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Root systems of homogeneous varieties

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Root Systems of Homogeneous Varieties*

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Let G be a complex semisimple group and let $H \subseteq G$ be the group of fixed points of an involutive automorphism of G. Then $X = G/H$ is called a symmetric variety. In [CP], De Concini and Procesi have constructed an equivariant compactification $\overline{X}$ which has a number of remarkable properties, some of them being:

i) The boundary is the union of divisors $D_1, \ldots, D_r$.

ii) There are exactly 2^r orbits. Their closures are the intersections $D_{i_1} \cap \ldots \cap D_{i_s}$ (even schematically). In particular, there is only one closed orbit.

iii) In case G is of adjoint type, all orbit closures are smooth.

It is called the *wonderful embedding of* X or a *complete symmetric variety* and is the foundation for most deeper results about X.

Independently, Luna and Vust developed in [LV] a general theory of equivariant compactifications of homogeneous varieties under a connected reductive group G. In particular, they realized the reason which makes symmetric varieties to behave so nicely: A Borel subgroup B has an open dense orbit in G/H. Varieties with this property are called *spherical.* Luna and Vust were able to describe all equivariant compactifications of them in terms of combinatorial data, very similar to torus embeddings which are actually a special case. They obtained in particular that every spherical embedding has only finitely many orbits (see also [Kn1] for a survey). Nevertheless, the reason for the existence of a compactification with properties i)-iii) remained mysterious.

Then Brion and Pauer established a relation with the automorphism group. They proved in [BP]: A spherical variety $X = G/H$ possesses an equivariant compactification with exactly one closed orbit if and only if $\operatorname{Aut}^G X = N_G(H)/H$ is finite. In this case there is a unique one which dominates all others: The wonderful compactification $\overline{X}$. They also showed that the orbits of $\overline{X}$ correspond to the faces of a strictly convex polyhedral cone $\mathcal{Z}$. Then properties i) and ii) above are equivalent to $\mathcal{Z}$ being simplicial.

*This note is essentially the introduction of my paper [Kn2]. A Postscript and a DVI file of that paper are available by anonymous ftp at daisy.math.unibas.ch (131.152.41.1) in /ftp/pub/knop.

This fact is much deeper and was proved by Brion in [Bri]. In fact he showed much more. Let Γ be the set of characters of B which are the character of a rational B-eigenfunction on X. This is a finitely generated free abelian group. Then the cone $\mathcal{Z}$ is a subset of the real vector space $\mathrm{Hom}(\Gamma, \mathbb{R})$. Brion showed that there is a finite reflection group W_X acting on Γ such that $\mathcal{Z}$ is one of its Weyl chambers. In case of a symmetric variety, W_X is its little Weyl group.

It remains property iii). In the same paper, Brion stated the following

Conjecture 1. *If the automorphism group of X is trivial, then all orbit closures in $\overline{X}$ are smooth.*

Actually, this is only part of his conjecture: Consider the Lie algebra $\mathfrak{h}$ of H as a point of the Graßmannian of $\mathfrak{g} = \mathrm{Lie}\, G$. Then $H = N_G(H)$ implies that H is the isotropy group of this point. Hence, the closure $\widetilde{X}$ of its orbit is a completion of G/H, the so-called Damazure embedding. Brion showed in [Bri] that $\overline{X}$ is the normalization of $\widetilde{X}$ and conjectured that $\widetilde{X} = \overline{X}$ is smooth.

The main result of [Kn2] is a proof of Conjecture 1. Unlike i) and ii), where only the combinatorial structure of $\mathcal{Z}$ matters, this is now a subtle problem of integrality. Each extremal ray of the dual cone $\mathcal{Z}^\vee \subseteq \Gamma \otimes \mathbb{R}$ is spanned by a unique primitive element of the lattice Γ. Let Σ be the set of these elements. Then one can show that $\overline{X}$ is smooth if and only if Σ generates the lattice Γ as a group (in which case, it is even a basis). Hence, Conjecture 1 follows from the following result which establishes a connection between Γ, W_X and the automorphism group of X:

Theorem 2. *There is a canonical inclusion* $\mathrm{Hom}(\Gamma/\langle\Sigma\rangle_\mathbb{Z}, k^*) \hookrightarrow \mathrm{Aut}^G X$.

There is a slightly different way to see this result which is closer to the theory of symmetric varieties. It is well known that the set $\Delta = W_X\Sigma \subseteq \Gamma$ is a root system with Σ as set of simple roots. Therefore, the theorem says, that if $\mathrm{Aut}^G X$ is trivial then Γ is the root lattice of a root system and W_X is its Weyl group. Hence we got almost a generalization of the restricted root system of a symmetric variety. I say "almost" because our root system is always reduced and doesn't have multiplicities.

For simplicity, we restricted ourself so far to spherical varieties. But all concepts generalize to arbitrary G-varieties. The trick is to put everything in relation to the field $k(X)^B$ of B-invariant rational functions, which is just k in the spherical case. For example, instead of taking all of $\mathrm{Aut}^G X$ one considers only the subgroup $\mathfrak{A}(X)$ of those automorphisms which induce the identity on $k(X)^B$. Therefore, we are able to attach a root system and a Weyl group to any variety with G-action.

Let me mention that for quasi-affine varieties X there is a very simple construction of its root system. For this, consider the isotypic decomposition of its algebra of global functions, $k[X] = \oplus_\chi R_\chi$, where χ runs through all dominant weights. This decomposition is usually not a gradation. To measure the deviation we define

$$\mathcal{M}' := \{\alpha \in \mathcal{X}(B) \mid \exists \chi, \eta \in \mathcal{X}(B) : \langle R_\chi R_\eta \rangle_k \cap R_{\chi+\eta-\alpha} \neq 0\}.$$

Let $\mathcal{M}$ be the saturated monoid generated by $\mathcal{M}'$, i.e., the intersection of the cone spanned by $\mathcal{M}'$ and the group generated by $\mathcal{M}'$.

Theorem 3. *The commutative monoid $\mathcal{M}$ is free and the set of free generators is the basis Σ of Δ_X.*

The proof of Theorem 2 is very indirect. A very brief synopsis follows. For every homomorphism $a : \Gamma \to k^*$ which vanishes on Σ we want to construct an automorphism φ of X. Consider the cotangent bundle $T^*_X \to X$. This bundle contains a certain open subset T^0_X which possesses a Galois covering $\widehat{T}_X$ with group W_X. Thus we get

$$\widehat{T}_X \twoheadrightarrow T^0_X \hookrightarrow T^*_X \twoheadrightarrow X.$$

We construct φ in several step by starting at $\widehat{T}_X$. There the whole torus $A = \mathrm{Hom}(\Gamma, k^*)$ acts in a natural way. Hence, A^{W_X} acts on T^0_X. By embedding A^{W_X} into a connected smooth group scheme we can show that the action of a extends to T^*_X in codimension one. The crucial condition is here that a is trivial on Σ. This step is the most technical part of [Kn2]. Then it is fairly easy to show that the automorphism actually extends to all of T^*_X and can be pushed down to X.

References

[Bri] Brion, M.: Vers une généralisation des espaces symétriques. *J. Algebra* **134** (1990), 115–143

[BP] Brion, M.; Pauer, F.: Valuation des espaces homogènes sphériques. *Comment. Math. Helv.* **62** (1987), 265–285

[CP] De Concini, C.; Procesi, C.: Complete symmetric varieties. In: *Invariant theory, Proceedings, Montecatini.* (F. Gherardelli, ed.) Lect. Notes Math. **996**, Heidelberg-Berlin-New York: Springer-Verlag 1983, 1–44

[Kn1] Knop, F.: The Luna-Vust theory of spherical embeddings. In: *Algebraic groups and its Applications, Hyderabad 1989.* (S. Ramanan, ed.) Madras: Manoj Prakashan 1991, 225–249

[Kn2] Knop, F.: Automorphisms, root systems, and compactifications of homogeneous varieties. To appear in: *J. AMS* (1994), 25 pages

[LV] Luna, D.; Vust, Th.: Plongements d'espaces homogènes. *Comment. Math. Helv.* **58** (1983), 186–245