\eta > 0$:*

- *(a)* **Base change and local freeness.** *$g_*\omega_{Y/S}^{\eta}$ is locally free of rank $r(\eta)$ and it is compatible with arbitrary base change.*
- *(b)* **Weak positivity.** *$g_*\omega_{Y/S}^{\eta}$ is weakly positive over S.*
- *(c)* **Weak stability.** *If $r(\nu) > 0$ and if $\eta \geq 2$, then there exists some $\iota > 0$ such that*

$$S^{\iota}(g_*\omega_{Y/S}^{\eta}) \otimes \det(g_*\omega_{Y/S}^{\nu})^{-1}$$

is weakly positive over S.

The proof of theorem 27 is easy, if S is non-singular or if the singular locus of S is proper. The general case, unfortunately, requires some technical constructions, which we are not able to include in this survey.

Sketch of the proof of 5 and 4. In 26 we obtained an ample invertible sheaf

$$\mathcal{A} = \lambda_{\mu\cdot\nu}^{r(\nu)} \otimes \lambda_{\nu}^{-r(\mu\cdot\nu)\cdot\mu}$$

on H. By 27 the sheaves λ_η, as determinants of weakly positive sheaves, are weakly positive and hence $\lambda_{\mu\cdot\nu}$ is ample. The weak stability in 27, for $\nu\cdot\mu$ instead of ν, implies that $f_*\omega_{Y/H}^{\eta}$ is ample, for $\eta > 1$. In particular, for $\eta = \nu$, one obtains that the sheaf $\mathcal{B}$ is ample. Moreover, λ_η is ample, whenever $\eta > 1$ and $r(\eta) > 0$.

We take $\mathcal{L} = \mathcal{B}$ and $\lambda = \lambda_\eta$ in 21 and 22. The sheaves $\mathcal{F}$ in 21 and Λ in 22 are given in the following way:

The lifting of the G-action $\sigma : G \times H \to H$ to $\sigma' : G \times \mathcal{Y} \to \mathcal{Y}$ allows to glue the pullback families $\mathcal{Y} \times_H U$ and $\mathcal{Y} \times_H V$ over $G \times H = U \cap V$ to a family $g : Y \to Z$. Then $\mathcal{F} = g_*\omega^\nu_{Y/Z}$ and $\Lambda = \det(g_*\omega^\eta_{Y/Z})$ satisfy the assumptions made in 21, b) and 22, d). We obtain that $H = H(\lambda_\eta)^s$ and 3 and 4 follow from 16 and 24. □

5 Allowing Singular Fibres

If one wants to study moduli of normal varieties X, with canonical singularities, one runs into the problem that the starting points, the local closedness and the boundedness of the corresponding moduli functors is not known. Assuming those two properties, the other results mentioned up to now remain true. For non normal schemes, the only results are known for curves and surfaces. Let us just indicate in the sequel the necessary assumptions and modifications of the statements.

Definition 28

1. A scheme X is called *$\mathbb{Q}$-Gorenstein of index* N_0 if X is Cohen-Macaulay and if the reflexive hull $\omega_X^{[N_0]} = (\omega_X^{N_0})^{\vee\vee}$ is invertible.
2. A morphism $f : X \to S$ of schemes is called a *family of $\mathbb{Q}$-Gorenstein schemes of index* N_0, if
   (a) $\omega_{Y/S}^{[N_0]}$ is invertible.
   (b) $\omega_{Y/S}^{[r]}$ is flat over S, for all $r > 0$.
   (c) $\omega_{Y/S}^{[r]}|_{f^{-1}(s)} = \omega_{f^{-1}(s)}^{[r]}$, for all $r > 0$.

If $\mathfrak{F}_h(k)$ is a set of connected projective equidimensional $\mathbb{Q}$-Gorenstein schemes X of index N_0, with $\omega_X^{[N_0]}$ ample and with $h(\nu) = \chi(\omega_X^{[N_0]\nu})$, then we define the moduli functor $\mathfrak{F}_h$ by $\mathfrak{F}_h(S) = \{g : Y \to S; g$ family of $\mathbb{Q}$-Gorenstein schemes of index N_0 and all $f^{-1}(s) \in \mathfrak{F}_h(k)\}/\cong$.

Definition 29 A normal variety X is said to have canonical singularities if X is $\mathbb{Q}$-Gorenstein of index N_0 and if for some (hence all) desingularizations $\tau : X' \to X$ one has $\tau_*\omega_{X'}^{N_0} = \omega_X^{[N_0]}$.

In the following discussion we will write $\tilde{\mathfrak{C}}_h(k)$ and $\tilde{\mathfrak{C}}_h$ instead of $\mathfrak{F}_h(k)$ and $\mathfrak{F}_h$, if $\mathfrak{F}_h(k)$ consists of normal varieties with at most canonical singularities. Since two-dimensional canonical singularities are rational double points, this definition is compatible with the notation in 4. For $\deg h \geq 3$, if $\tilde{\mathfrak{C}}_h(k)$ consists of all such varieties, the local closedness and boundedness remains an open problem. The other properties remain true, in particular 27, if one replaces $g_*\omega^\eta_{Y/S}$ by $g_*\omega^{[\eta]}_{Y/S}$ and, if one requires for the "Weak stability" that N_0 divides ν. A slight modification of the constructions indicated gives:

Theorem 30 *If $\widetilde{\mathfrak{C}}_h$ is a locally closed and bounded moduli functor of canonically polarized normal varieties with at most canonical singularities of index N_0, then there exists a coarse quasi-projective moduli scheme $\widetilde{C}_h$ for $\widetilde{\mathfrak{C}}_h$. Let η_0 be a positive integer with $H^0(X,\omega_X^{[\eta]}) \neq 0$ for all $X \in \widetilde{\mathfrak{C}}_h(k)$ and for all $\eta \geq \eta_0$. Then, for some $p \gg 0$, there is an invertible sheaf $\lambda_\eta^{(p)}$ on $\widetilde{C}_h$, such that for $g : Y \to S \in \widetilde{\mathfrak{C}}_h(S)$ and for the induced morphism $\varphi : S \to \widetilde{C}_h$ one has*

$$\varphi^* \lambda_\eta^{(p)} = \det(g_* \omega_{Y/S}^{[\eta]})^p.$$

For $\eta \geq$ Max $\{\eta_0, 2\}$ the sheaf $\lambda_\eta^{(p)}$ is ample.

If one tries to extend those methods to non-normal Gorenstein schemes, one has to add more and more assumptions.

Assumptions 31 Let $\mathfrak{F}(k)$ be a moduli problem of canonically polarized $\mathbb{Q}$-Gorenstein schemes of index N_0. Assume that:

1. $\mathfrak{F}(k)$ is closed under general cyclic coverings, i.e.:
   If $X \in \mathfrak{F}(k)$ and if $\omega_X^{[N_0]M\cdot N}$ is generated by global sections, for some $M, N > 0$, then the variety, obtained by taking the $N \cdot N_0$-th root out of a general section of $\omega_X^{[N_0]M\cdot N}$, belongs again to $\mathfrak{F}(k)$.
2. Each $X \in \mathfrak{F}(k)$ deforms to a normal variety, i.e.:
   there exists an non-singular curve C and a family $f : Y \to C \in \mathfrak{F}(C)$, whose general fibre is normal with at most canonical singularities and with $f^{-1}(c_0) \simeq X$, for some $c_0 \in C$.
3. For each $f : Y \to C \in \mathfrak{F}(C)$, as in 2), the total space Y has at most canonical singularities.

Those assumptions, influenced by those of stable surface in [6] allow to verify the first two properties in 27, the "Local freeness and base change" and the "weak positivity". Using 21 for the sheaf $\mathcal{L} = \mathcal{B} \otimes \lambda_{\nu\cdot\mu}$ one still obtains:

Theorem 32 *If $\mathfrak{F}(k)$ is a moduli problem satisfying the assumptions made in 31 and if for some $h \in \mathbb{Q}[T]$ the moduli problem*

$$\mathfrak{F}_h(k) = \{X \in \mathfrak{F}_h; h(\nu) = \chi(\omega_X^{[N_0]\nu}) \text{ for } \nu \in \mathbb{N}\}/\cong$$

is locally closed, bounded and separated, then there exists a coarse quasi-projective moduli scheme M_h for $\mathfrak{F}_h$. Moreover, using the same notation for the invertible sheaves, as in 30, the sheaf $\lambda_{\nu\cdot\mu}^{(r\cdot p)} \otimes \lambda_\nu^{(p)}$ is ample on M_h, for $p \gg \mu \gg \nu \gg 0$, with ν a multiple of N_0 and with $r = h(\nu \cdot N_0^{-1})$.

6 Projective Moduli via Algebraic Spaces

The only two examples, where the assumptions of the existence criterion 32 are known to hold true are those of stable curves and surfaces.

Example 33 (A. Mayer and D. Mumford)
One can to compactify the moduli scheme of curves of genus $g \geq 2$ by enlarging the moduli

problem, allowing "stable curves". A stable curve X is a connected, reduced curve with only ordinary double points as singularities and with an ample canonical sheaf. The latter condition is equivalent to the following one: If an irreducible component E of X is non-singular and isomorphic to $\mathbb{P}^1$, then E meets the closure of $X - E$ in at least three points. Let $\bar{\mathfrak{C}}(k)$ denote the moduli problem of stable curves. The properties asked for in 32 are well known for this moduli problem (see the references given in [8]) and by 32 we get another proof of the wellknown theorem of Mumford and Knudsen on the existence of a coarse quasi-projective moduli scheme $\bar{C}_g$ for stable curves of genus g, just with a more complicated proof and with an ample sheaf, not as nice as the ones obtained by Mumford.

The stable reduction theorem implies that the moduli problem $\bar{\mathfrak{C}}_g(k)$ is "complete" and hence that $\bar{C}_g$ is projective.

In [6] J. Kollár and N. I. Shepherd-Barron define "stable surfaces" and they verify most of the assumptions stated in 32. Let us recall their definitions.

Definition 34

1. A reduced connected scheme (or algebraic space) Z is called *semismooth* if the singular locus of Z is non-singular and locally (in the étale topology) isomorphic to the zero set of $z_1 \cdot z_2$ in $\mathbb{A}^{n+1}$ (double normal crossing points) or to the zero set of $z_1^2 - z_2^2 \cdot z_3$ in $\mathbb{A}^{n+1}$ (pinch points).

2. A proper birational map $\delta : Z \to X$ between reduced connected schemes (or algebraic spaces) is called a *semiresolution* if Z is semismooth, if for some open dense subscheme U of X with $\mathrm{codim}_X(X - U) \geq 2$ the restriction of δ to $\delta^{-1}(U)$ is an isomorphism and if δ maps each irreducible component of $\mathrm{Sing}(Z)$ birationally to the closure of an irreducible component of $\mathrm{Sing}(U)$.

3. A reduced connected scheme (or algebraic space) X is said to have at most *semi-log-canonical singularities*, if

   (a) X is Cohen-Macaulay.

   (b) $\omega_X^{[N_0]}$ is locally free for some $N_0 > 0$.

   (c) X is semismooth in codimension one.

   (d) For a semiresolution $\delta : Z \to X$ with exceptional divisor $F = \sum F_i$ one has

   $$\delta^* \omega_X^{[N_0]} = \omega_Z^{N_0}(-\sum a_i F_i),$$

   for $a_i \geq -N_0$.

The definition of semi-log-canonical singularities makes sense, since it has been shown in [5], 4.2, that the condition c) in 3) implies the existence of a semiresolution.

Example 35 (J. Kollár, N. I. Shepherd-Barron [6])
Let $\bar{\mathfrak{M}}^{N_0}(k)$ be the moduli problem of smoothable stable surfaces of index N_0. By definition, $\bar{\mathfrak{M}}_{N_0}(k)$ is the set of all schemes X with:

(a) X is a proper reduced scheme, equidimensional of dimension two.

(b) X has at most semi-log-canonical singularities.

(c) The sheaf $\omega_X^{[N_0]}$ is invertible and ample.

(d) For all $X \in \bar{\mathfrak{M}}^{N_0}(k)$ there exists a flat morphism $f : Y \to C$, for some irreducible curve C, such that

   i. All fibres $f^{-1}(c)$ are in $\bar{\mathfrak{M}}^{N_0}(k)$.

   ii. For some $c_0 \in C$ the fibre $f^{-1}(c_0)$ is isomorphic to X.

   iii. The general fibre of f is a normal surface with at most rational double points.

$\bar{\mathfrak{M}}^{N_0}(k)$ satisfies the assumptions made in 31. In fact, by [6], 5.5, semi-log-canonical singularities deform to semi-log-canonical singularities in a flat family $f : Y \to C$, provided $\omega_{Y/C}^{[N]}$ is invertible for some $N > 0$. The condition that $\omega_X^{[N_0]\cdot\nu}$ is very ample and coincides with a given polarization is locally closed, and the smoothability is obviously a closed condition.

The boundedness had been shown by J. Kollár before and the separatedness follows from the constructions in [6] or from the arguments used in the smooth case. The closedness of $\bar{\mathfrak{M}}^{N_0}(k)$ under finite coverings follows by arguments similar to the ones used for canonical singularities. The condition 2) in 31 holds true by definition and the last condition has been shown in [6], 5.1. Hence 32 implies that there exist a coarse quasi-projective moduli scheme for stable surfaces of index N_0 with Hilbert polynomial h.

The way we defined $\bar{\mathfrak{M}}^{N_0}(k)$ we are missing the main point. By [6], the moduli problem $\bar{\mathfrak{M}}(k) = \bigcup_{N_0} \bar{\mathfrak{M}}^{N_0}(k)$ is complete, i.e. for a given family $f_0 : Y_0 \to B_0 \in \bar{\mathfrak{M}}(B_0)$ over a curve B_0 there exists a finite covering C_0 of B_0, a projective curve C, containing C_0 and an extension of $Y_0 \times_{B_0} C_0 \to C_0$ to a family $f' : Y' \to C$. However, it was not at all clear, whether one can bound the index of the singularities of Y' in terms of invariants of the general fibre. This was settled recently by V. Alexeev in [1] and, without giving the exact form of his result, let us just state as a consequence:

Theorem 36 (Alexeev, Kollár) *The moduli problem $\bar{\mathfrak{M}}(k)$ can be written as a disjoint union of complete moduli problems of the form $\bar{\mathfrak{M}}_h^{N_0}(k)$. For each of those there exists a coarse projective moduli scheme $M_h^{N_0}$ and an ample invertible sheaf on $M_h^{N_0}$, using the notation from 30, is given by $\lambda_\nu^{(p)}$, for some multiple ν of N_0 and for p sufficiently large.*

With a different ample sheaf, theorem 36 follows from Alexeev's result and from 32. However, there is a more elegant way, due to J. Kollár, to handle complete moduli problems.

As sketched in [8], p. 172, it is easy to construct quotients in the category of algebraic spaces. Let us be more precise: A k-space is a sheaf of sets on the category *(Affine schemes over k)* for the étale topology. A scheme X gives rise to a k-space by taking $X(U) = \mathrm{Hom}(U, X)$. An equivalence relation $X_1 \rightrightarrows X_0$ in the category of *(k-spaces)* is given by an injection (of sheaves) $\delta : X_1 \hookrightarrow X_0 \times X_0$ such that $\delta : (U) : X_1(U) \to X_0(U) \times X_0(U)$ is an equivalence relation in the category of sets. In the category *(k-spaces)* quotients by equivalence relations exist, and a separated algebraic space is a k-space which is obtained as the quotient of an equivalence relation $\delta : X_1 \hookrightarrow X_0 \times X_0$ with X_1 and X_0 schemes, with δ an closed immersion and with $pr_1 \circ \delta$ and $pr_2 \circ \delta$ étale. A scheme is an algebraic space and one has

$$(Schemes) \longrightarrow (Algebraic\ spaces) \longrightarrow (k - spaces)$$

as full subcategories.

Most of the properties of schemes can be extended to algebraic spaces. One can define ample invertible sheaves on algebraic spaces and the existence of such implies that the algebraic space is, in fact, a scheme. If H is a scheme and G a finite group acting on H, then the image of $\psi : G \times H \to H \times H$ is an equivalence relation, but $pr_i|_{\psi(G\times H)}$ is not étale if G has fixed points. The quotient space is not necessarily an algebraic space (just a stack) but it can be represented coarsely by an algebraic space, in the following sense:

Definition 37 Let X_0 and X_1 be schemes and let $\delta : X_1 \to X_0 \times X_0$ be a morphims such that $\delta(X_1)$ is an equivalence relation. Then the quotient sheaf $\mathcal{G}$ of X_0 by $\delta(X_1)$ is coarsely represented by an algebraic space Z if there is a morphism of sheaves $\Theta : \mathcal{G} \to Z$ on the category *(Affine schemes)* with:

1.
$$\Theta(\operatorname{Spec} k) : \mathcal{G}(\operatorname{Spec} k) = X_0(k)/\delta(X_1(k)) \longrightarrow Z(k) = \operatorname{Hom}(\operatorname{Spec} k, Z)$$
is bijective.

2. If B is an algebraic space and if $\chi : \mathcal{G} \to B$ is a morphism of sheaves, then there is a unique morphism $\Psi : Z \to B$ with $\chi = \Psi \circ \Theta$.

Z is unique up to isomorphisms and we will call Z the quotient of H by G. Generalizing earlier results of P. Deligne on quotients of schemes by finite groups, D. Mumford and J. Fogarty showed in the second edition of [8]:

Theorem 38 *Let G be an algebraic group, acting properly on the scheme H, with finite stabilizers. Then the quotient of H by G exists in the category of algebraic spaces.*

For normal algebraic spaces there is a partial converse, due to M. Artin (see [5]):

Proposition 39 *If Z is a normal algebraic space, then there exists a scheme X and a finite group G, acting on X, such that Z is the quotient of X by G.*

Applying 38 to a Hilbert scheme, one obtains:

Corollary 40 *Let $\mathfrak{F}_h$ be a locally closed, bounded and separated moduli functor of canonically polarized $\mathbb{Q}$-Gorenstein schemes of index N_0. Then there exists a coarse separated algebraic moduli space M_h for $\mathfrak{F}_h$, of finite type over k.*

The definition of a coarse algebraic moduli space is the same as the one for a moduli scheme in 7, except that in the universal property b) one allows B to be an algebraic space.

J. Kollár realized, that the finite cover in 39 can be chosen such that it carries a "universal" family.

Theorem 41 *In corollary 40 M_h has a finite covering X such that:*

(a) X is a normal scheme.

(b) A finite group G acts on X and the quotient of X by G is isomorphic to the normalization $\widetilde{M}_h$ of M_h.

(c) There exists a family $f : Y \to X \in \mathfrak{F}_h(X)$ such that the morphism from $X \to M_h$ is the one induced by f.

Methods similar to those presented in sections 3 and 4 allow to prove that the sheaf

$$\det(f_* \omega_{Y/X}^{[\nu]})^{\alpha} \otimes \det(f_* \omega_{Y/X}^{[\nu \cdot \mu]})^{\beta}$$

are ample on X for $\beta \gg \alpha$ and $\mu \gg \nu \gg 0$. Hence one obtains that $\widetilde{M}_h$, as a quotient of a quasi-projective scheme, is a quasi-projective scheme. If the moduli problem is complete, the algebraic space M_h is compact. In this case, M_h is a projective scheme if and only if $\widetilde{M}_h$ is projective.

Theorem 42 (Kollár [5]) *Assume that the moduli problem in 40 is complete. Let ν be a multiple of N_0 chosen such that $\omega_X^{[N_0]\nu}$ is very ample and without higher cohomology, for all $X \in \mathfrak{F}_h(k)$. Assume moreover, that for projective curves C and for $f : Y \to C \in \mathfrak{F}_h(C)$ the sheaf $f_* \omega_{Y/C}^{[N_0]\cdot \nu}$ is weakly positive over C. Then there is a projective coarse moduli scheme M_h for $\mathfrak{F}_h$ and the sheaf $\lambda_{\nu \cdot N_0}^{(p)}$ is ample on M_h.*

The prove of 42 uses an ampleness criterion for invertible sheaves which is, in some way, an analogue of the stability criterion 21. In order to deduce 36 from 42 it remains to show that the sheaves $f_* \omega_{Y/C}^{[N_0]\cdot \nu}$ are weakly positive for families $f : Y \to C \in \mathfrak{M}(C)$ over curves C. One may assume that C is non-singular. If the general fibre is normal, this is a corollary of the positivity theorems of T. Fujita and Y. Kawamata and the quite technical machinery needed to prove 27 can be avoided. If the general fibre is non-normal, J. Kollár uses in [5] the specific properties of stable surfaces, to verify the assumptions made in 42.

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