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The Verlinde formula
(summary)

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The Verlinde formula (summery)

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The Verlinde formula, conjectured some 6 years ago by the mathematical physicist E. Verlinde [V], expresses the dimension of the vector spaces which appear in a certain type of Quantum Field Theories, the so-called Rational Conformal Field Theories (RCFT's). Following (roughly) Friedan and Schenker, a RCFT can be defined as follows. We are given an auxiliary finite set I with an involution (usually I will be a set of representations of the symmetry algebra of the theory). By a marked curve $(C, \vec{p}, \vec{\alpha})$ we mean a compact Riemann surface C with a distinguished finite subset $\vec{p} = (p_1, \cdots, p_n)$, and a "label" $\alpha_i \in I$ attached to each p_i. A RCFT associates to each marked curve $(C, \vec{p}, \vec{\alpha})$ a finite-dimensional vector space $V_C(\vec{p}, \vec{\alpha})$, in such a way that certain axioms are satisfied.

In order to compute the dimension of these spaces, the physicists have introduced an elegant device, the fusion ring of the RCFT. The dimension of $V_C(\vec{p}, \vec{\alpha})$ is easily determined in terms of the characters (= ring homomorphisms into $\mathbb{C}$) of the fusion ring. The Verlinde conjecture gives a way of computing these characters, using an action of $SL_2(\mathbb{Z})$ on $V_E(\emptyset)$ (E = an elliptic curve) deduced from the axioms.

An important example of RCFT is the Wess-Zumino-Witten (WZW) model, associated to a simple Lie algebra $\mathfrak{g}$ and a positive integer ℓ. The mathematical definition of this theory has been given by Tsuchiya, Ueno and Yamada [T-U-Y]; it involves the representation theory of Kac-Moody algebras. The key result is that the corresponding fusion ring is a quotient of the representation ring $\mathcal{R}(\mathfrak{g})$ of $\mathfrak{g}$ (this seems to be proved only when $\mathfrak{g}$ is of classical or G_2 type). Since the characters of $\mathcal{R}(\mathfrak{g})$ are well known, this allows to write down the complete list of characters of the fusion ring, hence to solve the dimension problem without appealing to the Verlinde conjecture. As an example, when $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$, we find for a curve C of genus g with no marked points

$$\dim V_C(\phi) = \left(\frac{\ell}{2}+1\right)^{g-1} \sum_{k=1}^{\ell+1} \frac{1}{\left(\sin \frac{k\pi}{\ell+2}\right)^{2g-2}} .$$

Since a RCFT can be viewed as a cohomology theory for marked Riemann surfaces, it is natural to ask for a more algebro-geometric definition of the WZW model. Such a definition has been proposed by Witten, and justified rigorously in [B-L] and [F]. Let me restrict for simplicity to the case $\mathfrak{g} = \mathfrak{sl}_r(\mathbb{C})$. Let $\mathcal{SU}_C(r)$ be the moduli space of semi-stable rank r vector bundles on C with trivial determinant. This space carries a naturally defined line bundle $\mathcal{L}$, the determinant bundle (actually every line bundle on $\mathcal{SU}_C(r)$ is a power of $\mathcal{L}$). Then for the WZW model associated to $\mathfrak{g} = \mathfrak{sl}_r(\mathbb{C})$ and ℓ, one has a canonical isomorphism

$$V_C(\phi) \xrightarrow{\sim} H^0(\mathcal{SU}_C(r), \mathcal{L}^\ell) .$$

More generally there is an interpretation of $V_C(\vec{p}, \vec{\alpha})$ in terms of the moduli space of parabolic vector bundles on C (this is due to C. Pauly).

Therefore the Verlinde formula in this case gives the dimension of the spaces $H^0(\mathcal{SU}_C(r), \mathcal{L}^\ell)$. This is an important step towards a better understanding of the projective geometry of the moduli space $\mathcal{SU}_C(r)$ (see e.g. [B1]).

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