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Geometric invariant theory (I)

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Geometric Invariant Theory

Introduction: Geometric Invariant Theory deals with the construction of orbit spaces in algebraic geometry when a (reductive) algebraic group operates on an algebraic variety. This is a subtle affair in algebraic geometry. This is a subtle affair in algebraic geometry and its importance stems from the fact that the construction of many moduli spaces rests on this theory. This theory was initiated by Mumford.

In this course we give an introduction to GIT. The material given here is at least 20 years old. One aim of this course is to outline GIT over an arbitrary base with some detail. Unfortunately, we may not have the time to go into the most interesting aspects of GIT, namely the computation of stable points in interesting concrete examples and the solution of at least one moduli problem using GIT.

References

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Chapter I – G I T over an algebraically closed field

§1. Geometric reductivity

We work over an algebraically closed base field k. By an algebraic G we mean always a reduced, *affine* algebraic group G over k. When we talk of an *action* of G on, say an algebraic scheme X (over k), we mean ~~an~~ a (left) action given by a morphism $G \times X \to X$ (called the action *morphism*) satisfying the usual properties. We have the notion of *G-morphisms* $f : X \to Y$ (i.e. equivariant G-morphisms, G acting on X and Y). When we talk of a finite dimensional G-module V (V a finite dimensional vector space over k), we mean that it is a *rational G-module* i.e. it is given by a homomorphism $G \to \operatorname{Aut} V$ of algebraic groups. We say that an infinite dimensional V over k is a *rational G-module* if it is the union of finite dimensional rational G-submodules. By *points* of algebraic schemes (over k), we always mean *closed points*, unless otherwise stated.

Definition 1: An algebraic group G is said to be *reductive* if ~~do radi~~ rad G (radical of G) is a *torus* i.e. a product of multiplicative groups (note that we have assumed that G is affine and reduced).

Definition 2: An algebraic group G is said to be *geometrically reductive* if for every ~~ration~~ finite dimensional rational G-module V and a G-invariant point $v \in V$, $v \neq 0$, there is a G-invariant homogeneous polynomial F on V of degree > 0 such that $F(v) \neq 0$.

<u>Remarks 1</u>: (a) We see that the definition of geometric reductivity is equivalent to supposing that there is a G-invariant polynomial F on V such that $F((0)) = 0$ and $F(v) \neq 0$ (V, G and v are as in Definition 2). In fact if X is the underlying affine space of V (i.e. $X = \operatorname{Spec} S(V^*)$, $S(V^*)$ being algebra on the dual V^* of V) and X_1 and X_2 are two ~~G-invariant~~ disjoint closed G stable subsets of X, then geometric reductivity is equivalent to saying that there is ~~a polynomial F on~~ G-invariant polynomial F on V such that $F(X_1) = 0$ and $F(X_2) = 1$. To prove this, observe first that there is polynomial g on V (not necessarily G-invariant) such that $g(X_1) = 0$ and $g(X_2) = 1$.

~~Observe that we have a canonical action of G on S(~~

Observe that $S(V^*)$ is canonically a G-module and it is a well-known fact that it is a <u>rational</u> G-module, in particular the G-span of every finite dimensional subspace of $S(V^*)$ is again finite dimensional. Hence the G-span of F in $S(V^*)$ is a finite dimensional G-module W. Let Y be the affine space $Y = \operatorname{Spec} S(W)$ and $f: X \to Y$ the canonical morphism induced by the canonical homomorphism ~~inclusion~~ $S(W) \hookrightarrow S(V^*)$. We see that f is a <u>G-morphism</u> ~~i.e. it is G-equivariant and~~

$f(X_1) = (0)$ and $f(X_2) =$ a non-zero G-invariant ~~function~~ point y of Y. By (a) there exists a G-invariant ~~polynomial~~ element F of $S(W)$ such that $F((0)) = 0$ and $F(y) \neq 0$. The pull-back $F_1 = f^*(F)$ has the required properties i.e. $F_1(X_1) = 0$ and $F_1(X_2) \neq 0$ and F_1 is G-invariant.

(b) Geometric reductivity is also equivalent to the following formulation. Let V, G be as in Def. 2. Then given a <u>semi-invariant</u> $v_1 \in V$, $v_1 \neq 0$ (i.e. $g \cdot v_1 = \chi(g) v_1$, where χ is a character of G i.e. a homorphism $G \to G_m$), there is a semi-invariant homogeneous polynomial F on V of deg > 0 such that $F(v) \neq 0$, equivalently given a G-invariant point $v_2 \in \mathbb{P}(V)$, $\exists$ G-invariant hypersurface F in $\mathbb{P}(V)$, not passing through v_2. Obviously geometric reducitivity implies this formulation. Conversely, given v_1, V as above, we can define a new action of G on V by multiplying this action by $\chi(g)^{-1}$ so that v_1 becomes G-invariant. Then the equivalence of this formulation with geometric reductivity follows.

(c) Let N be a normal algebraic subgroup of G. Then if N and G/N are geometrically reductive, G is geometrically reductive. To see this (with the notations as in Def. 2), by hypotheses there is an N-invariant homogeneous polynomial F_0 of deg > 0 such that $F_0(v) \neq 0$. Let W be the ~~G.~~ ~~N-span of F_0 in $S(V^*)$~~. G-span of F_0. Then the action of G on W goes down to an action of $H = G/N$. Then we get a canonical G-homomorphism $f : X \to Y$ where $X = \operatorname{Spec} S(V^*)$ and $Y = \operatorname{Spec} S(W)$ induced by the canonical homomorphism $S(W) \to S(V^*)$. Then $f(v)$ is ~~a G-invariant~~ H-invariant and we can find H-invariant F on W^* (homogeneous and $\deg > 0$) such that $F(f(v)) \neq 0$. Then the pull-back $f^*(F)$ is G-invariant and doesn't vanish on v.

It follows that if G_1 and G_2 are geometrically reductive, $G_1 \times G_2$ is geometrically reductive.

The following result, conjectured by Mumford, was proved by Haboush. It was proved earlier in particular cases.

Theorem 1: A reductive group G is geometrically reductive.

We give a proof of this theorem in the following sections.

§2 Linear reductivity

Definition 3: An algebraic G is said to be linearly reductive if for every finite dimensional G-module V and a G-invariant point $v \in V$, $v \neq 0$, there is a G-invariant linear form F on V such that $F(v) \neq 0$.

Proposition 1: G is linearly reductive if and only every finite dimensional G-module V is completely reducible (i.e. given a G-submodule V_1 of V, there exists a G-submodule V_2 of V such that $V = V_1 \oplus V_2$).

Proof: Let V be a G-module which is completely reducible. Then given $v \in V$, $v \neq 0$ and G-invariant, there is a G-stable linear subspace ~~V~~ of V not passing through v i.e. there is a semi-invariant linear L on V such that $L(v) \neq 0$. Now ~~$(g \cdot L)(v) =$~~ $g \cdot L = \chi(g) L$ (χ a character of G) and by the definition of action of G on V^*, we have $(g \cdot L)(v) = L(g^{-1}v)$. We have $L(g^{-1}v) = L(v)$. Hence $\chi(g) L(v) = L(v)$ and $L(v) \neq 0$, it follows that L is G-invariant. Thus complete reducibility implies linear reductivity (we could have also argued as in Remark 1, (b) above).

Conversely, let us prove complete reducibility assuming linear reductivity. Now if V_1, V_2 are G-modules, $\mathrm{Hom}(V_1, V_2)$ (the set of k linear maps of V_1 into V_2) acquires a canonical G-module structure, namely if $F \in \mathrm{Hom}(V_1, V_2)$, we define

$$(g \cdot F)(v) = g\, F(g^{-1}v), \; v \in V_1, \; g \in G.$$

We see that $\mathrm{Hom}(V_1, V_2)^G$ (the subspace of G-invariants in $\mathrm{Hom}(V_1, V_2)$) is ~~the~~ precisely the space of G-homomorphisms of V_1 into V_2. Let now W be a G-submodule of a finite dimensional G-module V. Then we have a canonical <u>surjective</u> G-homomorphism

$$\mathrm{Hom}(V, W) \xrightarrow{\varphi} \mathrm{Hom}(W, W), \; f \longmapsto f \circ i,$$

$$f \in \mathrm{Hom}(V, W), \; i - \text{the canonical inclusion } i : W \to V.$$

Now the identity homomorphism $\theta : W \to W$ is obviously in $\mathrm{Hom}(W, W)^G$. We claim that there exists $F \in \mathrm{Hom}(V, W)^G$ such that $\varphi(F) = \theta$ i.e. that F is a G-projection of V onto W, which would ~~so~~ prove complete reducibility. Now the claim is an immediate consequence of the dual version of linear reductivity, namely if $P \to Q$ is a surjection of finite dimensional G-modules and Q is a 1-dimensional G-module with ~~bas~~ trivial G-action, then there exists $p \in P^G$, $p \neq 0$, projecting onto a non-zero element of Q, q.e.d.

<u>Remark 2</u>: Let G be linearly reductive. Then if V is a finite dimensional G-module, the G-projection $p: V \to V^G$ is uniquely determined, since V^G is the isotypical component of V ~~with the~~ for the trivial ~~G~~ G-module structure. Now p is called the <u>Reynolds operator</u>. This operator is also <u>functorial</u> in the sense that if $f: V_1 \to V_2$ is a G-homomorphism of finite dimensional G-modules, we have the following commutative diagram

$$\begin{tikzcd} V_1 \arrow[r, "f"] \arrow[d, "p"'] \arrow[dr, phantom, "\swarrow"] & V_2 \arrow[d] \\ V_1^G \arrow[r, "f"] & V_2^G \end{tikzcd}$$

Further, if f is surjective, we see that f induces a surjection $V_1^G \to V_2^G$. These assertions are seen easily. Let now V be a rational G module, V not necessarily of finite dimension. Then by the above functoriality, we get also a unique Reynolds operator $p: V \to V^G$.

<u>Remark 3</u>: Let $k[G]$ be the coordinate ring of the algebraic group G i.e. $G = \operatorname{Spec} k[G]$. Then we have two canonical commuting actions of G on $k[G]$, called the <u>left regular</u> and <u>right regular</u> representations, namely if ~~$x \in$~~ x, g are in G, we set

$$(L_g f)(x) = f(g^{-1}x) \quad \text{- left regular representation}$$

$$(R_g f)(x) = f(xg) \quad \text{- right regular representation.}$$

(7) It is well-known that these give rational G-module structures on $k[G]$.

Let now V be a finite dimensional G-module. Then V^* becomes canonically a G-module and the linear pairing $\langle \; , \; \rangle$ on $V \times V^*$ has the invariance property

$$\langle g \cdot v, \; g \cdot v^* \rangle = \langle v, v^* \rangle; \quad v \in V, v^* \in V^*, g \in G.$$

Given v, v^* (in V and V^* respectively), we get a function (i.e. an element of $k[G]$) ~~u_{v,v^*}~~ u_{v,v^*} on G as follows:

$$u_{v,v^*}(g) = \langle v, \cancel{v^*} \, g v^* \rangle$$

~~We have then~~

Now u_{v,v^*} is called a matrix ~~function~~ coeffecient function (it is really ~~the~~ a suitable entry in the matrix of the representation $G \to \mathrm{Aut}\, V$, defining the G-module V, choosing v as ~~a basis~~ vector in a basis of V and v^* in the dual bases). Now we have:

$$u_{v,v^*}(h^{-1} g) = \langle v, h^{-1} g v^* \rangle$$

$$= \langle h v, g v^* \rangle = u_{hv}(g)$$

This computation shows that the map

$$\varphi : V \longrightarrow k[G]$$

defined by $\varphi(v) = u_{v,v^*}$ (fixing $v^* \in V^*$) is a homomorphism of G-modules (for the left regular representation on $k[G]$).

It follows then that given a finite dimensional ~~G~~-G-module V, we can find a G-homomorphism $\varphi : V \to k[G]$ such that $\varphi(v) \neq 0$ at a given $v \in V$ ($v \neq (0)$), for we have only to define the above φ, fixing

 $v^* \in V^*$ such that $\langle v, v^* \rangle \neq 0$. Suppose moreover that $v \in V^G$, $v \neq (0)$. Then $\varphi(v) \in k[G]^G$. But $k[G]^G$ is simply the one-dimensional subspace of V consisting of the constant functions. Thus ~~we~~ given a finite dimensional G-module V and $v \in V^G$, $v \neq (0)$, we can find a G-homomorphism $\varphi : V \to k[G]$ such that $\varphi(v) = 1$ (constant function 1).

The following theorem is due to H. Weyl (Weyl's result is stronger since it is formulated for Lie algebra modules).

<u>Theorem 2</u>: Let G be ~~a~~ reductive over k with $\mathrm{Char}\, k = 0$. Then G is linearly reductive (so that by Prop. 1 every finite dimensional G-module is completely reducible).

<u>Proof</u>: Let V be a finite dimensional G-module and $v \in V^G$, $v \neq (0)$. Then we have to produce a G-projection onto the 1-dimen-sional subspace of V spanned by v. Then we have seen above (Remark 3) that there is a G-homomorphism $\varphi : V \to k[G]$, such that $\varphi(v) = 1$. It suffices to suffice to produce a ~~G~~ G-projection of $k[G]$ ~~onto the one dimensional subspace of constant functions on G, for the "pull-back" of this projection~~

$$P : k[G] \to k \ (\approx \text{space of constant functions on } G)$$

for if we consider $(P \circ \varphi) : V \to k$, this is a G-invariant linear form on V and $(P \circ \varphi)(v) \neq 0$.

Now without ~~loss~~ loss of generality we can suppose that $k = \mathbb{C}$ - the field of complex numbers. Let U be a maximal compact subgroup of G. Let dg denote the <u>Haar measure</u> on U.

Then we set

$$P(f) = \int_U f\, dg \ , \quad f \in \mathbb{C}[G].$$

Obviously $P(1) = 1$ when we normalize dg so that the measure of U is 1. We have only to prove that P is a G-homomorphism i.e. we have ~~also~~ only to show that

$$\int_U f(h^{-1}g)\, dg = \int_U f(g)\, dg, \quad h \in G.$$

Set

$$F(h) = \int_U f(h^{-1}g)\, dg$$

It is not difficult to see that $F : G \to \mathbb{C}$ is a holomorphic function. By the invariance of Haar measure, we see that $F(h) = F(e)$ (e = identity element) when $h \in U$. Now in a complex coordinate system in a neighbourhood of e, U can be identified with the set of "real axes" (the Lie algebra of G is the complexification of the Lie algebra of U). Hence it follows that $F(h) = F(e)$ for all $h \in G$. This shows that P is a G-projection of $k[G]$ onto k, q.e.d.

Remark 4: The crucial idea of integrating over a maximal compact group of G is called the unitarian trick of H. Weyl.

Theorem 3: Let T be a torus group. Then T is linearly reductive (Char k is arbitrary).

Proof: By what we have seen above, we have only to find a T-projection $P : k[T] \to k$. Now we have

9) $T = \mathbb{G}_m \times \cdots \times \mathbb{G}_m$ (r times). Then we can write $f \in k[T]$ in the form

$$f = \sum_{(i)}' a_{(i)} t_1^{i_1} \cdots t_r^{i_r}, \quad (i) = (i_1, \ldots, i_r) \in \mathbb{Z}^r,$$

$t_i \in k^* \ (= k \setminus (0))$, the sum being finite.

We set $P(f) = a_{(0)}$ and we check immediately that P is a T-projection, q.e.d.

§ 3. Proof of Theorem 1

G is reductive (char k is arbitrary) and we are given a finite dimensional G-module and $v \in V^G$, $v \neq (0)$.
We can suppose that char $k > 0$.
Now rad G is a torus and $G/\text{rad}\, G$ is semi-simple.
Since rad G is linearly reductive (Th. 3) (in particular geometrically reductive), by Remark 1, (c), G would be geometrically reductive if $G/\text{rad}\, G$ is geometrically reductive. Hence we can assume that G is semi-simple. In fact, we can also suppose that G is also simply-connected (for if $\hat{G} \to G$ is the simply-connected cover, V becomes a $\hat{G}$-module and v is $\hat{G}$ invariant. Then we can find F which is of deg > 0 and $\hat{G}$-invariant, then a suitable power of F would be G-invariant). Thus we suppose that G is semi-simple and simply-connected.

Fix a maximal torus T, a Borel subgroup B, $B \supset T$, and take roots, weights etc. in the usual manner. Let G/B denote the generalized flag variety. It is a smooth projective variety. Let λ be a dominant weight. Then this defines a homomorphism

(11) $T \to \mathbb{G}_m$. ~~and this~~ This also defines ~~a~~ canonically a homomorphism $B \to \mathbb{G}_m$ (since $B/B^u \simeq T$, B^u = unipotent radical of B). Through this we get a $\mathbb{G}_m$-bundle and the corresponding line bundle is denoted by L_λ (to be technically more precise L_λ is the line bundle associated to the character $i(\lambda)$, ~~i = w~~ i-Weyl involution). Then $H^0(G/B, L_\lambda)$ is a G-module (for this purpose, we have supposed that G is simply-connected) and if char $k = 0$ this is an irreducible G-module with highest weight λ.

As we saw above (Remark 3), there is a G-homomorphism $V \to k[G]$ such that $v \mapsto$ constant function 1. Now consider $k[G]$ as a T-module through the <u>right regular</u> representation. Now we have a canonical T-projection (Th. 3)

$$k[G] \to k[G]^T = k[G/T]$$

~~It~~ Since this action of T and the (left regular) action of G on $k[G]$ commute, ~~this T~~ the above T-projection is in fact a homomorphism of G-modules. Hence we get a G-homomorphism

$$(*) \qquad V \to k[G/T], \quad v \mapsto \text{constant function } 1 \text{ on } G/T.$$

Let $\rho = \frac{1}{2}$ sum of positive roots and

$$W_{m\rho} = H^0(G/B, L_{m\rho})$$

Then we claim that

~~$k[G/T] = \bigcup_m P_m$, $P_m \subset P_{m+1}$, P_m finite dimensional G-submodule of~~

(12)

$$(**)\left\{\begin{array}{l} k[G/T] = \bigcup_m P_m,\ P_m \hookrightarrow P_{m+1},\ P_m \text{ finite dimensional } G\text{-sub} \\ \text{-module of } k[G/T] \text{ and } P_m \text{ is } G\text{-isomorphic to } W_{m\rho} \otimes W_{m\rho}. \end{array}\right.$$

To prove this claim, ~~consider the~~ we see that we have a well-defined morphism

$$j : G/T \longrightarrow G/B \times G/B$$

induced by $j' : G \to G \times G$, $g \mapsto (g, w_0 g w_0^{-1})$, where w_0 is the unique element of the Weyl group of maximal length (to define j', we have to fix a representative of w_0 in the normalizer $N(T)$ of T). Since $T = B \cap B^-$ (even scheme theoretically) with $B^- = w_0 B w_0^{-1}$ (opposite Borel subgroup), j' goes down to a ~~morphism~~ well-defined morphism j as above, ~~and~~ In fact, we see that j and the differential dj are injective. Further, since $\dim G/T = 2 \dim G/B$, it follows that j is an open immersion. Define

$$E = (G/B \times G/B) - j(G/T)$$

The essential point is to show that E is a divisor on $G/B \times G/B$ defining the line bundle $p_1^*(L_\rho) \otimes p_2^*(L_\rho)$ (p_i are the canonical projections of $G/B \times G/B$ onto G/B). This would imply that E is ~~a~~ ample and $U = G/T$ is affine with ~~a~~ coordinate ring equal to

$$\bigcup_m H^0(G/B \times G/B, L(mE))$$

$$= \bigcup H^0(G/B \times G/B, p_1^*(L_{m\rho}) \otimes p_2^*(L_{m\rho}))$$

$$= W_{m\rho} \otimes W_{m\rho}$$

There is a little fuss we have to make about G-actions. Note that j is <u>not</u> a G-morphism, taking the diagonal action of G on $G/B \times G/B$ and the left action of G on G. Now j': $G \to G \times G$ is a G-morphism for the left action of G on G and the action of G on $G\times G$ defined by

$$g\cdot(g_1, g_2) = (g g_1, w_0 g w_0^{-1} g_2)$$

Hence the G-module structure on $H^0(G/B\times G/B, p_1^*(L_{mp})\otimes p_2^*(L_{mp}))$ ~~is $W_{mp}\otimes W$~~ isomorphic to the G-module structure on $W_{mp}\otimes W_{mp}^{w_0}$, where $W_{mp}^{w_0}$ is the G-module structure on W_{mp} defined by

$$\cancel{g} \quad g\cdot x = (w_0 g w_0^{-1})x; \quad g\in G,\ x\in W_{mp}$$

~~H~~ However, $W_{mp}^{w_0} \approx W_{mp}$ as G-modules and we would be through ~~Thus,~~ (if one wants a more explicit identification, see). Thus, to prove the claim $(**)$ we have only to prove

$(**)$ $\begin{cases} E \text{ is a divisor on } G/B\times G/B, \text{ defining the line} \\ \text{bundle } p_1^*(L_\rho)\otimes p_2^*(L_\rho). \end{cases}$

Let Λ be the inverse image of E in $G\times G$ by the canonical morphism $G\times G \to G/B\times G/B$. Let α be the morphism

$$\alpha: G\times G\to G, \quad (g_1, g_2)\mapsto g_2^{-1} w_0 g_1$$

Let $D(\rho) = G - Bw_0B$. Then one knows that $D(\rho)$ is the pull-back of a divisor in G/B (complement of the "big cell"), which defines the line bundle $L(\rho)$ on G/B. We claim that

$$\Lambda = \alpha^{-1}(D(\rho))$$

~~Let~~ This is an easy computation. Let $\mathcal{V}$ be the inverse image in $G\times G$ of $\mathcal{U} = j(G/T)$ by the canonical morphism $G\times G \to G/B \times G/B$. In fact, we see that it suffices to show that

$$\mathcal{V} = \alpha^{-1}(B w_0 B)$$

Now $\mathcal{V}$ consists of points of the form

$$(g b_1,\ w_0 g w_0^{-1} b_2);\ g\in G,\ b_i \in B$$

We seen then that

$$(g_1, g_2) \in G\times G \text{ is in } \mathcal{V}$$

$$\iff \exists\, b_1, b_2 \text{ in } B \text{ such that } g_1 b_1^{-1} = \cancel{w_0^{-1} b_2^{-1}}\ w_0^{-1} g_2 b_2^{-1} w_0$$

$$\iff g_2^{-1} w_0 g_1 = b_2^{-1} w_0 b_1 \text{ for some } b_1, b_2 \text{ in } B \iff \alpha(g_1, g_2) \in B w_0 B$$

This proves the claim.

One knows that there is a regular function F on G vanishing precisely on $D(\rho)$ with multiplicity one on all its irreducible components and satisfying

$$F(b_1 g b_2) = \rho(b_1)^{-1} F(g) \rho(b_2),\ b_i \in B$$

ρ viewed as a character of T (F ~~corr~~ corresponds to the highest weight vector). Let F_1 be the pull-back by α of the function F i.e.

$$F_1(g_1, g_2) = F(g_2^{-1} w_0 g_1).$$

Then we check immediately that

$$F_1(g_1 b_1,\ g_2 b_2) = \rho(b_1)\rho(b_2)\ F_1(g_1, g_2)$$

$$(\text{for } LHS = F(b_2^{-1} g_2^{-1} w_0 g_1 b_1) = \rho(b_2)\rho(b_1) F(g_2^{-1} w_0 g_1) = RHS)$$

~~Now~~ This means that F_1 can be identified with a section of the line bundle $p_1^*(L_p) \otimes p_2^*(L_p)$. Further the above computations show also that the zero locus of F_1 is the divisor E. Thus the claim $(*{*}*)$ is proved.

Let us now finish the proof of the theorem. By $(*)$ and $(**)$, we can find a G-homomorphism (for all $m \gg 0$)

$$\varphi : V \to W_{mp} \otimes W_{mp}, \quad v \mapsto \text{non-zero element}$$

Let $\operatorname{char} k = p > 0$. Then by the work of Steinberg, it is known that if $m = p^\alpha - 1$, then W_{mp} is an irreducible G-module, which implies W_{mp}^* (dual of W_{mp}) is G-isomorphic to W_{mp}. ~~Hence~~ Take φ as above with $m = p^\alpha - 1$, $\alpha \gg 0$.

Now

$$W_{mp} \otimes W_{mp} \simeq \operatorname{Hom}(W_{mp}, W_{mp}) \text{ as } G\text{-modules}$$

and W_{mp} being irreducible, $\operatorname{Hom}(W_{mp}, W_{mp})^G$ is 1-dimensional ~~group~~ and spanned by ~~the i : W_{mp}~~ the identity map $i : W_{mp} \to W_{mp}$. Let D be the determinant function on $\operatorname{Hom}(W_{mp}, W_{mp})$ i.e.

$$D(f) = \det f, \quad f \in \operatorname{Hom}(W_{mp}, W_{mp}).$$

Obviously D is G-invariant

Now $\varphi(v) = (\text{non-zero scalar})\, i$ (we identify $W_{mp} \otimes W_{mp}$ with $\operatorname{Hom}(W_{mp}, W_{mp})$). Then consider the ~~poly~~ homogeneous polynomial function $F = \cancel{(D \circ f)}\ (D \circ \varphi)$. Then F is G-invariant and $F(v) \neq 0$, q.e.d.

<u>Remark 5</u>: If $\operatorname{char} k = 0$, then W_{mp} is irreducible and self dual (for all m). Then instead ~~we can~~ of D we can take the trace ~~func~~ function $tr : \operatorname{Hom}(W_{mp}, W_{mp})$. Then $(tr \circ \varphi)$

(16)

is a G-invariant linear form on V not vanishing on v.

This would give another proof of the linear reductivity of a reductive algebraic group when $\operatorname{Char} k = 0$ (provided we have not used complete reducibility in proving that W_{mp} is self-dual).

§ 4 Quotient spaces

Definition 4: Let G be an algebraic group acting on an algebraic X and $f : X \to Y$ a morphism of algebraic schemes. Then f is said to be a categorical quotient if the following properties hold:

(i) f is G-invariant i.e. for the trivial action of G on Y, f is a G-morphism

(ii) $\forall$ G-invariant morphism $g : X \to Z$, $\exists$ a unique morphism $h : Y \to Z$ such that $g = h \circ f$

$$\begin{tikzcd} X \arrow[d, "f"'] \arrow[dr, "g"] & \\ Y \arrow[r, dashed, "h"] & Z \end{tikzcd}$$

We see easily that a categorical quotient is uniquely determined upto isomorphism.

Definition 5: Let G be an (affine) algebraic group acting on an algebraic scheme and $f : X \to Y$ a G-morphism of algebraic schemes. Then f is said to be a good quotient if the following properties hold:

(i) f is a surjective, G-invariant affine morphism

(ii) $f_*(\mathcal{O}_X)^G = \mathcal{O}_Y$ ($\mathcal{O}_X, \mathcal{O}_Y$ structure sheaves of X, Y)

(iii) if Z is a closed G-stable subset of X, then $f(Z)$ is closed in Y; further if Z_1, Z_2 are two closed G-stable subsets of X such that $Z_1 \cap Z_2 = \emptyset$, then $f(Z_1) \cap f(Z_2) = \emptyset$.

Remark 6: (a) It is seen easily that (i), (ii) and (iii) are equivalent to (i), (ii) and (iii)$'$ where (iii)$'$ is the following:

(iii)$'$ if X_1, X_2 are two closed G-stable disjoint subsets of X, then the closures of their images are also disjoint i.e. $\overline{f(X_1)} \cap \overline{f(X_2)} = \emptyset$.

(b) the properties (i) and (ii) are equivalent to supposing that f is surjective, ~~G-invariant~~ G-invariant and there is an affine covering $\{Y_i\}$ of Y, $Y_i = \operatorname{Spec} B_i$ such that $f^{-1}(Y_i)$ is affine and if $f^{-1}(Y_i) = \operatorname{Spec} A_i$, then $B_i = A_i^G$. (we use the fact that forming invariants commutes with localization, see the proof in Th. 4 below).

(c) It is easily seen that the property of being a good quotient is local with respect to Y i.e. f is a good quotient if and only if there is an open covering ~~or~~ of Y such that the canonical morphisms $f^{-1}(Y_i) \to Y_i$ are good quotients.

(d) <u>good quotient $\Rightarrow$ categorical quotient</u>

$$\begin{tikzcd} X \arrow[d, "f"'] \arrow[dr, "g"] & \\ Y \arrow[r, dashed, "h"] & Z \end{tikzcd}$$

Let $g : X \to Z$ be a G-invariant morphism. Given $y \in Y$, $g(f^{-1}(y))$ consists only of one point (for otherwise $f^{-1}(y)$ ~~could~~ would contain two disjoint closed G-stable subsets, which would contradict that f is a good quotient). Hence we have a set theoretic map $h : Y \to Z$ with $h \circ f = g$. Now h is continuous for if C is closed in Z,

$g^{-1}(C)$ is closed, G-stable and $f(g^{-1}(C)) = h^{-1}(C)$ is closed. Let V be affine open in Z. Then $g^{-1}(V) = f^{-1}(U)$ where $U = h^{-1}(V)$.

The morphism 'g' being G-invariant, the canonical ~~homo~~ homomorphism

$$(*) \quad \Gamma(V, \mathcal{O}_V) \longrightarrow \Gamma(g^{-1}(V), \mathcal{O}_{g^{-1}(V)})$$

factors through $\Gamma(g^{-1}(V), \mathcal{O}_{g^{-1}(V)})^G$. But we have

$$\Gamma(g^{-1}(V), \mathcal{O}_{g^{-1}(V)})^G = \Gamma(f^{-1}(U), \mathcal{O}_{f^{-1}(U)})^G$$

$$= \Gamma(U, \mathcal{O}_U)$$

(by the property of good quotients).

Hence $(*)$ induces a homomorphism (of k-algebras)

$$\Gamma(V, \mathcal{O}_V) \rightarrow \Gamma(h^{-1}(V), \mathcal{O}_{h^{-1}(V)})$$

which shows that h is a morphism.

Definition 6: Let G be an (affine) algebraic group acting on an algebraic scheme X and $f : X \rightarrow Y$ a morphism of algebraic schemes. Then f is said to be a geometric quotient if the following properties hold:

(i) f is a good quotient

(ii) $\forall x_1, x_2 \in X$ (recall by our convention x_i are closed points), $f(x_1) = f(x_2)$ if and only $O(x_1) = O(x_2)$ where $O(x_i)$ is the G-orbit through x_i (equivalently because of (i), $\forall x \in X$, $O(x)$ is closed in X.

Intuitively, Y should be viewed as the "orbit space of X modulo G."

Proposition 2: Let $X = \operatorname{Spec} A$ be an algebraic scheme on which a reductive group G operates. Then ~~A^G~~ $Y = \operatorname{Spec} A^G$ is an algebraic scheme i.e. A^G is finitely generated as a k-algebra and the canonical morphism $f : X \to Y$ (induced by $A^G \hookrightarrow A$) is surjective.

Proof: In char $p > 0$, this result is due to Nagata who proved it assuming the geometric reductivity of G (Th. 1). ~~We shall prove this~~ This will follow later ~~when~~ (Chapter II) when we consider GIT over arbitrary base

In char. 0, this result is classical. We shall now quickly indicate its proof. Now G is linearly reductive (Th. 2). Then we have the Reynolds operator $p : A \to A^G$ which is a G-projection. Then by the functoriality of the Reynolds operator, we have the following commutative diagram:

$$\begin{array}{ccc} A & \xrightarrow{p} & A^G \\ {\scriptstyle m_g}\downarrow & & \downarrow{\scriptstyle m_g} \\ A & \longrightarrow & A^G \end{array}$$

$m_g : A \to A$ multiplication by $g \in A^G$.

Then we get

(I) $(I \cdot A) \cap A^G = I$, I being an ideal in A^G.

(for if $x \in A^G$, $p(x) = x$ and if $y \in (I \cdot A) \cap A^G$,

$$y = \sum_k{}' \, i_k \, a_k \; ; \; i_k \in I, a_k \in A \text{ and}$$

$$y = \pi(y) = \sum{}' \, i_k \, \pi(a_k) \in I \,)$$

(II) if J is a G-stable ideal in A, then the canonical homomorphism

$$A^G \longrightarrow (A/J)^G \text{ is surjective. (see Remark 2).}$$

Now from (1) it follows that $\operatorname{Spec} A \to \operatorname{Spec} A^G$ is surjective (we have only to take I to be a prime ideal in A^G).

To prove that A^G is a finitely generated k-algebra, we use

<u>Lemma 1!</u> Let $X = \operatorname{Spec} A$ be an affine algebraic scheme on which an ~~alg (affine)~~ algebraic G operates (here char k is arbitrary and G is a general affine algebraic group). Then we can find a finite dimensional G-module V and a G-surjection of k-algebras $S(V^*) \to A$ i.e. X can be imbedded as a G-closed subscheme of $\mathbb{A}^n = \operatorname{Spec} S(V^*)$ ~~on which~~

<u>Pf!</u> This is immediate by taking V^* as the G-span of a finite number of algebra generators of A.

By (ii) and the above lemma, we are reduced to the case when $A = S(V^*)$ or $S(V)$ where V is a finite dimensional G-module. Then A is graded

$$A = \bigoplus_{d \geq 0} A_d \text{ , } A_d \text{ finite dimensional } G\text{-module, and}$$

$$A^G = \oplus A_d^G$$

Let $A_+^G = \bigoplus_{d>0} A_d^G$. Let I be the ideal in A generated by A_+^G. Then I is finitely generated and homogeneous, so that we ~~have~~ can write

$$\begin{cases} I = (F_1, \dots, F_t), \ F_i \text{ are generators of } I, \ F_i \text{ are} \\ \text{homogeneous of deg} > 0 \text{ and } F_i \in A^G. \end{cases}$$

Let now $F \in A^G_d$, $d > 0$. Then we can write

$$F = \sum' F_i A_i$$ where A_i are homogeneous (not necessarily G-invariant).

Then applying the Reynolds operator p, we get

$$F = p(F) = \sum' F_i \, p(A_i) \, , \quad p(A_i) \in A^G$$

Now $\deg A_i < \deg F$ and it follows that $\deg p(A_i) < \deg F$. Then by induction on $\deg F$, we see easily that $\{F_i\}$ generate A^G as a k-algebra, q.e.d.

<u>Theorem 4:</u> Let G be a reductive algebraic group acting on an affine algebraic scheme $X = \operatorname{Spec} A$. Then $Y = \operatorname{Spec} A^G$ is an affine algebraic scheme and the canonical morphism $f: X \to Y$ is a good quotient. Let $X_1 = \{x \in X \mid$ the orbit morphism $\Psi_x : G \to X$, $g \mapsto g.x$ is <u>proper</u> i.e. $O(x)$ is closed and $\dim G = \dim O(x)\}$. Then X_1 is G-stable open in X, $Y_1 = f(X_1)$ is open in Y and the canonical morphism $f: X_1 \to Y_1$ is a geometric quotient.

<u>Proof:</u> We have seen above (Prop. 2) that A^G is finitely generated as a k-algebra and f is surjective. Further by Lemma 1, we can embed X as a closed ~~subsche~~ G-stable subscheme of an ~~the affine~~ $\cancel{\mathbb{A}^n}$ affine space $\mathbb{A}^n$ on which G acts through a linear representation. Then by Remark 1 (a), given two disjoint closed G-stable subsets X_1, X_2 of X, $\exists\, g \in A^G$ such that $g(X_1) = 0$ and $g(X_2) = 1$. This implies that

$\overline{f(X_1)} \cap \overline{f(X_2)} = \phi$ in Y. Let U be an affine open subset in Y given by $U = \operatorname{Spec}(A^G)_f$, $f \in A^G$. Since forming invariants commutes with localization (see Chap II for a more general formulation), we have

$$(A^G)_f = (A_f)^G$$

i.e. if $V = f^{-1}(U)$, we have

$$\Gamma(U, \mathcal{O}_U) = \Gamma(V, \mathcal{O}_V)^G$$

It follows then that ~~f is a good quotient (see Remark~~

$$f_*(\mathcal{O}_X)^G = \mathcal{O}_Y$$

so that f is a good quotient (see Remark 6 (a) also).

To prove the second assertion, let

$$X_0 = \{x \in X \mid \dim O(x) = \dim G\}$$

Then we see that X_0 is open and G-stable in X. That X_0 is G-stable is immediate. Consider the morphism $\varphi: G \times X \to X \times X$, $(g, x) \mapsto (gx, x)$. Then the set of points of $G \times X$ at which φ is quasi-finite (i.e. the fibre of φ through ~~this point~~ this point is of dimension zero) is open in $G \times X$. We see that X_0 is the image in X of this open subset by the canonical projection $G \times X \to X$. Hence X_0 is open. Set $Y_1 = f(X - X_0)$. This is closed in Y. Then we see easily that

$$X_1 = f^{-1}(Y - Y_1)$$

(for $\Psi_x : G \to X$ is not proper if and only if

~~either dim O(x) < G or if dim ...~~

either $\dim O(x) < \dim G$ or if $\dim G = \dim O(x)$, then $O(x)$ is not closed in X, in which case $\overline{O(x)}$ contains an orbit of $\dim < \dim G$. From this it follows immediately that Ψ_x is not proper if and only if $f(x) \in Y_1$, which proves $X_1 = f(Y - Y_1))$, q.e.d.

<u>Definition 7:</u> An action of a reductive group G on $\mathbb{P}(V)$ (V finite dimensional $/k$) is said to be <u>linear</u> if this action comes from a G-module structure on V. Let X be a closed subscheme of $\mathbb{P}(V)$. Then a G-action on X is ~~linear~~ said to be <u>linear</u> if this action comes from a linear action on ~~from~~ $\mathbb{P}(V)$ making X a closed ~~an~~ G-submanifold of $\mathbb{P}(V)$. We call this a linear action on $(X, \mathbb{P}(V))$. More intrinsically, if L is the very ample line bundle on X induced from the ample tautological bundle on $\mathbb{P}(V)$, a linear action of G on $(X, \mathbb{P}(V))$ is simply an action of G on X which can be lifted to the line bundle L (rather the sheaf associated to L). We ~~can~~ write this as an ~~Let us take a linear action~~ action on (X, L).

Let us take a linear action of G on $(X, \mathbb{P}(V))$ as above. Then we have an action of G on the affine space $\mathbb{A}^{n+1} = \operatorname{Spec} S(V^*)$ ($\dim V = n+1$) which is the "cone over $\mathbb{P}(V)$". Let I be the graded ~~a~~ ideal in $S(V^*) \simeq k[X_0, \dots, X_n]$ defining X. Then I is a G-stable ideal. Then we have $X = \text{Proj } R$, where $R = S(V^*)/I$. We denote by $\hat{X}$ - the cone over X i.e. $\hat{X} = \text{Spec } R$ and by (0) the vertex of the cone $\hat{X}$. The action of G on X lifts to an action on $\hat{X}$. This action and the canonical action G_m on $\hat{X}$ commute.

(Note that

$$X \text{ is also Proj } R' \text{ where } R' = \bigoplus_{n \geq 0} H^0(X, L^n))$$

<u>Definition 8</u>: We are given a linear action of a reductive group G on $(X, \mathbb{P}(V))$ as in Def 7 above. A (closed) point $x \in X$ is said to be <u>semi-stable</u> if for some $\hat{x} \in \hat{X}$ (cone over X), $\overline{O(\hat{x})}$ (orbit closure in $\hat{X}$) does not pass through (0). We denote these set of points by X^{ss}. A point $x \in X$ is said to be <u>stable</u> (or <u>properly stable</u> according to Mumford) if the orbit morphism $\psi_{\hat{x}} : G \to \hat{X}$ is proper for some $\hat{x}$ over x. We denote ~~these~~ this set of points by X^s

<u>Remark 7</u>: (a) Since the action of G and G_m (homothecy action) on $\hat{X}$ commute, we see that the definition of stable and semi-stable points is independent of the particular of the point $\hat{x}$ over x.

(b) To ~~be~~ be more intrinsic, the definition of stable and semi-stable points on the (very) ample line bundle L - induced from the projective embedding of X in $\mathbb{P}(V)$. ~~B~~ This is a consequence, ~~of~~ for example, of the result to be proved below (Th. 5). Hence it is appropriate to write $X^{ss}(L)$ and $X^{s}(L)$. Also we have

$$X^{ss}(L) = X^{ss}(L^n), \quad X^s(L) = X^s(L^n), \; n > 0$$

Theorem 5: We are given a linear action of a reductive group ~~on~~ G on $(X, \mathbb{P}(V))$ with notations as in Def. 7 above. Recall that $X = \operatorname{Proj} R$ ($R = S(V^*)/I$, I homogeneous ideal of X in $\mathbb{P}(V)$). ~~Let $\varphi : X \to$~~ Let $Y = \operatorname{Proj} R^G$ and $\varphi : X \to Y$ the canonical rational morphism induced by the inclusion $R^G \hookrightarrow R$ of graded rings. Then we have the following:

(i) $x \in X^{ss} \Leftrightarrow \exists\, s \in R^G$ homogeneous of deg > 0 s.t. $s(x) \neq 0 \Leftrightarrow \exists\, s \in \Gamma(X, L^n)^G$ for some $n \geq 1$ s.t. $s(x) \neq 0$. In particular X^{ss} is open (and G-stable).

(ii) φ is a morphism in X^{ss} and $\varphi : X^{ss} \to Y$ is a good quotient

(iii) $x \in X^s \Leftrightarrow$ the orbit morphism $\psi_x : G \to X^{ss}$ is proper. Further X^s is open and G stable, $f(X^s) = Y^s$ is open and $f : X^s \to Y^s$ is a geometric quotient.

Proof: The assertion (i) is immediate (Note that $R_d = H^0(X, L^d)$ for $d \gg 0$). That φ is a morphism in X^{ss} is also immediate. Now X^{ss} is covered by affine open subsets of the form

$$X_f = \{x \in X \mid f(x) \neq 0,\ f \in R^G_d,\ d \geq 1\}$$

Set $Y_f = \{y \in Y \mid f(y) \neq 0,\ f \in R^G_d,\ d \geq 1\}$

Then we have

$$X_f = \mathrm{Spec}(R^0_f)$$ (elements of degree 0 in the locali-sation R_f)

$$Y_f = \mathrm{Spec}\ (R^G)^0_f$$

and φ induces a morphism $\varphi_f : X_f \longrightarrow Y_f$. Note that

$$(R^G_f)^0 = (R^0_f)^G$$

i.e. the coordinate ring of Y_f is precisely the subring of G-invariants of that of X_f. ~~Note that Y_{f_λ} also give a covering of~~ Hence $\varphi_f : X_f \to Y_f$ is a good quotient. Since $\{Y_f\}$ cover Y and being a good quotient is local with respect to the base, we see that $\varphi : X \to Y$ is a good quotient.

Let $x \in X^{ss}$. Choose $f \in R^G_d$ $(d \geq 1)$ such that $f(x) \neq 0$.

Then we see that $\psi_x : G \to X^{ss}$ is proper if and only if $\psi_x : G \to X_f$ (X_f as above) is proper (for $\overline{O(x)} \subset X_f$ for any $x \in X^{ss}$ and $f \in R^G_d$, $d \geq 1$, $f(x) \neq 0$). Now we have $\mathrm{Spec}\, R^0_f = \mathrm{Spec}\,(R/(f-1))$ so that X_f can be identified with a G-closed subset of the cone $\hat{X}$ over X. Choosing a representative $\hat{x}$ over x such that $f(\hat{x}) = 1$, we deduce that

$$\psi_x : G \to X^{ss} \text{ is proper} \Leftrightarrow \psi_{\hat{x}} : G \to \hat{X} \text{ is proper}$$

Now by Th. 4, $X^s \cap X_f$ is G-stable open and its image in Y_f is open. From these considerations (and Th. 4) it follows easily that X^s is G-stable open, $\varphi(X^s) = Y^s$ is open and $\varphi : X^s \to Y^s$ is a geometric quotient, q.e.d.

~~§ 4. Preliminaries~~