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Schubert varieties and Demazure modules

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These are preliminary lecture notes, intended only for distribution to participants

Schubert Varieties and Demazure modules

(V. LAKSHMIBAI)

Let k be an algebraically closed field of arbitrary characteristic. Let V be an n dimensional k-vector space. Identifying V with k^n, let $\{e_i\}_{1\le i\le n}$ denote the standard basis for V. Fix an integer d, $1\le d\le n$. Let $G_{d,n}$ denote the Grassmannian variety

$$G_{d,n} = \{d \text{ dim'l subspaces of } V\}.$$

Then $G_{d,n}$ has a canonical structure of a projective variety and we have the canonical projective embedding (the <u>Plücker embedding</u>)

$$i : G_{d,n} \hookrightarrow \mathbb{P}(\wedge^d V),$$

where $\wedge^d V$ is the d^{th} exterior power of V, namely, $i(W) = [u_1\wedge\cdots\wedge u_d]$, where $\{u_1,\ldots,u_d\}$ is a basis for W. Let

$$I_{d,n} = \{(i)=(i_1,\ldots,i_d) \mid 1\le i_1<\cdots<i_d\le n\}$$

For $(i)\in I_{d,n}$, let $e_{(i)} = e_{i_1}\wedge\cdots\wedge e_{i_d}$. Then $\{e_{(i)}, (i)\in I_{d,n}\}$ is a basis for $\wedge^d V$. Let $\{p_{(j)}, (j)\in I_{d,n}\}$ denote the basis of $(\wedge^d V)^*$ (the linear dual of $\wedge^d V$) dual to $\{e_{(i)}\}$. Then $\{p_{(j)}, (j)\in I_{d,n}\}$ gives a set of projective co-ordinates on $\mathbb{P}(\wedge^d V)$ (the so-called <u>Plücker co-ordinates</u>).

For the Plücker embedding, let R denote the homogeneous co-ordinate ring of $I_{d,n}$

n explicit description ~~of~~ R:

Let $X = \underbrace{V \oplus \cdots \oplus V}_{d}$, and let

$\pi : X \longrightarrow \wedge^d V$ be the map

$(v_1, \ldots, v_d) \longmapsto v_1 \wedge \cdots \wedge v_d$

Then $\pi(X) = \hat{G}_{d,n}$. Thus we obtain an inclusion $R \hookrightarrow k[x_{ij}]_{\substack{1 \le i \le n \\ 1 \le j \le d}}$ $(= k[X])$, and as a subring of $k[x_{ij}]$ R is generated by $\{p(i), (i) \in I_{d,n}\}$, $p(i)$ being the $d \times d$ minor of (x_{ij}) with row indices given by $i_1, \ldots, i_d$.

A presentation for the k-algebra R.

R is the free associative, commutative k-algebra with 1 on the generators $\{p(i), (i) \in I_{d,n}\}$ modulo the quadratic relations given as follows: fix r, $1 \le r \le d$. Then given $(i), (j) \in I_{d,n}$, quadratic relation is given by the corresponding

$$\sum_\sigma (\operatorname{sign}\sigma)\, p_{\sigma(i)}\, p_{\sigma(j)} = 0$$

where σ runs over all permutations of $\{i_r, \ldots, i_d, j_1, \ldots, j_r\}$, and $\sigma(i)$ (resp $\sigma(j)$) is the d-tuple obtained by adjoining the first $d+1-r$ (resp. the last r) entries in

$\sigma(i_r,\dots,i_d,j_1,\dots,j_r)$ to $\{i_1,\dots,i_{r-1}\}$ (resp. $\{j_{r+1},\dots,j_d\}$) ③

(here, if there is a repetition in $\sigma(i)$ (resp. $\sigma(j)$), then $p_{\sigma(i)}$ (resp. $p_{\sigma(j)}$) is understood to be 0. Also, if τ is a permutation of $\{i_1,\dots,i_d\}$, then $p_{\tau(i)}$ is identified with $(\text{sign}\,\tau)\,p_{(i)}$. (A similar remark applies to $p_{(j)}$))).

$G_{d,n}$ as a homogeneous space: For the canonical action of $G = GL_n(k)$ on $\mathbb{P}(\wedge^d V)$, the isotropy at $[e_1\wedge\dots\wedge e_d]$ is given by

$$P_d = \left\{ \left(\begin{array}{c|c} * & * \\ \hline O_{n-d\times d} & * \end{array}\right) \in G \right\}$$

while the orbit through $[e_1\wedge\dots\wedge e_d]$ is precisely $G_{d,n}$. Thus we get an identification

$$G_{d,n} \approx G/P_d$$

Let $T = \{$diagonal matrices in $G\}$. Then the T-fixed points in $G_{d,n}$ are precisely $[e_{i_1}\wedge\dots\wedge e_{i_d}]$, $(i)\in I_{d,n}$.

Let $B = \{$upper triangular matrices in $G\}$.

Schubert varieties in $G_{d,n}$. Take a T-fixed point $[e_{(i)}]$ in $G_{d,n}$, and take the B-orbit $B[e_{(i)}]$. Then $X_{(i)} :=$ Zariski closure of $B[e_{(i)}]$, endowed with the canonical reduced structure is the Schubert variety associated to (i).

(classical definition of $X_{(i)} := \{W \in G_{d,n} \mid \dim(W \cap V_{i_t}) \geq t,\ 1 \leq t \leq d\}$

where V_{i_t} = span of $\{e_1, e_2, \ldots, e_{i_t}\}$).

Thus we have a bijection

$$\{\text{Schubert varieties in } G_{d,n}\} \xleftrightarrow{bij} I_{d,n}$$

The partial order on $\{$Schubert varieties in $G_{d,n}\}$ given by inclusion induces a partial order on $I_{d,n}$, viz, for $(i), (j) \in I_{d,n}$, $(i) \geq (j)$ $\Longleftrightarrow$ $i_t \geq j_t$, $1 \leq t \leq d$.

From now on, we shall denote the elements of $I_{d,n}$ either by Greek or Roman letters. For $w \in I_{d,n}$, say $w = (i_1, \ldots, i_d)$, let $e_w = e_{i_1} \wedge \cdots \wedge e_{i_d}$, and V_w = B submodule of $\wedge^d V$ generated by e_w. Then we have

Fact 1. $\mathbb{P}(V_w)$ is the smallest linear subspace of $\mathbb{P}(\wedge^d V)$ containing X_w (and is called the Demazure module corresponding to w). Under

$$X_w \hookrightarrow G_{d,n} \hookrightarrow \mathbb{P}(\wedge^d V)$$

let R_w denote the homogeneous co-ordinate ring of X_w. Hodge gave a basis for $(R(w))_m$ (the m^{th} graded piece of $R(w)$) in terms of certain monomials in the Plücker co-ordinates p_τ's, which we shall describe now.

<u>Definition 1</u>. A monomial $F = p_{\tau_1} \cdots p_{\tau_m}$ is called <u>standard</u> if

$$(*) : \tau_1 \geq \tau_2 \geq \cdots \geq \tau_m$$

<u>Definition 2</u>. Such a monomial is said to be standard on a Schubert variety $X(\tau)$, if in addition to $(*)$, we have $\tau \geq \tau_1$.

<u>Theorem</u> (Hodge). Monomials of degree m standard on $X(\tau)$ form a basis of $(R(\tau))_m$.

Hodge proved this result in characteristic 0. In positive characteristics, this theorem was proved independently by Musili, Hochster (1972).

The above result has been generalized to Schubert varieties in G/B, G being a semi-simple algebraic group, in the series Geometry of G/P.

Let G be a semi-simple, simply connected algebraic group over k. Let T be a maximal torus, B a Borel subgroup, $B \supset T$.

Let W $(= N_G(T)/T)$ be the Weyl group of G relative to T. Let

$$X(T) = \mathrm{Hom}_{alg\cdot gp}(T, G_m) \quad (\text{the character group of } T).$$

Let $\lambda : T \to G_m$. Then λ induces

$$\lambda : B \to G_m$$

Let L_λ be the associated line bundle on G/B (L_λ is simply the line bundle on G/B associated to the principal B-bundle $G \to G/B$, for the action of B on G_a given by λ). Thus we obtain

$$X(T) \longrightarrow \mathrm{Pic}(G/B)$$

which is in fact an isomorphism (since G is simply connected).

Facts. 1. $H^0(G/B, L_\chi) = \{ f : G \to k \mid f(gb) = \chi(b^{-1}) f(g) \}$

2. $H^0(G/B, L_\chi) \neq 0 \iff L_\chi$ is ample
$\iff L_\chi$ is very ample
$\iff \chi$ is dominant, integral
(i.e., $\chi = \sum a_i \omega_i$, $a_i \in \mathbb{Z}^+$, ω_i being the fundamental weights)

(Here, we suppose χ to be regular, i.e., $a_i \neq 0$, $\forall i$. In case $a_i = 0$ for some i's, then we should replace B by the corresponding parabolic subgroup).

3. The G-module $H^0(G/B, L_\chi)$ is indecomposable

4. In char 0 (G being completely reducible) we obtain from (3) that $H^0(G/B, L_\chi)$ is irreducible. Thus in char 0 $\{H^0(G/B, L_\chi), \chi$ being dominant, integral$\}$ gives <u>all</u> finite dimensional, irreducible G module.

For $w \in W$, let $X(w) = \overline{BwB} \pmod B$ be the <u>Schubert variety</u> in G/B associated to w. Let us now fix a dominant, integral weight λ. Let $V(\lambda)$ be the irreducible G-module over $\mathbb{Q}$ with highest weight λ. Let us fix a highest weight vector, e. For $w \in W$, let $e_w = we$, the extremal weight vector in $V(\lambda)$ of weight $w(\lambda)$. Let $U_{\mathbb{Z}}^+$ be the Kostant $\mathbb{Z}$-form for U^+, namely the $\mathbb{Z}$-sub algebra of U generated by $\{E_{\alpha_i}^{(n)}, n \in \mathbb{N}, \alpha_i$ simple$\}$

Let $V_{\mathbb{Z},w} = U_{\mathbb{Z}}^+ e_w$, and $V_w(\lambda) = V_{\mathbb{Z},w} \otimes_{\mathbb{Z}} k$.

Then we have

<u>Fact 5</u>: $\mathbb{P}(V_w)$ is the smallest linear subspace of $\mathbb{P}((H^0(G/B, L_\lambda))^*)$ containing $X(w)$ (under $X(w) \hookrightarrow G/B \hookrightarrow \mathbb{P}((H^0(G/B, L_\lambda))^*)$);

(Again, we suppose λ is regular; if λ is not regular, we shall work inside G/P for a suitable parabolic subgroup P.)

V_w: = <u>the Demazure module associated to $X(w)$</u>

FACT 6: $V_w = (H^0(X(w), L_\lambda))^*$

Explicit bases have been constructed for $H^0(X(w), L_\lambda)$ in the series Geometry of G/P (generalizing Hodge's result). This has led to very many geometric and representation-theoretic consequences.

<u>Geometric Consequences.</u>

1) <u>Vanishing theorems</u>;
$H^i(X(w), L_\lambda) = 0$, $i \geq 1$, λ being dominant, integral

2) <u>Arithmetic Cohen-Macaulayness</u> and <u>arithmetic normality</u> for Schubert varieties (using deformation techniques)
(Of course, one now has a proof of (1), (2) using Frobenius splitting (thanks to the work of Mehta, Ramanathan, Ramanan, Kempf etc).

3) <u>An explicit knowledge of the ideal theory</u> of Schubert varieties

4) <u>Determination of singular locus</u> of a Schubert variety

5) <u>Results in classical invariant theory</u>

6) <u>Cohen Macaulayness</u> of Variety of complexes, the variety $XY=0, YX=0$, where X (resp Y) is an $m \times n$ (resp. $n \times m$) matrix

7) <u>Determination of the multiplicity at a singular</u> point on a Schubert variety

Representation-theoretic consequences:

1. Character formula. We obtain an explicit, positive-termed character formula for the G-module $H^0(G/B, L_\chi)$ (and also for the T-modules $H^0(X(w), L_\chi)$)
2. Explicit bases for all finite dimensional irreducible G-modules in ch 0.
3. Decomposition formula for $V_{\chi_1} \otimes V_{\chi_2}$ as a direct sum of irreducible G-modules (in ch0 where V_{χ_i} is the irreducible G module with highest weight χ_i).
4. Restriction rule giving the direct sum decomposition into irreducible L-modules (in ch 0) for the L-module V_χ, L being a Levi subgroup of G
5. "Good filtration" for $H^0(G/B, L_{\chi_1}) \otimes H^0(G/B, L_{\chi_2})$

(The results in (3), (4), (5) are due to Littelmann)

In the process of generalizing the above results for Kac-Moody groups, we (L-Seshadri) gave a conjectural formula for $H^0(G/B, L_\lambda)$. We shall denote $(H^0(G/B, L_\lambda))^*$ by V_λ and describe the formula for V_λ. (This conjectural formula has now been shown to be true by Littelmann).

Recall,

Recall

Weyl's character formula

$$\operatorname{ch} V_\lambda = \frac{\sum_\sigma (\operatorname{sign} \sigma)\, e^{\sigma(\lambda+\rho)}}{\sum_\sigma (\operatorname{sign}\sigma)\, e^{\sigma(\rho)}}$$

Demazure character formula: Let $\{s_i, 1 \le i \le n\}$ denote the simple reflections in W. Let

$$L_i : \mathbb{Z}X \to \mathbb{Z}X \quad (X \text{ being the weight lattice})$$

$$L_i(e^\mu) = \frac{e^{\mu+\rho} - e^{s_i(\mu+\rho)}}{1-e^{-\alpha_i}} \cdot e^{-\rho}$$

where $\rho = \frac{1}{2}$ sum of positive roots. Then

$$L_i(e^\mu) = \begin{cases} \sum_{j=0}^{m} e^{\mu - j\alpha_i}, & \text{if } (\mu, \alpha_i^*) = m \ge 0 \\ 0, & \text{if } (\mu, \alpha_i^*) = -1 \\ -\sum_{j=1}^{m} e^{\mu + j\alpha_i}, & \text{if } (\mu, \alpha_i^*) = -(m+1) \le -2 \end{cases}$$

Demazure character formula:

$$\operatorname{ch} V_\lambda = L_{i_1} \circ \cdots \circ L_{i_N}(e^\lambda)$$

$$\operatorname{ch} V_w = L_{j_1} \circ \cdots \circ L_{j_\ell}(e^\lambda)$$

where $s_{i_1} \circ \cdots \circ s_{i_N}$ (resp $s_{j_1} \circ \cdots \circ s_{j_\ell}$) is a reduced expression for w_0 (resp. w) (here w_0 is the longest element in the Weyl group.

Remarks 1. Both $L_{i_1} \cdots L_{i_N}$ and $L_{j_1} \cdots L_{j_\ell}$ are independent of the reduced expression chosen

2. Both Weyl and Demazure character formulas involve massive cancellations.

We shall now describe a character formula for $ch.V_\lambda$ which is positive-termed (and is the only positive-termed ~~character~~ formula in the literature for $ch\,V_\lambda$).

Consider a chain

$$\underline{c}: \tau_0 > \tau_1 > \cdots > \tau_r$$

i.e., $\tau_i > \tau_{i+1}$, and $l(\tau_i) = l(\tau_{i+1}) + 1$

Then one knows, $\tau_{i+1} = \tau_i s_{\beta_i}$, for some positive root β_i. Let $m_\lambda(\tau_{i+1}, \tau_i) = (\lambda, \beta_i^*)$. We call $m_\lambda(\tau_{i+1}, \tau_i)$ as the <u>λ-multiplicity</u> of τ_{i+1} in τ_i

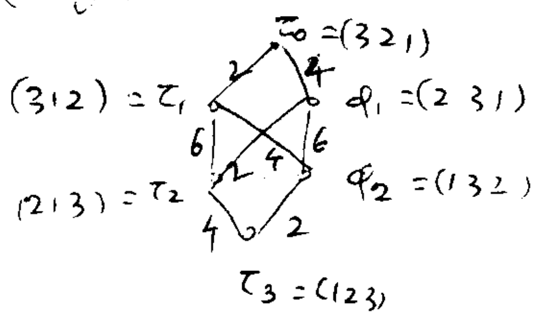
Let us consider $(\underline{c}, \underline{n})$ where $\underline{c}$ is a chain as above and $\underline{n} = \{n_1, \ldots, n_r\}$, $n_i \in \mathbb{Z}^+$.

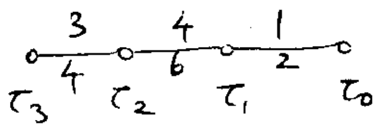
<u>Definition</u>: $(\underline{c}, \underline{n})$ is <u>admissible</u> if

$$1 \geq \frac{n_r}{m_r} \geq \cdots \geq \frac{n_1}{m_1} \geq 0$$

(where $m_i = m_\lambda(\tau_i, \tau_{i-1})$)

(e.g.) $\lambda = 4\omega_1 + 2\omega_2$, $G = SL_3$.





et us denote the distinct values in $\{\frac{n_1}{m_1}, \ldots, \frac{n_r}{m_r}\}$ by $a_1, \ldots, a_s$ so that

$$1 \geq a_s > a_{s-1} > \quad > a_1 \geq 0$$

Let $\{i_0, \ldots, i_s\}$ be defined by

$$i_0 = 0, \quad i_s = r, \quad \frac{n_j}{m_j} = a_t, \quad i_{t-1}+1 \leq j \leq i_t$$

(e.g.) $\underset{\tau_3}{(\circ)} \frac{3}{4} \underset{\tau_2}{(\circ)} \frac{3}{6} \underset{\tau_1}{\circ} \frac{1}{2} \underset{\tau_0}{(\circ)}$

$\{(\tau_0, \tau_2, \tau_3); (\frac{3}{4}, 1/2)\}$

Set $D_{\underline{c}, \underline{n}} = \{(\tau_{i_0}, \ldots, \tau_{i_s}) : (a_1, \ldots, a_s)\}$.

Let $I_\lambda = D_{\underline{c}, \underline{n}} / \sim$, where

$(\underline{c}, \underline{n}) \sim (\underline{c}', \underline{n}')$, if $D_{\underline{c}, \underline{n}} = D_{\underline{c}', \underline{n}'}$

(e.g.) $\underset{\tau_3}{\circ} \frac{3}{4} \underset{\tau_2}{\circ} \frac{1}{2} \underset{\varphi_1}{\circ} \frac{2}{4} \underset{\tau_0}{\circ} \quad \sim \quad \underset{\tau_3}{\circ} \frac{3}{4} \underset{\tau_2}{\circ} \frac{3}{6} \underset{\tau_1}{\circ} \frac{1}{2} \underset{\tau_0}{\circ}$

Remark 1. $m_\lambda(\tau_{i+1}, \tau_i)$ has a geometric meaning, namely, it is the multiplicity with which ~~the divisor~~ $[X(\tau_{i+1})]$ occurs in $[X(\tau_i)] \cdot [p_{\tau_i} = 0]$, p_{τ_i} being the extremal weight vector in $H^0(G/B, L_\lambda)$ of weight $-\tau_i(\lambda)$.

Let $\pi \in I_\lambda$, say $\pi = \{(\mu_0, \ldots, \mu_s); (a_1, \ldots, a_s)\}$.

Then π may be thought of as a piecewise linear path $\pi : [0,1] \longrightarrow X \otimes \mathbb{R}$, namely, say $a_i \le t \le a_{i+1}$, $0 \le i \le r$ (here $a_0 = 0$, $a_{r+1} = 1$).

$$\pi(t) = \sum_{j=0}^{i-1} (a_{j+1} - a_j)\mu_j(\lambda) + (t - a_i)\mu_i(\lambda)$$

Set

$$\nu(\pi) = \sum_{j=0}^{r} (a_{j+1} - a_j)\mu_j(\lambda), \quad \phi(\pi) = \mu_0,$$

$$I_\tau(\lambda) = \{ \pi \mid \phi(\pi) \le \tau \}.$$

<u>L-S (Lakshmibai-Seshadri) character formula:</u>

$$\mathrm{ch}\, V_\lambda = \sum_{\pi \in I(\lambda)} e^{\nu(\pi)}, \quad \mathrm{ch}\, V_\tau = \sum_{\pi \in I_\tau(\lambda)} e^{\nu(\pi)}$$

Note that the above also gives the formula for the multiplicity of a weight χ in V_λ.

<u>Littelmann's operators e_α, f_α:</u>

Let $\mathcal{T}\pi = \{ \pi : [0,1] \to X \otimes \mathbb{R},$ piecewise linear paths such that $\pi(0) = 1 \}$.

<u>Definition</u>: Given π_1, π_2, the concatenation of π_1 and the shifted path $\pi_1(1) + \pi_2$ will be denoted by $\pi_1 * \pi_2$, i.e.,

$$\pi_1 * \pi_2(t) = \begin{cases} \pi_1(2t), & 0 \le t \le 1/2 \\ \pi_1(1) + \pi_2(2t-1), & 1/2 \le t \le 1. \end{cases}$$

Definition For α simple, $s_\alpha \pi(t) := s_\alpha(\pi(t))$

Define $h_\alpha : [0,1] \longrightarrow \mathbb{R}$

$$t \longmapsto (\pi(t), \alpha^*).$$

Let $Q = \min \operatorname{Im} h_\alpha \quad (\leq 0)$.

Let q be minimal such that $h_\alpha(a_q) = Q$

Let y be maximal such that $h_\alpha(t) \geq Q+1$, $t \leq a_y$

(NOTE; $y \leq q$).

We have, $\pi = \pi_1 * \pi_2 * \pi_3$, where

$$\pi_1(t) = \pi(ty)$$
$$\pi_2(t) = \pi(y + t(q-y)) - \pi(y)$$
$$\pi_3(t) = \pi(q + t(1-q)) - \pi(q)$$

Define
$$e_\alpha \pi = \begin{cases} 0, & \text{if } Q = 0 \\ \pi_1 * s_\alpha(\pi_2) * \pi_3, & \text{otherwise} \end{cases}$$

Definition of f_α

Let p be maximal such that $h_\alpha(a_t) = Q$

Let x be minimal such that $h_\alpha(t) \geq Q+1$, $t \geq a_x$

We have, $\pi = \pi_1 * \pi_2 * \pi_3$, where

$$\pi_1(t) = \pi(tp)$$
$$\pi_2(t) = \pi(p + t(x-p)) - \pi(p)$$
$$\pi_3(t) = \pi(x + t(1-x)) - \pi(x)$$

Define
$$f_\alpha(\pi) = \begin{cases} 0, & \text{if } P = 0 \\ \pi * s_\alpha(\pi_2) * \pi_3, & \text{otherwise} \end{cases}$$

Facts. 1. $I_\lambda \cup \{0\}$ is stable under e_α, f_α, α simple.

2. $\pi' = e_\alpha \pi \iff \pi = f_\alpha \pi'$.

3. $I_\lambda = \{\pi \mid \pi = f_{\alpha_1} \circ \cdots \circ f_{\alpha_s} \pi_\lambda\}$, α_i simple, π_λ being the line segment 

Lemma. Let $\tau \in W$, and $\tau = s_{i_1} \cdots s_{i_r}$ (reduced expression). Let $\pi \in I_\tau(\lambda)$. Then π has an unique expression

$$\pi = f_{i_1}^{n_1} \cdots f_{i_r}^{n_r} \pi_\lambda, \quad n_i \geq 0.$$

Theorem: Let the notations be as in the above Lemma. Set

$$G_\pi = F_{i_1}^{(n_1)} \cdots F_{i_r}^{(n_r)} e \qquad (F_\alpha^{(n)} = F_\alpha^n / n!$$

Then

(a) $\{G_\pi, \pi \in I_\tau(\lambda)\}$ is a basis for V_τ

(b) Let $w \leq \tau$. Then $\{G_\pi \mid \phi(\pi) \leq w\}$ is a basis for V_w (Bruhat-order compatibility)

Remark. The above results hold even in the context of quantum groups (in which case $F_\alpha^{(n)}$ will denote the "quantum divided" powers. The transition matrix from $\{G_\pi, \pi \in I_\tau(\lambda)\}$ to Kashiwara's lower global basis for V_τ ($\equiv$ Lusztig's canonical basis for V_τ) is upper triangular with diagonal entries $= 1$.

Announcement of results on the singular locus of a Schubert variety.

As above for a dominant integral weight λ, we set

$$V_\lambda = H^0(G/B, L_\lambda)^*$$

$$V_w(\ell) = V_{\mathbb{Z},w} \otimes k \quad \text{(the Demazure module associated to } w\text{)}.$$

Let $T_{w,\mathrm{Id}}$ be the Zariski tangent space to $X(w)$ at e_{Id}. Let

$$N_w = \{\beta \in R^+ \mid \mathrm{mult}_{\ell-\beta} V_w(\ell) = \mathrm{mult}_{\ell-\beta} V_\ell\}.$$

Theorem. $\dim T_{w,\mathrm{Id}} = \# N_w$

N_w has a particularly simple description for G classical which we give below

Type A_n or D_n: $\quad \beta \in N_w \iff w > s_\beta$

Type C_n. We have

$$R^+ = \{ \epsilon_i + \epsilon_j, 1 \le i < j \le n, \ 2\epsilon_l, \ 1 \le l \le n \}.$$

(1) Let $\beta = \epsilon_i - \epsilon_j$ or $2\epsilon_i$ or $\epsilon_k + \epsilon_l$, $l \neq n$

Then $\beta \in N_w \iff w \geq s_\beta$.

(2) Let $\beta = \epsilon_i + \epsilon_n$. Then

$\beta \in N_w \iff w \geq$ either $s_{\epsilon_i + \epsilon_n}$ or $s_{2\epsilon_i}$

(here, note that $s_{\epsilon_i + \epsilon_n} = s_n s_{n-1} \cdots s_i s_{i+1} \cdots s_n$, and

$s_{2\epsilon_i} = s_i s_{i+1} \cdots s_n s_{n-1} \cdots s_i$).

Type B_n. We have

$$R^+ = \{ \epsilon_i \pm \epsilon_j, 1 \le i < j \le n, \ \epsilon_l, \ 1 \le l \le n \}$$

(1) Let $\beta = \epsilon_i - \epsilon_j$ or ϵ_i or $\epsilon_k + \epsilon_l$, $l \neq n$.

Then $\beta \in N_w \iff w \geq s_\beta$

(2) Let $\beta = \epsilon_i + \epsilon_n$. Then

$\beta \in N_w \iff w \geq$ either $s_{\epsilon_i + \epsilon_n}$ or s_{ϵ_i}

(here again $s_{\epsilon_i + \epsilon_n} = s_n \cdots s_i \cdots s_n$,

$s_{\epsilon_i} = s_i \cdots s_n \cdots s_i$