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I.C.T.P., P.O. BOX 586, 34100 TRIESTE, ITALY, CABLE: CENTRATOM TRIESTE

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A good model for the complement of hyperplanes

C. Procesi
Dipartimento di Matematica
Istituto 'G. Castelnuovo'
Università di Roma 'La Sapienza'
P.le A. Moro 2
00185 Roma
Italy

A GOOD MODEL FOR THE COMPLEMENT OF HYPERPLANES.

C. PROCESI - ROMA

I will discuss some joint work with DE CONCINI. LET V be a finite dimensional vector space over $\mathbb{C}$. Let us give some set of hyperplanes given by linear equations $\{\alpha_i\}_{i\in X}$. In the study of the homotopy group of the complement of these hyperplanes, say V^0, plays an important role a formal connection. One constructs the 1-form $\Omega = \sum t_i \, d\log(\alpha_i)$ where t_i are non commutative indeterminates subject to known commutation relations which imply the flatness of the corresponding connection.

Integrating this connection one finds a function G on the universal cover of the space V^0 with values in the completion of the universal enveloping algebra of the Lie algebra generated by the t_i's.

G is unique up to multiplicative constant and it can be chosen in the following way. One picks a neighborhood U of 0 in $\mathbb{C}^n$ with $n = \dim V$ and a map $\pi : U \to V$ so that the preimage of the hyperplane arrangement in U is the set of coordinate hyperplanes $u_i = 0$, $i = 1, \dots, n$,

$$\pi^* \Omega = \sum_{i=1}^{n} \frac{A_i}{u_i} + \Omega^0(u_1, \dots, u_n)$$

with the A_i commuting with each other and Ω^0 holomorphic on U. Then there is a unique solution G of the form $f(u_1, \dots, u_n) \prod u_i^{A_i}$ with f holomorphic and $f(0) = 1$.

With this discussion in mind it is clear the interest for the following problem.

Find a smooth variety Y containing V^0 such that:

a) The identity of V^0 extends to a proper mapping $Y \longrightarrow V$

b) The complement of V^0 in Y is a divisor with normal crossings.

c) Find also a covering of Y by explicit charts $U_i \subset \mathbb{C}^n$ so that the complement of V^0 intersects U_i in the coordinate hyperplanes.

We present now a solution to this problem.

i) Any intersection of the hyperplanes $\alpha_i = 0$ will be called a stratum.

ii) Given a stratum W let $\alpha_1, \dots, \alpha_k$

be the linear equations in our set which vanish on W. We say W is "reducible" if we can divide these equations in two sets, $\alpha_1 - \alpha_s$ $\alpha_{s+1} - \alpha_k$ so that the two sets span two subspaces which form direct sum, otherwise say W irreducible.

For every irreducible W let P_W be the projective space associated to V/W, we have a map $V^0 \to P(V/W)$ for every W since $W \cap V^0 = \phi$.

Let $j : V^0 \longrightarrow V \times \prod_{W \text{ irreducible}} P(V/W)$ the induced product map.

Theorem Setting $Y = \overline{j(V^0)}$ one solves the previous problem.

Remark The fact that $Y \supset V^0$ and maps properly to V is clear. In order to prove the rest we have to describe a covering of Y.

We need a further definition.

A family $\{W_i\}$ of irreducible strata is "nested" if given $W_1, \ldots, W_k$ in this family and not comparable their intersection is transversal and if $\alpha_i = 0$ contains $W_1 \cap \ldots \cap W_k$ then it must contain one of the W_i's.

Theorem a) A maximal nested set has cardinality n.

b) Given a maximal nested set $W_1, \ldots, W_n$ there is a basis $\alpha_1, \ldots, \alpha_n$ of the dual V^* extracted from the $\{\alpha_i\}_{i \in X}$, such that each α_i is orthogonal to W_i and not orthogonal to the intersection of the W_j's properly containing W_i.

c) Given W_i in a nested set the W_j's contained in it are linearly ordered.

d) For every i set $u_i = \alpha_i$ if W_i is minimal otherwise $u_i = \alpha_i / \alpha_{i'}$ when $W_{i'}$ is the maximal element of the W_j contained properly in W_i then the u_i give a local coordinate chart in Y provided we remove some hypersurfaces not passing through the origin, (corresponding to the other hyperplanes $\alpha_j = 0$), the equations are explicitly obtained by the change of coordinates. These charts cover Y and the complement of V^0 meets the chart U in the coordinate hyperplanes.

Some geometric features of Y.

The irreducible boundary divisors of Y are indexed by the irreducible strata and a set of boundary divisors meets if and only if the corresponding strata form a nested set. Call D_w the divisor indexed by w. The cohomology algebra of Y is generated by the dual classes c_w of the irreducible divisors D_w of the boundary subject to two relations.

i) If $w_1 \dots w_k$ are not nested then $c_{w_1} \dots c_{w_k} = 0$

ii) For every hyperplane α_i we have $\sum_{\alpha_i(w)=0} c_w = 0$.

<u>Example</u> The hyperplanes given by roots in an irreducible root system.
In this case the Weyl group W acts on Y. The irreducible strata up to W conjugacy correspond to connected subset of the Dynkin diagram. A family is nested if and only if non comparable subsets are connected components of their union.
For the case A_n one may also index the maximal nested sets by "non associative" monomials in $n+1$ variables (each variable appearing once). This can be used to link with the theory of Drinfeld's quasi Hopf algebras and the computation of the corresponding monodromy representation of the Braid group.