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I.C.T.P., P.O. BOX 586, 34100 TRIESTE, ITALY, CABLE: CENTRATOM TRIESTE

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Singular Noether-Horikawa surfaces and differentiable invariants

F. Catanese
Dipartimento di Matematica
Università di Pisa
Via Buonarroti 2
56127 Pisa
Italy

Singular Noether-Horikawa Surfaces and Differentiable invariants

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§ 0 Motivations and general considerations

General problems are:

i) Moduli-Spaces $\mathcal{M}$ of Surfaces of general type S with fixed topological type have several connected components: there is a map

$\pi_0(\mathcal{M}) \longrightarrow$ Diff. types of 4-manifolds.

Is this map injective?

iii) Study some beautiful geometry and algebra

ii) Calculate explicitly the Donaldson differentiable invariants of some ~~algebr~~ Diff. 4-manifolds.

Remark. We shall assume throughout $\pi_1(S)=0$.

Lot of work has been successfully done in the case of elliptic surfaces (not of general type) (long list of people.

The simplest series of Surfaces of general type are the N-H surfaces

Noether's ineq. $K^2 \geq 2p_g - 4$ for a minimal model S and the surfaces for which $K^2 = 2p_g - 4$ are the N-H surfaces, which belong to 2 types:

Type C (= Connected branch locus):

Ex. 1 Double cover of $\mathbb{P}^1 \times \mathbb{P}^1$ branched on B of bidegree $(6, 2m)$ (~~and~~ with their limits get the whole family)

Equation: $z^2 = F_{6,2m}(x,y)$

$K^2 = 4(n-2)$ $\quad p_g = 2(m-1)$

Odd Intersection Form

Type-N (Non-connected branch locus)

Ex. 2.

$$S \longrightarrow \mathbb{F}_{2k+2} \qquad \begin{array}{l} \text{——— } \Delta_\infty \\ \text{——— } \Delta_0 \quad \text{cont. 2 sbdles} \end{array}$$

$$\shortparallel$$

$$\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(2k+2))$$

$$\Delta_\infty^2 = -(2k+2)$$

$$\Delta_0^2 = 2k+2 \qquad \Delta_\infty \cdot \Delta_0 = 0$$

Double cover branched on $B = \Delta_\infty + B'$, $B' \in |5\Delta_0|$

$p_g = 4k+2$, $K^2 = 8k$

Intersection - Form odd if k even
even if k odd

So, for $8 | K^2$, $16 \nmid K^2$ ~~th~~ they have different top. type.

Thm. (Horikawa, Freedman) For $16 | K^2$ ~~there~~ there are two connected components.

Question: Are then these surfaces diffeomorphic? (Particular case of i)).

They are both of simple type, i.e. (more or less) they both contain a tight Riemann Surface of genus $g \geq 2$.

Def. Σ is tight if $2g-2 = \Sigma^2$.

Example: If $S/_{\mathbb{R}}$ and $S(\mathbb{R})$ has a smooth orientable component Σ, Σ is tight.

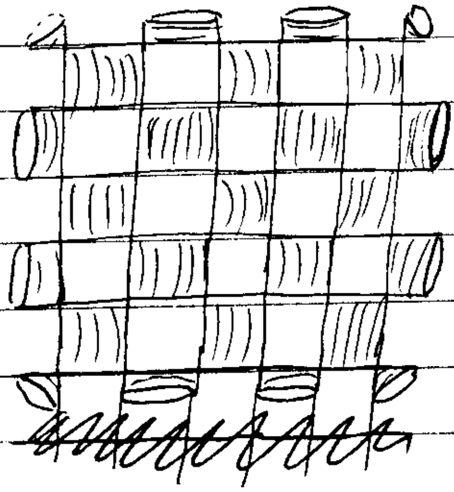
Idea of Proof. Mult by i yields an isom $(T_\Sigma)^\vee$ with N_Σ ∎

E.g. in type C :

Singular Surface X $\quad z^2 = f_6(x)\, g_{2m}(y)$

f, g with real roots.

Picture for $m = 3$



We get S by varying ~~p~~ the branch curve a bit, so nodes are replaced by

and here $S(\mathbb{R})$ is then a conn. smooth. Riem. surf. of genus $5m - 2$.

Kronheimer - Mrowka - Theorem

$\mathcal{M}_U(c, p) =$ Moduli space of A.S.D. connections (for a generic metric g) on a $U(2)$ bdle E with $c_1 = c$,

$$c_1^2 - 4c_2 = p$$

||2

$\mathcal{M}_{PU}(w, p) =$ ... on a $PU(2) \cong SO(3)$ - bdle $P = P(E)$ with

$$w_2 = w = c \pmod 2 \qquad w \in H^2(S, \mathbb{Z}/2)$$

$$p_1 = p$$

These spaces have real dimension

$$2d = -2p - 6(p_g + 1)$$

and define a polynomial $q_d(w)$ of degree $\left(-3(p_g+1) - p\right) = d$ on $H_2(S, \mathbb{Z})$, which is a $\mathcal{C}^\infty$ - invariant

(Donaldson - polynomials)

THM (K-M) If S contains Σ tight (or...)
then, varying p, if you sum these polynomials you build a power series $\mathcal{D}^w$ on $H_2(S, \mathbb{Z})$ s.t.

1) $\exists$ $K_1, \dots K_s \in H^2(S, \mathbb{Z})$ s.t. $K_i \equiv K_S$ (mod 2)
$\beta_1, \dots, \beta_s$ constants $\in \mathbb{Q}$, with

2) $\mathcal{D}^w = \exp(Q/2) \sum_{r=1}^{s} (-1)^{(w^2 + K_r w)/2} \beta_r e^{K_r}$

Problem: what are $K_1, \dots K_r, \beta_1, \dots \beta_r$?

Remark 1 If S has big monodromy (F-M-M), what is not true for N-H surfaces, then all the K_r are rational multiples of K_S.

Remark 2. If $d = 0$ we get a constant

$\mathcal{D}^w(0) = \sum (\pm 1) \beta_r$, depending a priori on w.

§ 1 Statement of our result.

Thm. Let S be a H-N surface of type c). Then there ~~is a~~ are some $w \in H^2(S, \mathbb{Z}/2)$ such that for $d = 0$ we get the constant $2^{2(m-2)}$

Comments

1) A similar result was done by Kronheimer for K-3 surfaces (then the $\# = 1$)

2) The case $m = 3$ ($\# = 4$) was done by Kametani (independent of us)

3) Both Kronheimer and (after him) Kametani used this calculation to show that the surface S admits then ∞-non equivalent Diff. Structures. The same could be done here, I believe.

4) I'm trying to choose other w's such that $\# = 0$ (this would imply : there are at least two K-M classes K_2), the problem is reduced to some cohomological calculation.

5) The method employed applies also to surfaces of type $\mathcal{N}$, for which calculations are more involved (they rely on a sing. model introduced by U. Persson)

§2 The method : Kronheimer's method of orbifolds marked $SO(3)$-representations.

"ORBIFOLD"

X = a normal singular surface with $p_1, \dots, p_s$ as only singular points, R.D.P.'s, i.e. locally $\cong \mathbb{C}^2/G$ $\quad G \subset SL(2,\mathbb{C})$

$\boxed{\text{Eg. } A_n \text{ sings, } G = \mathbb{Z}/n+1, \ g = \begin{pmatrix} \zeta & \\ & \zeta^{-1} \end{pmatrix}, \ \zeta^{n+1} = 1}$

$$X^{\#} = X - \{p_1, \dots, p_s\}$$

$\rho : \pi_1(X^{\#}) \longrightarrow SO(3)$, $P^{\#}_{\rho}$ the associated $PU(2)$-flat bdle

A marking μ_i at p_i : a choice of a local lift of $\rho : \pi_{1,loc}(X^{\#}, p_i) \longrightarrow PU(2) = SO(3)$

$$\begin{tikzcd} \pi_{1,loc}(X^{\#}, p_i) \arrow[r] & PU(2) = SO(3) \\ G \arrow[u, phantom, "\cong" rotate=90] \arrow[r, "\mu_i"'] & U(2) \arrow[u] \end{tikzcd}$$

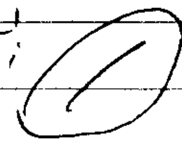
Kronheimer's correspondence : $(\rho, \mu_i) \longmapsto P$ on S

where $S \xrightarrow{\pi} X$ is the min. resolution of singularities.

If p_i is such that $P^\#$ is non trivial in a nbhd. of p_i (TWISTED)

Example (n=1, node) st $\pi^{-1}(p_i) = A_i$, $A_i^2 = -2$

then patch $\pi^*(P^\#)$ on a nbhd U_i'



of A_i with $\mathbb{P}(\mathcal{O}_{U_i'} \oplus \mathcal{O}_{U_i'}(-\frac{1}{2} A_i))$

Changing the factors, the $\mathbb{P}^1$ - bdle is changed to $\mathbb{P}(\mathcal{O}_{U_i'} \oplus \mathcal{O}_{U_i'}(\frac{1}{2} A_i))$, and w_2 is changed by A_i.

The Dimension of moduli space d is given by

$$d = \delta^- - 6(p_g + 1), \quad \delta^- = \# \text{ of twisted nodes}$$

for a bundle P above, one can reverse the construction, define δ^- and

Conversely, if $-2p = \delta^-$, $P^\#$ is induced, for a Hodge metric, by an orbifold representation ρ (Kobayashi-Hitchin correspondence).

Thus we need surfaces with many singular points in order to get in this way moduli spaces of dimension 0.

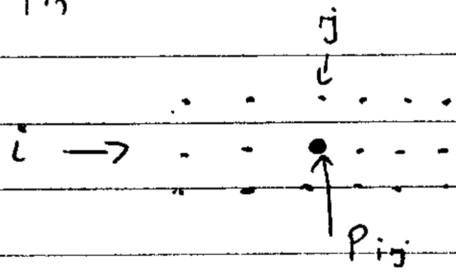
Ex. (Generalized Kummers) $\{z^2 = f_6(x)\, g_{2m}(y)\} = X$

$\delta = 12m$

we get $d = 0 \iff \delta^+ = \delta - \delta^- = 6$.

We need to classify all such representations ρ, their markings, calculate ~~their~~ the $w_2(P)$, ... as follows.

1) We calculate $\pi_1(X^\#)$, we let p_{ij} the node lying on the position i,j



We let ε_{ij} be a standard loop conjugate to a generator of $\pi_{1,loc}(X^\#, p_{ij}) \cong \mathbb{Z}/2$.

Notice that $X = C_1 \times C_2 / \mathbb{Z}/2$, where $C_1 = \{u^2 = f(x)\}$, $C_2 = \{w^2 = g(y)\}$ are hyperelliptic curves, $\mathbb{Z}/2$ acts by $(u,w) \to (-u,-w)$, so $z = uw$ gives the quotient map.

Thus $C_1 \times C_2 - \{q_{ij}\} \underset{def}{=} M^\# \longrightarrow X^\# = X - \{p_{ij}\}$ is an unramified double cover, and we find that

$$\pi_1(X^\#) = [\underbrace{\pi_1(C_1) \times \pi_1(C_2)}_{\Gamma_1 \times \Gamma_2}] \rtimes \mathbb{Z}/2 \quad , \mathbb{Z}/2 \text{ spanned}$$

by ε_{11}, which conjugates some standard generators of Γ_1, Γ_2, $\varepsilon_{11}\varepsilon_{h1}$, $\varepsilon_{11}\varepsilon_{1k}$ to their inverses.

2) To classify the representations $\rho : \pi_1(X^\#) \to SO(3)$ the basic idea is that either $c_{ij} = \rho(\varepsilon_{ij})$ is always non trivial, or we may assume $c_{11} = 1$. Thus ρ is induced from $\Gamma_1 \times \Gamma_2$ and $a_h = \rho(\varepsilon_{11}\varepsilon_{h1})$, $b_k = \rho(\varepsilon_{11}\varepsilon_{1k})$ are elements of order 2 s.t.

each a_ℓ and b_k commute.

Then the classification is reduced to a linear algebra problem.

We get different types of representations:

1) Vertical (resp. horizontal), i.e., $c_{ij} = c_j \ \forall i$. These give a positive dimensional moduli space (of dimension bigger than the expected dimension)

2) Monohooked: $b_k = 1$ or a fixed element b (or similar condition for the a_ℓ's). These also give a pos. dim. moduli space.

3) Kleinian, i.e. ρ maps to a conjugate of the Klein group $\mathcal{K}$ of diagonal matrices in $SO(3)$, $\mathcal{K} \cong \mathbb{Z}/2$, and ρ is not monohooked. Then the moduli space of representations has a single point, since $w_2(P^\#_\rho)$ determines ρ.

Analyzing the markings, one has to determine the code $K' = K \cong \otimes \mathbb{Z}/2 A_{ij}$, given by the set $K' = \{\sum k_{ij} A_{ij} \mid \sum k_{ij} A_{ij} \equiv 0 \ (2) \text{ in } H^2(S,\mathbb{Z})\}$.

One can explicitly describe this code

$$K' = \{(k_{ij}) \mid k_{ij} + k_{rs} + k_{is} + k_{rj} \equiv 0, \ \sum k_{1j} \equiv \sum k_{i1} \equiv 0\}$$

REFERENCES

P.B. Kronheimer: "Instanton invariants and flat connections on the Kummer surface", DUKE Math. J. 64 (1991) 229-24

Y. Kametani. "Simple invariant and differentiable structures on a Horikawa surface", preprint (Jan. '94?)

P.B. Kronheimer – T.S. Mrowka: "Embedded Surfaces and the structure of Donaldson's polynomial invariants", preprint (July '9

F. Catanese, "Generalized Kummer surfaces and differentiab invariants of Noether-Horikawa surfaces I", to appear in Proc. of Conf. for Calabi's 70th birthday, Symposia Math, Cambridge Press