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Affine n-space, $\mathbb{C}^+$-actions and related problems

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These are preliminary lecture notes, intended only for distribution to participants

AFFINE n-SPACE, $\mathbb{C}^+$-ACTIONS AND RELATED PROBLEMS

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ABSTRACT. We describe some basic unsolved problems concerning the algebraic-geometric structure of the complex affine n-space $\mathbb{A}^n$ and its symmetries. They are all related in some way to problems arising from algebraic group actions on $\mathbb{A}^n$. We show that some new results about unipotent group actions and in particular actions of the additive group $\mathbb{C}^+$ on affine n-space lead to interesting constructions and might even produce counterexamples to some of these problems.

§1. Introduction

There is no doubt that the complex affine n-space $\mathbb{A}^n$ $(:= \mathbb{A}^n_{\mathbb{C}})$ is the most fundamental object in affine algebraic geometry. However, surprisingly little is known about its algebraic-geometric properties and in particular about its symmetry group. Following are some of the basic unsolved problems in this context. (We will always work over the field $\mathbb{C}$ of complex numbers.)

- **Characterization Problem.** Find an algebraic characterization of $\mathbb{A}^n$.
- **Embedding Problem.** Is every closed embedding $\mathbb{A}^k \hookrightarrow \mathbb{A}^n$ equivalent to the standard embedding?
- **Jacobian Problem.** Is every polynomial morphism $\varphi\colon \mathbb{A}^n \to \mathbb{A}^n$ of maximal rank an isomorphism?
- **Cancellation Problem.** Does an isomorphism $Y \times \mathbb{A}^k \simeq \mathbb{A}^{n+k}$ imply that Y is isomorphic to $\mathbb{A}^n$?
- **Automorphism Problem.** Give an algebraic description of the group of (polynomial) automorphisms of $\mathbb{A}^n$. E.g. is every automorphism tame?
- **Linearization Problem.** Is every automorphism of $\mathbb{A}^n$ of finite order linearizable?
- **Fixed Point Problem.** Does every reductive group action on $\mathbb{A}^n$ have fixed points?

These problems are clearly not unrelated. For instance, a positive solution of the *Linearization Problem* would imply a positive solution of the *Cancellation Problem*. In fact, if $Y \times \mathbb{A}^k$ is isomorphic to an affine space $\mathbb{A}^N$ consider the action of the cyclic group of order 2 on $\mathbb{A}^N \simeq Y \times \mathbb{A}^k$ induced by $(y, z) \mapsto (y, -z)$. Then $Y \times \{0\}$ is the fixed point set, hence is isomorphic to $\mathbb{A}^{N-k}$ in case the action is linearizable.

Another obvious remark is that all problems above can be formulated in terms of the polynomial ring $\mathbb{C}[x_1, \dots, x_n]$, considered as the algebra of regular functions

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on $\mathbb{A}^n$. For instance, the *Embedding Problem* for the line into the plane solved by ABHYANKAR-MOH and SUZUKI (see §2) has the following equivalent formulation:

If $\varphi, \psi \in \mathbb{C}[t]$ are two polynomials generating the polynomial ring $\mathbb{C}[t]$ and with $\deg \varphi \leq \deg \psi$ *then the degree of ψ has to be a multiple of the degree of φ.*

Finally, all the problems above have an obvious positive solution in dimension $n = 1$. This follows immediately from the following facts:

(a) $\mathbb{A}^1$ is the only smooth algebraic curve which is acyclic,
(b) Every non-constant morphism $\varphi: \mathbb{A}^1 \to \mathbb{A}^1$ is a finite (ramified) covering,
(c) The automorphisms of $\mathbb{A}^1$ are affine transformations.

Since the analogs of these assertions do not hold in higher dimension the situation becomes much more complicated for $n > 1$.

In the next paragraph we give a short account on the present situation of the different problems. Since our previous report [Kr89b] there was some interesting progress in the *Linearization Problem* and the *Fixed Point Problem*. Nevertheless, the problems formulated above are still far from being solved. We are convinced that they will finally have a negative solution, at least for large dimension. Our recent work on unipotent group actions even suggests possible ways to construct counterexamples. We will discuss this in the last two sections 3 and 4.

§2. Description of some basic problems for $\mathbb{A}^n$

Characterization Problem. This problem is solved for $n = 2$ due to fundamental work of FUJITA, MIYANISHI and SUGIE (see [Su89]). One of their main results is the following.

Theorem. *Let Y be an affine smooth surface. Assume that Y is factorial and that there is a dominant morphism $\mathbb{A}^N \to Y$ for some N. Then Y is isomorphic to $\mathbb{A}^2$.*

It is an open question whether this also holds in higher dimension. This would have a number of interesting consequences. In fact, it is easy to see that this characterization immediately implies a positive solution of the *Cancellation Problem*: If $Y \times \mathbb{A}^k \simeq \mathbb{A}^{n+k}$ then Y is affine, smooth and factorial and the projection gives a surjective morphism $\mathbb{A}^{n+k} \to Y$.

There is also a *topological* characterization of $\mathbb{A}^2$ due to RAMANUJAM [Ra71]:

An affine smooth surface which is contractible and simply connected at infinity is isomorphic to $\mathbb{A}^2$. In particular, every normal affine surface which is homeomorphic to $\mathbb{A}^2$ is isomorphic to $\mathbb{A}^2$.

This result does not hold in dimension ≥ 3. In fact, RAMANUJAM has also constructed a contractible smooth affine surface R which is not isomorphic to $\mathbb{A}^2$. It follows now from H-cobordism theory that $R \times \mathbb{A}^1$ is homeomorphic to $\mathbb{A}^3$, but not isomorphic to $\mathbb{A}^3$, because of the positive solution of the *Cancellation Problem* in dimension 2.

Embedding Problem. There is a famous result about the embedding of the line into the plane due to ABHYANKAR-MOH [AM75] and SUZUKI [Suz74].

Theorem. *Every closed embedding of* $\iota: \mathbb{A}^1 \hookrightarrow \mathbb{A}^2$ *is equivalent to a coordinate line, i.e., there is a (polynomial) automorphism* φ *of* $\mathbb{A}^2$ *such that the composition* $\varphi \circ \iota$ *is the map* $x \mapsto (x, 0)$.

Recently, SUZUKI has given an elegant new proof of this result based on a careful analysis of the singularity of the embedded line $\mathbb{A}^1$ at infinity. The study of the singularities of plane curves at infinity plays an important role in the investigation of the *Jacobian Problem* in dimension 2.

Concerning generalisations of this theorem, the following general result is known:

Two closed embeddings of a smooth affine variety Z *into* $\mathbb{A}^N$ *are equivalent if* $N \geq 2 \dim Z + 2$.

This result is due to NORI (see [Sr90]) and in some special cases to JELONEK [Je87]. In particular, the only open case for the affine line $\mathbb{A}^1$ is the embedding into $\mathbb{A}^3$.

Jacobian Problem. The really exciting new development here is an example by SERGUEY PINCHUCK (May 1994) which shows that the *real* Jacobian Conjecture is false:

There is a polynomial morphism $\varphi: \mathbb{R}^2 \to \mathbb{R}^2$ *whose jacobian is everywhere strictly positive but which is not an isomorphism.*

Cancellation Problem. As already mentioned above this problem is solved for $\dim Y = 2$ using the algebraic characterization of $\mathbb{A}^2$ given by FUJITA, SUGIE and MIYANISHI, see [Su89]. On the other hand there is a beautiful example of DANIELEWSKI [Da89] which shows that an obvious generalisation of the *Cancellation Problem* does not hold, even not in dimension 2.

Theorem. *Let* $Y_n \subset \mathbb{C}^3$ $(n \in \mathbb{N})$ *denote the smooth affine surface defined by the equation* $x^n y + z^2 = 1$.

(a) *The varieties* $Y_n \times \mathbb{A}^1$ *are all isomorphic.*
(b) *The varieties* Y_n *are pairwise non-homeomorphic. More precisely, the fundamental group of* Y_n *at infinity is* $\mathbb{Z}/2n$.

The second statement is due to FIESELER; the original result of DANIELEWSKI was weaker. For more details and further results in this direction we refer to the recent paper [Fi94] of FIESELER where he studies and classifies $\mathbb{C}^+$-actions on affine surfaces (see also [tDK94]).

Automorphism Problem. There is an old result about the structure of the automophism group of $\mathbb{A}^2$. In order to describe it and for the following discussion we have to introduce some notation. First we recall that a polynomial morphism $\varphi = (\varphi_1, \dots, \varphi_n): \mathbb{A}^n \to \mathbb{A}^n$, $\varphi_i \in \mathbb{C}[x_1, \dots, x_n]$, is an isomorphism (i.e., has a polynomial inverse) if and only if it is bijective. This is also equivalent to the condition that the φ_i generate the polynomial ring $\mathbb{C}[x_1, \dots, x_n]$.

We denote by $\mathcal{G}_n$ the group all polynomial automorphism of $\mathbb{A}^n$ and define the two subgroups $\mathcal{A}_n$ of *affine transformation* and $\mathcal{J}_n$ of *triangular transformation* in the following way:

$$\mathcal{A}_n := \{\varphi = (\varphi_1, \dots, \varphi_n) \in \mathcal{G}_n \mid \varphi_i \text{ linear for all } i\},$$
$$\mathcal{J}_n := \{\varphi = (\varphi_1, \dots, \varphi_n) \in \mathcal{G}_n \mid \varphi_i \in \mathbb{C}[x_1, \dots, x_i] \text{ for all } i\}.$$

Clearly, $\mathcal{A}_n$ is the semidirect product of GL_n with the subgroup $\mathcal{T}_n$ of translations.

In dimension 2 the structure of $\mathcal{G}_2$ is given by the following theorem which goes back to VAN DER KULK [Ku53].

Theorem. *The automorphism group $\mathcal{G}_2$ is the amalgamated product $\mathcal{A}_2 *_{\mathcal{B}_2} \mathcal{J}_2$ where $\mathcal{B}_2 := \mathcal{A}_2 \cap \mathcal{J}_2$.*

This fundamental result solves the *Linearization Problem* and the *Fixed Point Problem* in dimension 2:

Every algebraic subgroup G of $\mathcal{G}_2$ is conjugate to a subgroup of $\mathcal{A}_2$ or of $\mathcal{J}_2$. In particular, every reductive subgroup of $\mathcal{G}_2$ is conjugate to a subgroup of GL_2.

It is known that a similar amalgamated product structure does not exist in dimension $n \geq 3$. For instance, consider the following two automophism of $\mathbb{A}^3$:

$$\sigma(x, y, z) = (y, x, z) \quad \text{and} \quad \tau(x, y, z) = (x, y, z + x^2).$$

Then $\sigma \in \mathcal{A}_3$, $\tau \in \mathcal{J}_3$ and $\sigma, \tau \notin \mathcal{A}_3 \cap \mathcal{J}_3$, and the composition $\sigma \circ \tau \circ \sigma$ maps (x, y, z) to $(x, y, z + y^2)$. Hence $\sigma \circ \tau \circ \sigma \in \mathcal{J}_3$ which contradicts the fact that in an amalgamated product $A *_B C$ every element has a decomposition of the form $a_1 c_1 a_2 c_2 \cdots a_r c_r$ $(a_i \in A,\ c_i \in C)$ which is unique modulo relations of the form $ac = (ab^{-1})(bc)$ $(a \in A,\ b \in B,\ c \in C)$. Another way to see that $\mathcal{G}_3$ cannot be an amalgamated product of the form above follows from an example of BASS [Ba85]. We will discuss it in the next paragraph.

The subgroup of $\mathcal{G}_n$ generated by $\mathcal{A}_n$ and $\mathcal{J}_n$ is called the group of *tame* automorphisms. It is an open problem whether every automorphism of $\mathcal{G}_n$ is tame.

Linearization Problem. This problem was originally formulated for reductive group actions on affine space [Ka79]. (It is not difficult to see that for every non-linearly reductive group there exists an action on affine n-space without fixed points. See [KP85].) We refer to [Kr89b, §5] for more details and further references.

In 1989 SCHWARZ discovered the first counterexamples, namely non-linearizable actions of the orthogonal group O_2 on $\mathbb{A}^4$ and of SL_2 on $\mathbb{A}^7$ ([Sch89], see also [KS92]). Using these results KNOP showed that every connected reductive group which is not a torus admits a faithful non-linearizable action on some affine space $\mathbb{A}^n$ [Kn91]. Using a different approach MASUDA, MOSER-JAUSLIN and PETRIE produced more examples and discovered the first non-linearizable action of finite groups, e.g., for dihedral groups of order ≥ 12 on $\mathbb{A}^4$ (see [MP91] and [MMP91]).

So far, all examples of non-linearizable actions have been obtained from non-trivial G-vector bundles on representation spaces V of G by using an idea of BASS and HABOUSH (see [Kr89a]). Since every vector bundle on V is trivial by the famous theorem of QUILLEN and SUSLIN and hence has a affine space as its total space, the G-vector bundles provide us with interesting G-actions on affine space. Some of these turned out to be non-linearizable. In a recent paper MASUDA, MOSER-JAUSLIN and PETRIE showed however that this approach cannot work for *commutative* reductive groups [MMP94]:

Theorem. *Let G be a commutative reductive group (i.e. a product of a torus and a finite commutative group) and let V be a representation of G. Then every G-vector bundle on V is trivial.*

(A G-vector bundle on V is *trivial* if it is isomorphic to a bundle of the form $\Theta_W := V \oplus W \xrightarrow{\text{pr}} V$ where W is a G-representation.)

Fixed Point Problem. There are a number of results from topological transformation groups which can be applied to algebraic situation. E.g., for every action of a *torus* on $\mathbb{A}^n$ the fixed point set is an acyclic smooth subvariety and in particular non-empty. One also knows that every *finite cyclic group* acting on $\mathbb{A}^n$ has fixed points (see [PR86]). For more details we refer again to the surway [Kr89b, §3].

Substantial progress was made recently by FANKHAUSER in his thesis [Fa94]. He was able to extend some results of HSIANG and STRAUME [HS86] about compact Lie group actions on acyclic manifolds to the algebraic setting. Among other things he shows that *there are always fixed points provided the algebraic quotient $\mathbb{A}^n /\!\!/ G$ has at most dimension 3 or is small compared with the rank of G.*

§3. $\mathbb{C}^+$-actions on affine n-space

We have seen above that in the last few years most of the progress concerning the problems formulated in the introduction was made for reductive group actions. These studies are motivated by several issues. On one hand one hopes to achieve a better understanding of the automorphism group $\mathcal{G}_n$ and of the algebraic-geometric properties of the affine space $\mathbb{A}^n$. On the other hand the results here might serve as a model for more general situations and in particular for the study of group actions and quotient spaces.

It became clear that one should also consider group actions of more general groups, i.e., of non-reductive groups and in particular unipotent groups. Some work in this direction has been done by FAUNTLEROY [Fau85,88]. Let us give two examples where group actions of unipotent groups appear in some general context.

Examples. (1) In some work of GRUNEWALD about affine cristallographic groups the following question appears: *Does every unipotent group U has an affine structure such that left-multiplication with elements of U become affine transformation?* A necessary condition is that every unipotent group U has a faithful representation of dimension $\dim U + 1$.

(2) The following question is discussed by SNOW in [Sn89]. Given an algebraic group G and a closed subgroup $H \subset G$ a necessary condition for the homogeneous space G/H to be affine is the following: *For every reductive subgroup M of G the intersection with the unipotent radical of H is trivial: $M \cap H_u = \{e\}$.* Snow realized that this condition would also be sufficient if one could show that for every free triangular action of a unipotent group U on $\mathbb{A}^n$ the orbit space exists as an affine variety. (We will see later that this is not the case in general.)

Let us now start with the simplest unipotent group, the additive group $\mathbb{C}^+$ of complex numbers. Any (algebraic) action of $\mathbb{C}^+$ on $\mathbb{A}^n$ or more generally on a variety X determines a (algebraic) *vector field* ξ in the usual way:

$$\xi_x := d\varphi_x(1) \in T_x(X) \quad \text{for all } x \in X$$

where $\varphi_x : \mathbb{C}^+ \to X$ is the orbit map $t \mapsto t.x$ and $d\varphi_x$ its differential. For $X = \mathbb{A}^n$ we have

$$\xi_x = \lim_{t \to 0} \frac{t.x - x}{t} \in T_x\mathbb{A}^n = \mathbb{C}^n.$$

The vector field ξ, considered as a *derivation* of the coordinate ring $\mathcal{O}(X)$ is *locally nilpotent.* (This is an obvious consequence of the representation theory of $\mathbb{C}^+$.) It is well known and easy to prove that for every affine variety X there is a bijection

$$\{\mathbb{C}^+\text{-actions on } X\} \longleftrightarrow \{\text{locally nilpotent vector fields on } X\}.$$

For example, the action on $\mathbb{A}^2$ corresponding to the standard representation $\mathbb{C}^+ = \{\begin{pmatrix} 1 & t \\ & 1 \end{pmatrix}\} \subset \mathrm{GL}_2$ is given by the vector field $x\frac{\partial}{\partial y}$.

Remark 1. The *zero set* of the vector field ξ is exactly the *fixed point set* of the corresponding action of $\mathbb{C}^+$: $X^{\mathbb{C}^+} = \{x \in X \mid \xi_x = 0\}$. Similarly, one sees that a function $f \in \mathcal{O}(X)$ is $\mathbb{C}^+$*-invariant* if and only if $\xi f = 0$.

In dimension 2 we have the following complete description of $\mathbb{C}^+$-action on $\mathbb{A}^2$ which follows immediately from the structure theorem of VAN DER KULK (see §2, *Automorphism Problem*). A direct proof was given by RENTSCHLER [Re68].

Proposition. *Every action of $\mathbb{C}^+$ on $\mathbb{A}^2$ is equivalent to an action of the form $t.(x,y) = (x, y + tf(x))$ for some polynomial $f(x) \in \mathbb{C}[x]$. Equivalently, every locally nilpotent vector field on $\mathbb{A}^2$ is equivalent to one of the form $f(x)\frac{\partial}{\partial y}$.*

Corollary. *(a) The fixed point set of a $\mathbb{C}^+$-action on $\mathbb{A}^2$ is smooth.*
(b) A fixed point free action of $\mathbb{C}^+$ on $\mathbb{A}^2$ is equivalent to the translation $t.(x.y) = (x, y + t)$, i.e., $\mathbb{A}^2$ is equivariantly isomorphic to $\mathbb{C}^+ \times \mathbb{A}^1$.

This leads to the following definition.

Definition. An action of G on X is called *equivariantly trivial* if X is equivariantly isomorphic to $G \times Y$ for some variety Y. It is called *locally equivariantly trivial* if this holds locally (in Zariski-topology).

Clearly, equivariantly trivial means that the orbit space X/G exists as a variety and that the orbit map $X \to X/G$ is a trivial G-bundle.

Remark 2. A $\mathbb{C}^+$-action on X is equivariantly trivial if and only if there is an equivariant function $f : X \to \mathbb{C}^+$. In terms of the corresponding vector field this means that $\xi f = 1$.

A famous theorem of ROSENLICHT states that for every action of an algebraic group on a variety X there exists an open set $U \subset X$ which admits a geometric quotient. (The definition of a geometric quotient is given below). For $\mathbb{C}^+$-actions this is an easy consequence of the remark above:

There is always an open $\mathbb{C}^+$-stable subset $W \subset X$ such that the action on W is equivariantly trivial.

In fact, choose a function f such that $q := \xi f \neq 0$ but $\xi q = 0$. Then q is an invariant and $\xi \frac{f}{q} = 1$. Hence $\frac{f}{q}$ defines a trivialization on $U := X_q := \{x \in X \mid q(x) \neq 0\}$.

The proposition above can also be stated in the following way: Every $\mathbb{C}^+$-action on $\mathbb{A}^2$ is *triangularizable*, i.e., every subgroup $\mathbb{C}^+ \subset \mathcal{G}_2$ is conjugate to a subgroup of $\mathcal{J}_2$. This result does not hold in higher dimension as shown by the following example due to BASS. This answers a question of SHAFAREVICH (see [Po87]).

Example 1. The vector field $\xi := (xz + y^2)(x\frac{\partial}{\partial y} - 2y\frac{\partial}{\partial z})$ is locally nilpotent and defines an action of $\mathbb{C}^+$ on $\mathbb{A}^3$ which cannot be triangularized. (In fact, the fixed point set given by $\{xz + y^2 = 0\}$ is not of the form $\mathbb{A}^1 \times Y$ since it has an isolated singularity at zero.)

This example shows again that the automorphism group $\mathcal{G}_3$ is not an amalgamated product of $\mathcal{A}_3$ and $\mathcal{J}_3$ since such a structure implies that every algebraic subgroup of $\mathcal{G}_3$ is conjugate to one of the factors. (This was remarked by WRIGHT who showed that an algebraic subgroup would be of bounded length [Wr].)

On the other hand SNOW showed that every *free triangular action* of $\mathbb{C}^+$ on $\mathbb{A}^3$ is equivariantly trivial, i.e., equivalent to a translation. (This result was also obtained by PANYUSHEV.) More general, one can prove the following.

Proposition. *A separated action of* $\mathbb{C}^+$ *on* $\mathbb{A}^3$ *is equivariantly trivial.*

DEVENEY and FINSTON showed this under the stronger assumption that the action is proper [DF94a]. ("Separated" and "proper" actions will be defined below.) The proof uses essentially a fundamental result of MIYANISHI [Mi85] which states that *a two-dimensional invariant ring of a polynomial ring under a unipotent group is a polynomial ring in two variables.*

Definition. An action of G on X is called *separated* if the set $\{(x, g.x) \mid x \in X, G \in G\}$ is closed in $X \times X$. This means that the orbit space X/G with the induced (strong) topology is a separated topological space. The action is called *proper* if the morphism $G \times X \to X \times X$, $(g, x) \mapsto (g.x, x)$ is finite. In more topological term this means the following: Given sequences $\{x_i\}$ in X and $\{g_i\}$ in G such that $\lim_{i\to\infty} x_i = x$ and $\lim_{i\to\infty} g_i.x_i = y$ then a subsequence of $\{g_i\}$ converges to some $g \in G$ and $y = g.x$.

We remark that a proper action of a unipotent group is separated and *free*, (i.e., all stabilizers are trivial), and that a separated action has all orbits of the same dimension.

Conjecture. *Every free action on* $\mathbb{A}^3$ *is equivariantly trivial.*

The proposition above and in particular the conjecture do not hold in dimension ≥ 4 This follows from the following example given by M. SMITH (see [Sn89]). Another example was found by WINKELMANN [Wi90].

Example 2. The vector field $\xi = x\frac{\partial}{\partial y} + y\frac{\partial}{\partial z} + (1+y^2)\frac{\partial}{\partial w}$ determines a free triangular action of $\mathbb{C}^+$ on $\mathbb{A}^4$ which is not equivariantly trivial. More precisely, this action is even not separated.

We have seen above that for an equivariantly trivial action the quotient $\pi: X \to X/G$ exists as a trivial G-bundle. More generally, one has the following concept of an "algebraic orbit space".

Definition. A *geometric quotient* $\pi: X \to Y$ is a morphism with the following properties:

(1) The fibers of π are the orbits in X;
(2) Y carries the quotient (Zariski-) topology;
(3) π is affine and $\mathcal{O}_Y = (\pi_* \mathcal{O}_X)^G$.

Remark 3. If a geometric quotient $\pi: \mathbb{A}^n \to Y$ exists for the action of a unipotent group U on $\mathbb{A}^n$ then the following holds: (a) The action is proper (hence separated). (b) Y is an open subset of $\operatorname{Spec} \mathcal{O}(\mathbb{A}^n)^U$, and (c) invariant functions separate the orbits. (This follows essentially from the factoriality of $\mathbb{A}^n$, cf. [Fau85,88].)

We also remark that an action of $\mathbb{C}^+$ (or any unipotent group U) on $\mathbb{A}^n$ which is locally equivariantly trivial admits a geometric quotient $\pi: \mathbb{A}^n \to Y$, and π is a principal U-bundle which locally trivial in Zariski-topology.

We now come to the next example which shows that even under strong assumption, like properness, a geometric quotient need not exist. It is due to DEVENEY anbd FINSTON [DF94b].

Example 3. The action of $\mathbb{C}^+$ on $\mathbb{A}^5$ given by the vector field $\xi = x_1 \frac{\partial}{\partial y_1} + x_2 \frac{\partial}{\partial y_2} + (1 - x_1 y_2^2) \frac{\partial}{\partial z}$ is proper, but does not have a geometric quotient.

There remains the question if a geometric quotient of $\mathbb{A}^n$ by the action of $\mathbb{C}^+$ is always affine. This would imply that a locally equivariantly trivial action of $\mathbb{C}^+$ on $\mathbb{A}^n$ is always (globally) equivariantly trivial, because every principal $\mathbb{C}^+$-bundle over an affine variety is trivial. Again, there is an example which shows that this is not the case in general. It is due to WINKELMANN [Wi90].

Example 4. Consider the following action of $\mathbb{C}^+$ on $\mathbb{A}^5$:

$$t(x_1, x_2, y_1, y_2, z) = (x_1, x_2, y_1 + tx_1, y_2, tx_2, z + t(1 + x_1 y_2 - x_2 y_1)).$$

The corresponding vector field is $\xi = x_1 \frac{\partial}{\partial y_1} + x_2 \frac{\partial}{\partial y_2} + (1 + x_1 y_2 - x_2 y_1) \frac{\partial}{\partial z}$. This action has a geometric quotient $\pi: \mathbb{A}^5 \to Y$ where Y is an open subset of the affine 4-dimensional quadric with a complement of codimension 2. Moreover, π is a (nontrivial) principal $\mathbb{C}^+$-bundle.

At this point there remain two questions:

Question. (a) Given a $\mathbb{C}^+$-action on $\mathbb{A}^n$ which admits a geometric quotient $\pi: \mathbb{A}^n \to Y$, does it follow that Y is smooth? (This would imply that π is in fact a principal $\mathbb{C}^+$-bundle.)
(b) Are all proper action on $\mathbb{A}^4$ locally (or even globally) equivariantly trivial?

The examples of this section clearly show that for $\mathbb{C}^+$-actions on affine spaces there are no general results concerning the existence of quotients and their structure. If fact, it seems that every kind of strange behavior is possible. We will show in the last section how one could use these exotic actions to produce negative answers to some of the problems mentioned in the introduction.

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