INTERNATIONAL ATOMIC ENERGY AGENCY
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INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS
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SMR.770/12

ADVANCED WORKSHOP ON ALGEBRAIC GEOMETRY

(15 - 26 August 1994)

Representations of reductive groups in cohomology spaces of vector bundles

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These are preliminary lecture notes, intended only for distribution to participants

REPRESENTATIONS OF REDUCTIVE GROUPS IN COHOMOLOGY SPACES OF VECTOR BUNDLES

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The following notes are a version of my paper "Représentations des groupes réductifs dans des espaces de cohomologie" to appear in the Mathematische Annalen, 1994. Some of the statements given here are stronger than in the original paper.

0. Introduction and statement of the results.

Consider a connected reductive algebraic group G over $\mathbb{C}$, acting on an algebraic variety X. Let E be a G-vector bundle over X; then any cohomology group $H^i(X, E)$ is a rational G-module, see [6]. Therefore, this group decomposes into a direct sum of rational, simple G-modules, with (finite or infinite) multiplicities. In the case where $X = G/B$ is the flag variety of G, it follows easily from Bott's theorem [1] that all multiplicities in $H^i(X,E)$ are at most equal to the rank of E. We generalize this result as follows.

<u>Theorem A</u>. Let X be a spherical G-variety. Then there exists a constant $C(X)$ such that for any G-vector bundle E over X, all multiplicities in $H^i(X,E)$ are at most $C(X)\,\mathrm{rank}(E)$.

Recall that a G-variety X is <u>spherical</u> if it is normal and if a Borel subgroup of G has an open orbit in X. The proof of Theorem A is given in the next three sections; it leads to an effective value of $C(X)$, see page 10 below. With the notation of Theorem A, the multiplicity of any simple, rational G-module V in the Euler characteristic of E makes sense. Denote by Λ the set of isomorphism classes of simple, rational G-modules V_λ. Identify Λ with the set of dominant weights of G, i.e. with the intersection of the Weyl chamber C, and the lattice of weights. Then the Euler characteristic of E defines a function

$$\chi(X,E): \Lambda \longrightarrow \mathbb{Z}$$
$$\lambda \longrightarrow \sum_{i\geq 0}(-1)^i[\text{mult } V_\lambda \text{ in } H^i(X,E)].$$

<u>Theorem B</u> Notation being as above, there exists a finite collection of convex polyhedra $(P_i)_{i\in I}$ in C, and of integers $a_i \in \mathbb{Z}$ such that $\chi(X,E) = \sum_{i\in I} a_i \, 1_{P_i}$.

The proof of Theorem B is given in [4]. Finally, the boundedness of multiplicities in cohomology groups of G-vector bundles, fails for non-spherical varieties. Namely, we have:

<u>Proposition.</u> Let X be a G-variety where no Borel subgroup has a dense orbit. Then there exists a smooth G-variety $\tilde{X}$ which is G-birational to X, and a G-line bundle L over $\tilde{X}$, such that the multiplicity of the trivial G-module in $H^0(\tilde{X},L)$ or in $H^1(\tilde{X},L)$ is infinite.

The proof of this result is given in [4].

1. Local cohomology sheaves of vector bundles over spherical varieties.

Consider a spherical G-variety X, a G-orbit Y in X, and a locally free G-sheaf $\mathcal{E}$ on X. Then we have local cohomology sheaves $\mathcal{H}^i_Y(X, \mathcal{E})$ for all $i \geq 0$. Before studying these sheaves, recall three results on spherical varieties.

(i) [7], [3] Any spherical variety has rational singularities. In particular, it is Cohen-Macaulay.

(ii) [2] Any spherical variety contains only finitely many G-orbits, and these orbits are spherical. In particular, Y contains an open B-orbit Y_0.

(iii) [3] Denote by P the set of all $g \in G$ such that $gY_0 = Y_0$ (a parabolic subgroup of G), and by P_u its unipotent radical. Then there exists a Levi subgroup L of P and an affine, L-stable subvariety $Z \subset X$ such that the map

$$P_u^* \times Z \longrightarrow X : \quad (g, z) \longmapsto g \cdot z$$

is an open immersion. Moreover, $Y \cap Z$ is a unique L-orbit, and the derived subgroup (L, L) acts trivially on $Y \cap Z$.

<u>Proposition 1</u>. Denote by d the codimension of Y in X and by $\mathcal{I}_Y$ the sheaf of ideals of Y. Then we have: $\mathcal{H}^i_Y(X, \mathcal{E}) = 0$ for all $i \neq d$. Moreover, the associated graded sheaf of $\mathcal{H}^d_Y(X, \mathcal{E})$ (for a canonical increasing filtration) satisfies:

$$gr\, \mathcal{H}^d_Y(X, \mathcal{E}) \hookrightarrow \mathcal{H}om_{\mathcal{O}_Y}\Big(\bigoplus_{m=0}^{\infty} \mathcal{I}_Y^m / \mathcal{I}_Y^{m+1}, \ \mathcal{E}xt^d_{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{E}) \Big).$$

<u>Proof</u>. Because X is Cohen-Macaulay and $\mathcal{E}$ is locally free, we have $\mathcal{H}^i_Y(X, \mathcal{E}) = 0$ for $i < d$; see [5] 3.8.

On the other hand, we have by [5] 2.8:

$$\mathcal{H}^d_Y(X, \mathcal{E}) = \varinjlim_m \mathcal{E}xt^d_{\mathcal{O}_X}(\mathcal{O}_X / \mathcal{I}_Y^m, \mathcal{E}).$$

Consider the exact sequence

$$\cdots \to \mathcal{E}xt^{d-1}_{\mathcal{O}_X}(\mathcal{I}_Y^m / \mathcal{I}_Y^{m+1}, \mathcal{E}) \to \mathcal{E}xt^d_{\mathcal{O}_X}(\mathcal{O}_X / \mathcal{I}_Y^m, \mathcal{E}) \to$$
$$\to \mathcal{E}xt^d_{\mathcal{O}_X}(\mathcal{O}_X / \mathcal{I}_Y^{m+1}, \mathcal{E}) \to \mathcal{E}xt^d_{\mathcal{O}_X}(\mathcal{I}_Y^m / \mathcal{I}_Y^{m+1}, \mathcal{E}) \to \cdots$$

We claim that $\mathcal{E}xt^{d-1}_{\mathcal{O}_X}(\mathcal{I}_Y^m / \mathcal{I}_Y^{m+1}, \mathcal{E}) = 0$ and that

$$\mathcal{E}xt^d_{\mathcal{O}_X}(\mathcal{I}_Y^m / \mathcal{I}_Y^{m+1}, \mathcal{E}) \simeq \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{I}_Y^m / \mathcal{I}_Y^{m+1}, \mathcal{E}xt^d_{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{E})).$$

Namely, there is a spectral sequence

$$\mathcal{E}xt^p_{\mathcal{O}_Y}(\mathcal{I}_Y^m / \mathcal{I}_Y^{m+1}, \mathcal{E}xt^q_{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{E})) \Rightarrow \mathcal{E}xt^{p+q}_{\mathcal{O}_X}(\mathcal{I}_Y^m / \mathcal{I}_Y^{m+1}, \mathcal{E}).$$

But $\mathcal{I}_Y^m/\mathcal{I}_Y^{m+1}$ is locally free on Y, hence an isomorphism

$$\mathcal{H}om_{\mathcal{O}_Y}(\mathcal{I}_Y^m/\mathcal{I}_Y^{m+1}, \mathcal{E}xt^q_{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{E})) \simeq \mathcal{E}xt^q_{\mathcal{O}_X}(\mathcal{I}_Y^m/\mathcal{I}_Y^{m+1}, \mathcal{E}).$$

Furthermore, $\mathcal{E}xt^q_{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{E}) = 0$ for $q<d$, hence our claim.

It follows that $\mathcal{H}^d_Y(X, \mathcal{E})$ has an increasing filtration with associated graded: $\mathrm{gr}\, \mathcal{H}^d_Y(X,\mathcal{E}) \hookrightarrow \mathcal{H}om_{\mathcal{O}_Y}(\bigoplus_{m=0}^{\infty} \mathcal{I}_Y^m/\mathcal{I}_Y^{m+1}, \mathcal{E}xt^d_{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{E}))$.

Finally, we check that $\mathcal{H}^i_Y(X, \mathcal{E}) = 0$ for $i>d$.

With the notation of (iii) above, it suffices to check that

$$H^i_{P_u^* \times (Y\cap Z)}(P_u \times Z, \mathcal{E}) = 0 \qquad \text{for } i>d.$$

Denote by $\pi : P_u \times Z \to Z$ the second projection. Then π is affine and $\pi^{-1}(Y\cap Z) = P_u \times (Y \cap Z)$. Therefore, we have by [5] 5.5:

$$H^i_{P_u^* \times (Y\cap Z)}(P_u \times Z, \mathcal{E}) \simeq H^i_{Y\cap Z}(Z, \pi_* \mathcal{E}).$$

Because $\mathcal{E}$ is G-linearized, there exists an L-sheaf $\mathcal{F}$ on Z such that $\mathcal{E}|_{P_u \times Z} = \pi^* \mathcal{F}$. Therefore, we have:

$$\begin{aligned} H^i_{Y\cap Z}(Z, \pi_* \mathcal{E}) &= H^i_{Y\cap Z}(Z, \mathcal{F} \otimes \pi_* \mathcal{O}_{P_u \times Z}) \\ &= H^i_{Y \cap Z}(Z, \mathcal{F}) \otimes H^0(P_u^*, \mathcal{O}_{P_u}). \end{aligned}$$

So, replacing X, Y, G by $Z, Y\cap Z, L$, we may assume that X is affine. Then $H^i_Y(X, \mathcal{E}) \simeq H^{i-1}(X \setminus Y, \mathcal{E})$ for $i \geq 2$.

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By a corollary of Luna's slice theorem, there exists a reductive subgroup $H \subset G$ and an affine H-variety S with a fixed point 0, such that $X = G \times_H S$ and that $Y = G \times_H (0) \cong G/H$. Denote by $p : X \setminus Y \longrightarrow G/H$ the projection. Then $R^i p_* \mathcal{E}$ is the G-sheaf on G/H associated to the H-module $H^i(S \setminus (0), \mathcal{E}|_S)$. It follows that $R^i p_* \mathcal{E} = 0$ for $i > \dim S = d$. By the Leray spectral sequence, this implies: $H^i(X \setminus Y, \mathcal{E}) = 0$ for $i > d$.

2. Local cohomology groups of vector bundles over spherical varieties.

We keep the notation of § 1. We denote by W the Weyl group of G, and by l the length function of W. We set:
$\omega(i) = \# \{ w \in W \mid l(w) = i \}$ and $\omega = \max_{i \geq 0} \omega(i)$.
We denote by $r(Y)$ the <u>type</u> of the Cohen-Macaulay ring $\mathcal{O}_{X,Y}$; then $r(Y)$ is the rank of the $\mathcal{O}_Y$-module $\mathrm{Ext}^d_{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{O}_X)$.

<u>Proposition 2</u>. All multiplicities in $H^i_Y(X, E)$ are at most $\omega \cdot r(Y) \cdot \mathrm{rank}(E)$.

Proof. Recall the spectral sequence ([5] 1.4):

$$H^p(X, \mathcal{H}^q_Y(X,\mathcal{E})) \Rightarrow H^{p+q}_Y(X,\mathcal{E}).$$

By Proposition 1, it degenerates into an isomorphism:

$$H^i_Y(X,\mathcal{E}) \simeq H^{i-d}(Y, \mathcal{H}^d_Y(X,\mathcal{E})).$$

Denote by H the isotropy group of some point of Y. Then there exists a parabolic subgroup Q of G such that $H \subset Q$ and that Q/H is affine. We denote by $\pi: Y = G/H \longrightarrow G/Q$ the canonical morphism. Then π is affine, and therefore:

$$H^{i-d}(Y, \mathcal{H}^d_Y(X,\mathcal{E})) \simeq H^{i-d}(G/Q, \pi_* \mathcal{H}^d_Y(X,\mathcal{E})).$$

Moreover $\pi_* \mathcal{H}^d_Y(X,\mathcal{E})$ is the G-sheaf on G/Q associated to the Q-module $\Gamma(Q/H, \mathcal{H}^d_Y(X,\mathcal{E}))$.

Choose any Levi subgroup M of Q. Then we claim that all multiplicities in the M-module $\Gamma(Q/H, \mathcal{H}^d_Y(X,E))$ are at most $r(Y)$. rank(E). This claim implies easily Proposition 2 by computing cohomology on G/Q with Bott's theorem (see [4] §2 for more details).

By Proposition 1, it is enough to prove the claim for the M-module $\Gamma(Q/H, \mathcal{H}om_{\mathcal{O}_Y}(\bigoplus_{n=0}^{\infty} \mathcal{I}_Y^n/\mathcal{I}_Y^{n+1}, \mathcal{E}xt^d_{\mathcal{O}_X}(\mathcal{O}_Y,\mathcal{E})).$

Choose a Borel subgroup B_M of M. Denote by Q^{op} the parabolic subgroup of G such that $Q \cap Q^{op} = M$. Then $Q_u^{op} B_M$ is a Borel subgroup of G, and moreover the map $Q_u^{op} \times Q/H \longrightarrow G/H$ is an open immersion. It follows that B_M has an open orbit in Q/H; denote this orbit by $\mathcal{O}$. Then

$\Gamma(Q/H, \mathcal{F}) \hookrightarrow \Gamma(\mathcal{O}, \mathcal{F})$ where $\mathcal{F} := \mathcal{H}om_{\mathcal{O}_Y}\left(\bigoplus_{m=0}^{\infty} \mathcal{I}_Y^m/\mathcal{I}_Y^{m+1}, \mathcal{E}xt^d_{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{E})\right)$.

So, to prove the claim, it is enough to check that the multiplicities of all B_M-eigenspaces of $\Gamma(\mathcal{O}, \mathcal{F})$ are at most $r(Y) \cdot \operatorname{rank}(E)$.

Observe that $Q_u^{op} \times \mathcal{O} \xrightarrow{\sim} Q_u^{op} \mathcal{O} = Y_0$ (the open B-orbit in Y). So we are reduced to show that all multiplicities of B-eigenvectors in $\Gamma(Y_0, \mathcal{F})$ are at most $r(Y) \cdot \operatorname{rank}(E)$.

Now, as in the proof of Proposition 1, we may assume that X is affine and that Y is a fixed point. Then

$\Gamma(Y_0, \mathcal{F}) \simeq \operatorname{Hom}_{\mathbb{C}}\left(\Gamma(X, \mathcal{O}_X), \operatorname{Ext}^d_X(\mathbb{C}, \mathcal{E})\right)$.

But the G-module $\Gamma(X, \mathcal{O}_X)$ is multiplicity-free; on the other hand, $\dim_{\mathbb{C}} \operatorname{Ext}^d_X(\mathbb{C}, \mathcal{E}) = r(Y) \operatorname{rank}(E)$. Now it is easy to check that the tensor product of a multiplicity-free G-module by an n-dimensional G-module, has all its multiplicities at most n. This concludes the proof.

3. Cohomology groups of vector bundles over spherical varieties.

Theorem 1. Consider a spherical variety X and a G-vector bundle E over X. For any G-orbit Y in X, denote by $r(Y)$ the type of the (Cohen-Macaulay) ring $\mathcal{O}_{X,Y}$. Denote by ω the maximum of the numbers of elements of the Weyl group of G with a given length. Then all multiplicities in the G-modules $H^i(X, E)$ are at most $\sum_{Y \subset X} r(Y) \cdot \omega \cdot \operatorname{rank}(E)$.

Proof. This follows at once from Proposition 2, using the long exact sequence $\dots \to H^i_Y(X,E) \to H^i(X,E) \to H^i(X \setminus Y, E) \to \dots$

Theorem 2. Consider a (possibly non-normal) spherical variety X with a locally free G-sheaf $\mathcal{E}$ on X, and a coherent G-sheaf $\mathcal{F}$ on X. Then there exists a constant $C(X, \mathcal{F})$ such that all multiplicities in the G-modules $H^i(X, \mathcal{F} \otimes \mathcal{E})$ are at most $C(X, \mathcal{F}) \operatorname{rank}(\mathcal{E})$.

The proof of Theorem 2 is quite uneffective; it would be interesting to have an explicit value for $C(X, \mathcal{F})$.

Proof. Call a coherent G-sheaf $\mathcal{F}$ good if the statement of Theorem 2 holds for $\mathcal{F}$. Observe that given a long exact sequence $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \cdots \to \mathcal{F}_n \to 0$ of coherent G-sheaves, such that $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_{i-1}, \mathcal{F}_{i+1}, \ldots, \mathcal{F}_n$ are good, then $\mathcal{F}_i$ is good, too. In particular, if $\mathcal{F}$ has a finite resolution by locally free G-sheaves, then $\mathcal{F}$ is good by Theorem 1. Therefore, Theorem 2 holds in the case where X is smooth and quasi-projective.

In the general case, we can find a finite filtration

$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_n = \mathcal{F}$$

by coherent G sheaves, such that all subquotients have an irreducible support. Hence we may assume that $\mathrm{Supp}(\mathcal{F}) := Y$ is irreducible. Now we argue by induction on $\dim(Y)$.

If $\dim(Y)$ is minimal, then Y is a closed G-orbit in X, and hence $\mathcal{F}|_Y$ is locally free on Y. So Theorem 1 applies to $\mathcal{F} \otimes \mathcal{E}|_Y$.

If $\dim(Y)$ is arbitrary, there exists a smooth, quasi-projective spherical variety $\tilde{Y}$ together with a proper, birational G-morphism

$\pi : \tilde{Y} \to Y$. Observe that the natural map $\mathcal{F} \to \pi_*(\pi^* \mathcal{F})$ is an isomorphism at the generic point of Y. Therefore, there is an exact sequence $\quad 0 \to \mathcal{F}_1 \to \mathcal{F} \to \pi_*(\pi^*\mathcal{F}) \to \mathcal{F}_2 \to 0$ where $\mathrm{supp}(\mathcal{F}_1)$ and $\mathrm{supp}(\mathcal{F}_2)$ have smaller dimension.

So it is enough to show that $\pi_*(\pi^*\mathcal{F})$ is good.

Set $\tilde{\mathcal{F}} = \pi^*\mathcal{F}$. By the first part of the proof, $\tilde{\mathcal{F}}$ is good. Moreover, $R^q \pi_* \tilde{\mathcal{F}}$ vanishes at the generic point of Y for any $q \geq 1$, so this sheaf is good by the induction hypothesis.

Now consider the Leray spectral sequence

$$H^p(Y, \mathcal{E} \otimes R^q\pi_* \tilde{\mathcal{F}}) \Rightarrow H^{p+q}(\tilde{Y}, \pi^*\mathcal{E} \otimes \tilde{\mathcal{F}}).$$

Then the statement of Theorem 2 holds for all $E_2^{p,q}$ terms with $q \geq 1$, and hence for all $E_n^{p,q}$ terms ($n \geq 2$, $q \geq 1$). On the other hand, by induction over n, we see that $E_n^{p,0}$ is a quotient of $E_2^{p,0} = H^p(Y, \mathcal{E} \otimes \pi_* \tilde{\mathcal{F}})$ by a G-module for which Theorem 2 holds. Finally, Theorem 2 holds for the abutment $H^*(\tilde{Y}, \pi^*\mathcal{E} \otimes \tilde{\mathcal{F}})$ and hence for $E_n^{p,0}$ with $n >> 0$.

This concludes the proof.

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