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Geometric invariant theory (II)

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Chap. II - GIT over an arbitrary base

Here we follow the treatment in [ ]. It turns out that there are no essential difficulties in carrying out GIT over an arbitrary base. However, the point is to find the basic definitions, hypotheses and formulations in this task. ~~are not~~ In this exposition we will insist on these rather than detailed proofs.

§1. Preliminaries:

For simplicity we work over a base scheme S which is affine, $S = \operatorname{Spec} R$ (R noetherian ring with 1).

Let $G = \operatorname{Spec} R[G]$ be an affine group scheme over S. Let V be an R-module. Then V is said to be a G-S module (or G-R module or shortly G-module) if for every R-algebra A, we are given a homomorphism (of groups)

$$\varphi_A : G(A) \longrightarrow \operatorname{Aut}_{A\text{-mod}}(V \otimes_R A)$$

which is functorial in A. When $g \in G(A)$ and $v \in V \otimes_R A$, we often simply write gv instead of $\varphi_A(g)v$. It is easily seen that V is a G-module if and only if we are given an R-linear map

$$V \longrightarrow V \otimes_R R[G]$$

which makes V into a comodule under the coalgebra (or bigebra) $R[G]$ over R.

The R-algebra $R[G]$, considered as an R-module, has two natural G-module structures called the left regular and right regular representations respectively. ~~Then~~ These can be seen as follows: To define $G(A)$ module structures on the A-module $R[G]\otimes_R A$ $(= A[G\otimes_R A])$. An element $F \in R[G]\otimes_R A$ is simply giving maps:

$$F_{A'} : G(A') \to A', \quad A' \text{ an } A\text{-algebra}$$

which are functorial in A'. We have canonical homomorphisms

$$\rho_{A'} : G(A) \to G(A')$$

Then if $s \in G(A)$, it is clear that the left and right regular representations are defined respectively, as follows: Let $g \in G(A')$.

(i) $L_s F_{A'}(g) = F_{A'}(\rho_{A'}(s^{-1})g)$ or simply $F_{A'}(s^{-1}g)$

(ii) $R_s F_{A'}(g) = F_{A'}(gs)$.

Let V_1, V_2 be two G-R modules. We have the notion of a <u>G-homomorphism</u> $\varphi : V_1 \to V_2$, namely φ is R-linear and $\varphi \otimes_A R$ is a $G(A)$-homomorphism for every R-alg A (functorial in A). ~~If V_1 is an R-module of~~ If V_1, V_2 are two G-R modules and V_1 is an R-submodule of V_2 such that the inclusion is a G-homomorphism, we say that V_1 is a <u>G-submodule</u> of V_2. If, moreover, the canonical homomorphism is injective $\forall$ R-algebra A, we say that V_1 is a <u>pure</u> G-submodule of V_2. If V_1 is a direct summand of V_2 (as an R-module) and V_1 is a G-submodule of V_2, then V_1 is a pure G-submodule of V_2. Any R-module has a <u>trivial</u> G-module structure, namely we define

$gv = v$, if $g \in G(A)$ and $v \in V \otimes_R A$. The structure morphism $G \to S$ is given by $R \to R[G]$ ($= R \otimes_R R[G]$). It is easy easy that this gives the trivial G-module structure on R (constants

An element $v \in V$, where V is a G-module, is called <u>G-invariant</u> if $\forall$ R-algebra $G(A)v = v$ (to be strict v denotes the element $v \otimes 1$ in $V \otimes_R A$), or equivalently, the R-homomorphism $R \to V$, defined by $1 \mapsto v$, is a G-homomorphism, or ~~under~~ equivalently under the comodule structure $V \to V \otimes_R R[G]$, $v \mapsto v \otimes 1$. The set of G-invariants is an R-submodule of V, denoted as V^G. In fact V^G is a G-submodule of V (V^G endowed with the trivial G-module structure). An element $v \in V \otimes_R A$ (A R-algebra, which is $G \otimes_R A$ invariant is sometimes called for shortness, a G-invariant element of $V \otimes_R A$.

Let V be a G-module. Then $\forall$ R-algebra A, the A-module $\mathrm{Hom}_A(V \otimes_R A, A)$ has a canonical $G(A)$ module structure (contragradient to the $G(A)$ action on $V \otimes_R A$). However, these data need not define a G-module on $V^* = \mathrm{Hom}_R(V, R)$. Suppose that V is <u>free of finite rank</u> over R. Then we get a canonical G-module structure on V^*, for we have

$$V^* \otimes_R A \simeq \mathrm{Hom}_A(V \otimes_R A, A)$$

Let $\langle\, ,\, \rangle$ denote the canonical pairing between $V \otimes_R A$ and $V^* \otimes_R A$. Then we have

$$\langle v, v^* \rangle = \langle gv, gv^* \rangle; \quad v \in V \otimes_R A, \quad v^* \in V^* \otimes_R A,$$
$$g \in G(A).$$

Given $v \in V$ and $v^* \in V^*$, we can define the "matrix coeffecient function" u_{v,v^*} (as in Remark 3, Chap. I), namely define

$$u_{v,v^*,A} : G(A) \to A, \quad A \text{ an } R\text{-algebra}$$
$$u_{v,v^*,A}(g) = \langle v, gv^* \rangle, \quad g \in G(A)$$

We check that this is functorial in A so that these data define an element $u_{v,v^*} \in R[G]$. Fixing v^*, we get an R-linear map

$$\delta_{v^*} : V \longrightarrow R[G]$$

which is checked to be a G-homomorphism for the left regular representation of G on $R[G]$. If $V \to V \otimes_R R[G]$ is the comodule structure, we check that δ_{v^*} is obtained by contracting this with respect to v^*.

Let X be an S-scheme. An <u>action</u> (or <u>operation</u>) of G on X (say on the left) is to give an action of $G(A)$ on $X(A)$ ($\forall$ R-algebra A) which is functorial in A. The action of G on X can be equivalently defined by a morphism

$$G \times_S X \to X$$

satisfying the usual axioms. Let X, Y be S-schemes on which G operates. Then an S-morphism $\varphi : X \to Y$ is called a G-morphism if $\forall$ R-algebra, $\varphi_A : X(A) \to Y(A)$ is $G(A)$ (i.e. $G(A)$ equivariant map). We say that φ is a <u>G-immersion</u> (resp. <u>a closed G-immersion</u>) if φ is an immersion (resp. a closed immersion) and a G-morphism. We then refer to the subscheme (resp. closed subscheme)

$\varphi(X)$ of Y as a G-stable subscheme (resp. closed subscheme) of Y. Let X_1, X_2 be two S-schemes on each of which G operates. Then we get a canonical action of $G \times_S G$ on $X_1 \times_S X_2$ and restricting this ~~to~~ the diagonal of $G \times_S G$ which is identified with G, we get an action of G on $X_1 \times_S X_2$, called the diagonal action of G.

Let B be an R-algebra. We call B a <u>G-R algebra</u> (or simply a G-algebra) if B is a G-module for the underlying R-module structure and $\forall$ R-algebra A, the elements of $G(A)$ induce ~~Gf~~ A-algebra automorphisms of $B \otimes_R A$ (functorial in A). Equivalently, this means that we are given an action of ~~on of~~ G on $X = \operatorname{Spec} B$. We see also that a G-R algebra <u>structure on B is equivalently given by</u> an <u>R-algebra homomorphism</u>

$$B \longrightarrow B \otimes_R R[G]$$

making B into a comodule under the coalgebra $R[G]$. We ~~denote by B^G~~ see that B^G (the G-invariant submodule of B as defined above) is indeed an R-subalgebra of B.

If V_1, V_2 are two G-R modules, then on $V_1 \oplus V_2$ and $V_1 \otimes_R V_2$, we get canonical structures of G-R modules. If V is a G-module, we get a canonical structure of a <u>graded G-R algebra</u> (defined in the obvious way) on the symmetric algebra $S(V)$ of V.

Let V be a free module of finite rank endowed with a G-module structure. Let $X = \operatorname{Spec} S(V^*)$. Then since

$X(A) \simeq V \otimes_R A$ (A an R-algebra), we get an action of G on ~~the~~ the affine scheme $X = \mathbb{A}^r_S$ ($r = \operatorname{rk} V$). We call such an action of G on $\mathbb{A}^r_S$ a linear action.

<u>Proposition 1:</u> Let V be a G-R module, free of finite rank over R. Then we can find a G-homomorphism

$$\varphi : V \to \bigoplus_{1 \le i \le r} R(G) \quad (r\text{-fold direct sum})$$

such that φ identifies V as a pure submodule of $\bigoplus_{1 \le i \le r} R(G)$. In particular, if $\theta : R \to R'$ is a k-algebra homomorphism and $v \in V$ is such that if v_0 is the canonical image of v in $V \otimes_R R'$, $v_0 \neq 0$ then there is a G-homomorphism

$$\Psi : V \to R[G]$$

~~such that~~ $(\Psi \otimes_R R')(v_0) \neq 0$.

<u>Proof:</u> Let $\{v_i\}$, $1 \le i \le r$, be a basis of V over R and $\{v_i^*\}$ be the dual basis. If we set ~~$\varphi = \oplus \delta_{i,v}$~~

$\varphi = \oplus \delta_{v_i^*}$ (with notations as above), it is clear that φ is injective, as well as that $\varphi \otimes_R A$ is injective (A an R-algebra). The last assertion is immediate.

In the sequel, we assume the following (consequence of a result due to Raynaud).

<u>Lemma 1:</u> Let G be smooth over S with connected geometric fibres. Then $R[G]$ is <u>projective</u> over R (as an R-module).

Let $X = \operatorname{Spec} B$ be an affine S-scheme on which G acts. We say that a B-module M is a <u>G-B module</u> (or a <u>quasi-coherent G-$\mathcal{O}_X$-module</u>) if ~~the G-module~~ on the underlying R-module of M we are given a G-module structure, compatible with the action of G on X i.e. $\forall$ R-algebra A, we have

$$g(a\cdot m) = g(a)\, g(m)\,; \quad a \in B \otimes_A R\,,\ m \in M \otimes_R A.$$

If M is coherent over B and B Noetherian, we also call M a <u>coherent G-$\mathcal{O}_X$-module</u>. If M_1, M_2 are two G-B modules, we see easily that $M_1 \oplus M_2$, $M_1 \otimes M_2$ and $S(M_1)$ ~~become~~ have canonical G-B module structures.

The following proposition shows that we have to make some hypotheses to operate as freely as we do over a base field.

<u>Proposition 2</u>: Let $X = \operatorname{Spec} B$ with a G-S (i.e G-R) action. Then we have:

(i) Suppose that G is <u>flat</u> over S. Then the category of G-R modules (resp. G-B modules) is Abelian

(ii) Suppose that G is <u>flat</u> over S. Then if $f: V \to W$ is a G-homomorphism and W_1 is a G-submodule of W such that $f(V) \subset W_1$, then $f: V \to W_1$ is a G-homomorphism.

(iii) Suppose that $R[G]$ is R-<u>projective</u> (in particular, G smooth over S with connected geometric fibres). Let M be a G-B module and $I = \operatorname{Ann} M$ (annihilator of M considered as a B-module). Then I is a G-<u>stable ideal</u> in B.

(iv) Suppose that G is <u>smooth over S</u> with connected <u>geometric fibres</u> and B an <u>integral</u> domain. Then the <u>torsion submodule</u> $T(M)$ of M (considered as a B-module) is a G-B submodule of M.

Proof: The proofs are not difficult and we refer to [ ]. For example, if $f: V \to W$ is a G-homomorphism of G-R module and $V_1 = \operatorname{Ker} f$ (as an R-module), it is not clear that we would have a canonical homomorphism

$$\operatorname{Ker} f \longrightarrow \operatorname{Ker} f \otimes_R R[G]$$

defining a comodule (or ~~⊗~~ G-module) structure on $\operatorname{Ker} f$. However, if $R[G]$ is flat over R, we have:

$$\begin{array}{ccccccccc} 0 \to & \operatorname{Ker} f & \to & V & \to & \operatorname{Im} f & \to 0 & \text{exact} \\ & \downarrow & & \downarrow & & \downarrow & & \\ 0 \to & \operatorname{Ker} f \otimes_R R[G] & \to & V \otimes_R R[G] & \to & \operatorname{Im} f \otimes_R R[G] & \to 0 & \text{exact} \end{array}$$

and we see that the canonical homomorphism $V \to V \otimes_R R[G]$ factors through $\operatorname{Ker} f \to \operatorname{Ker} f \otimes_R R[G]$. It is not difficult to see that this gives a comodule structure on $\operatorname{Ker} f$, etc.

Corollary: Let G be flat over S and G act on $X = \operatorname{Spec} B$ (B an R-algebra). Let $Y = \operatorname{Spec} C$ be a closed G-stable subscheme of X i.e. the canonical homomorphism $B \to C$ is a G-R algebra homomorphism. This is equivalent to saying that $I = \operatorname{Ker}(B \to C)$ is a G-stable ideal in B. Further, all the powers I^m ~~also acquire canonical~~ are also canonically G-stable ideals in B.

Proof: The first assertion follows from Prop 2, (i). As for the second consider, for example, I^2. Now ~~by~~ for the diagonal action of G on $B \otimes_R B$, the R-algebra homomorphism $j : B \otimes_R B \to B$, $j(b_1 \otimes b_2) = b_1 b_2$ (diagonal morphism) is a homomorphism of G-Ralgebras. The canonical inclusion $I \hookrightarrow B$ induces a G-B homomorphism $I \otimes_R I \xrightarrow{i} B \otimes_R B$. Then $(j \circ i)$ is a G-B homomorphism $\operatorname{Im}(j \circ i) = I^2$. Again by Prop. 2, we see that I^2 is a G-stable ideal in B.

__Proposition 3:__ Suppose that $R[G]$ is R-projective (in particular G smooth over S with connected geometric fibres). Let V be a G-R module. Then every finitely generated R-submodule of V is contained in a G-stable submodule of V, finitely generated over R.

__Proof!__ The proof runs on the same lines as in Mumford's book [8, Chap. 1, Sect. 1] or see Prop. 3, [ ].

__Corollary 1:__ Let G be as in Prop. 3 above. Then we have an increasing filtration

$$\bigcup_i P_i = R[G]; \quad P_i \subset P_{i+1}$$

of G-submodules of $R[G]$ (say for the left regular representation) such that P_i is an R-module of finite type.

__Corollary 2:__ Let H be a $\mathbb{Z}$-group scheme such that $\mathbb{Z}[H]$ is free over $\mathbb{Z}$ (in particular H is smooth over $\mathbb{Z}$ with connected geometric fibres) and $G = H \times_{\operatorname{Spec} \mathbb{Z}} S$. Then we have an increasing filtration

$$\bigcup_i P_i = R[G], \quad P_i \subset P_{i+1}$$

such that P_i is ~~free over~~ a G-submodule of $R[G]$ and free and finitely generated over R. Further, we could also ~~Proof! The first Corollary is immediate. & As~~ suppose that P_i is a pure submodule of $R[G]$.

Proof! ~~The~~ Now Cor. 1 is immediate. As for ~~the~~ Cor. 2, because of base change, it suffices to prove it when $R = \mathbb{Z}$. Then since $\mathbb{Z}[G]$ is free over $\mathbb{Z}$, P_i is a finitely generated and torsion free over $\mathbb{Z}$ and hence free over $\mathbb{Z}$. To prove the last assertion, consider $G \otimes_{\mathbb{Z}} \mathbb{Q}$ and a filtration

$$\bigcup_i Q_i = \mathbb{Q}[G], \quad Q_i \subset Q_{i+1}$$

where Q_i are $G \otimes_{\mathbb{Z}} \mathbb{Q}$ modules, free of finite rank over $\mathbb{Q}$.

Now define

$$P_i = Q_i \cap R[G]$$

Then we see that P_i is a $\mathbb{Z}$-direct summand in $R[G]$ and acquires a canonical G-submodule structure of $R[G]$..

§ 2 Geometric reductivity

Definition 1: An (affine) group scheme G over S ($S = \operatorname{Spec} R$, R noetherian) ~~if~~ is said to be reductive if (i) $G \to S$ is smooth (in particular $G \to S$ is flat and of finite type) (ii) the geometric fibres of $G \to S$ are connected and are reductive algebraic groups (see Def. 1, Chap. I). It is said to be moreover split if we have a <u>maximal torus</u> ~~subsch~~ <u>subgroupscheme</u> of G which is <u>split</u>. A torus group scheme is said to be <u>split</u> if

$$T = \prod_{r \text{ fold}} G_{m,R}, \quad G_{m,R} = \operatorname{Spec} R[X, X^{-1}].$$

The basic results that we assume are

(i) if $G \to S$ is split, reductive, then it is obtained as base change of a split, reductive group scheme over $\mathbb{Z}$.

(ii) if $G \to S$ is reductive, then given $s \in S$, there is a neighbourhood U of s and an <u>étale</u> <u>surjective</u> map $U' \to U$ such that the base change $G \times_S U'$ is split, reductive over U'. In particular, we note that if R is a local ring, then we can find a morphism $S' \to S$ ($S = \operatorname{Spec} R$) which is (faithfully) flat and finite type, such that $G \times_S S'$ is split over S'.

<u>Proposition 4</u>: Let G be a split reductive group scheme over S. Then we have

(i) $$\bigcup_i P_i = R[G], \quad P_i \subset P_{i+1}$$

where P_i is a G-submodule of $R[G]$, free of finite rank over R.

(ii) Take a representation $G = H \times_{\operatorname{Spec} \mathbb{Z}} S$, where H is split, reductive over $\mathbb{Z}$. Then given a G-R module V, free of finite rank over R and $v_0 \in V \otimes_R k$, $v_0 \neq 0$ (through an R-algebra homomorphism $R \to k$, k a field),

$\exists$ a homomorphism φ of G-modules

$$\varphi : V \longrightarrow P$$

such that $(\varphi \otimes k)(v_0) \neq 0$ and P is the base change of an ~~H-m~~ H-submodule Q of $\mathbb{Z}[H]$ such that Q is free of finite rank over $\mathbb{Z}$.

Proof: The first assertion follows from Cor. 1 of Prop. 3. We have a representation $G = H \times_{\operatorname{Spec} \mathbb{Z}} S$, as in (i) above. Then (i) follows from Cor. 2 of Prop. 3. Now by Prop. 1, $\exists$ a G-homomorphism $\psi : V \to R[G]$ such that $(\psi \otimes k)(v_0) \neq 0$. Again by Prop. 3, Cor 2, ~~it follows~~ we have a factorisation

$$V \xrightarrow{\varphi} P \xhookrightarrow{i} R[G], \quad \psi = i \circ \varphi$$

φ G-homomorphism

(here we use (ii) of Prop. 2)

where P is the base change of an H-submodule ~~of~~ Q of $\mathbb{Z}[H]$, free of finite rank over $\mathbb{Z}$. It follows that ~~$\varphi\otimes$~~ $(\varphi \otimes k)(v_0) \neq 0$, q.e.d.

Let T be a split torus group scheme. Then we have

$$T = \operatorname{Spec} R[X_1, \ldots, X_r ; X_1^{-1}, X_2^{-1}, \ldots, X_r^{-1}]$$
$$(= \operatorname{Spec} R[T]).$$

Then we see that $R[T]$ is a free module and that

$$R[T] = \bigoplus_{m = (m_1, \ldots, m_r)} \psi_m \qquad m_i \in \mathbb{Z}$$

where Ψ_m is the T-module such that the underlying R-module is free of rank one, associated to character $\chi_m \in \mathrm{Hom}(T, \mathbb{G}_m)$, χ being the character canonically associated to $m = (m_1, \dots, m_r)$, $m_i \in \mathbb{Z}$. We get a canonical T-projection $R[T] \to \Psi_m$, in particular a canonical T-projection $R[T] \to R[T]^T \approx \Psi_0 \approx R$. If $R[T]^{(k)}$ denotes the k-fold direct of $R[T]$, then we have

$$R[T]^{(k)} = \bigoplus_m \Psi_m^{(k)}, \quad m = (m_1, \dots m_r), \ m_i \in \mathbb{Z}.$$

Note that for distinct m, $\Psi_m^{(k)}$ are non-isomorphic T-modules.

Let us now assume for simplicity that $R = ~~\mathbb{Z}~~$ is a <u>discrete valuation ring</u>. This suffices for our purpose. Then note that if W is a T-submodule of $R[T]^{(k)}$, we have

$$W = \bigoplus_m (W \cap \Psi_m^{(k)})$$

In particular, we have a canonical T-projection

$$W \to W^T \text{ and } W^T \text{ is free over } R$$

<u>Proposition 5</u>: Let T be a split torus group scheme over R with $R = \mathbb{Z}$ or a discrete valuation ring. Let V be a T-module, which is free over R. Then we have a canonical T-projection

$$V \to V^T.$$

<u>Proof</u>: By Prop. 3, V is the union of finitely generated T-modules and since R is ~~a~~ as above, these submodules

can also supposed to be free over R. Hence it suffices to prove the proposition when V is also supposed to be free ~~over~~ R. and finitely generated over R. Then by Prop. 1, we can find a a (pure) ~~G~~-T-embedding

$$V \hookrightarrow \bigoplus_k R[T] = R[T]^{(k)} \text{ (k fold direct sum)}$$

Then by the preceding remarks, we get a canonical T-projection $V \to V^T$, q.e.d.

The following lemma says that taking invariants, commutes with flat change and is crucial for our purpose.

Lemma 1: Let V be a G-R module (G any affine group scheme over R) and $R \to R'$ be a flat extension. Then we have

$$V^G \otimes_R R' = (V \otimes_R R')^{G \otimes_R R'}$$

Proof: Let $\varphi : V \to V \otimes_R R[G]$ be the comodule structure on V. We have seen that

$$0 \to V^G \longrightarrow V \xrightarrow{\ker(\varphi - I)} V \otimes_R R[G], \text{ exact sequence of } R \text{ modules}$$

where I is the R-linear map

$$I : V \longrightarrow V \otimes_R R[G], \quad v \mapsto v \otimes 1.$$

We have only to tensor the above sequence by R', q.e.d.

Theorem 1: Let G be a reductive group scheme over S ($=$ Spec R) and V a G-R module which is a free finitely generated R-module of rank n. Take the canonical (linear) action of G on

$$X = \operatorname{Spec} S(V^*) = \mathbb{A}^n_S$$

Suppose that $x_0 \in X(k)$ [$X(k)$ (k-valued points of X) $= V \otimes k$, through an R-algebra homomorphism $R \to k$, k being a field] is a <u>non-zero $(G \otimes_R k)$ invariant point</u>. Then $\exists$ a homogeneous G-invariant element F of $S(V^*)$ of degree > 0 such that $F(x_0) \neq 0$.

<u>Proof:</u> By Lemma 1, we see that we can suppose that R is <u>local</u>, since ~~$R \to R$~~ $R \to k$ ~~#~~ factors through a local R'. Now if R is local, by the structure of reductive group schemes, we can ~~suppose that~~ find $R \to R'$ (faithfully) flat such that $G \otimes_R R'$ is <u>split</u> over R'. We can then find a commutative diagram

$$\begin{tikzcd} R \arrow[r] \arrow[d] & R' \arrow[d] \arrow[dl] \\ k \arrow[r] & k' \end{tikzcd}$$

k' a field

Now replacing R ~~and~~ by R' and k by k', we can suppose that G is further <u>split</u> over R (again applying Lemma 1).

Now if G is split over S, we have $G = H \times_{\mathbb{Z}} S$, where H is split reductive over $\mathbb{Z}$. Then by Prop. 4, we can find a G-homomorphism

$$\varphi : V \to P$$

where P is the base change by S of an H-$\mathbb{Z}$ module

Q, free of finite rank over $\mathbb{Z}$ and such that

$$(\varphi \otimes k)(x_0) \neq 0 \quad (x_0 \in X(k) = V \otimes k)$$

~~Now~~ Now we have $(\mathbb{Z} \to R \to k)$

$$P \underset{R}{\otimes} k = (Q \otimes_{\mathbb{Z}} R) \otimes_R k = Q \otimes_{\mathbb{Z}} k$$

so that $(\varphi \otimes k)(x_0)$ can be identified with y_0, a non-zero $H \otimes k$ invariant point of $Q \otimes k$. Obviously, it suffices to find $F_1 \in S(Q^*)^H$ homogeneous of deg > 0 such that $F_1(y_0) \neq 0$, for the base change of F_1 by R achieves the purpose.

Thus we can suppose that $R = \mathbb{Z}$ and G is split reductive over $\mathbb{Z}$. Then $\mathbb{Z} \to k$ factors through a discrete valuation ring $A \to k$ and by the usual arguments, by taking a suitable $\mathbb{Z} \to A$ (flat over $\mathbb{Z}$), we can reduce to the ~~su~~ situation where A is a discrete valuation with an algebraically closed residue ~~fel~~ field. Thus we can assume that G is reductive and split over R where R is a discrete valuation ring with an algebraically closed residue field.

Now by Prop. 5, we have a canonical T projection

$$p: \quad R[G] \longrightarrow R[G]^T$$

where we take the action of T to be the restriction to T of the right regular G action on $R[G]$. We have $R[G]^T = R[G/T]$ (where G/T is the quotient space of G modulo T, this ~~is an~~ can also be constructed by using the considerations of Prop. 5 etc.) and p becomes a G-morphism for the canonical left action a of G on G/T and the left regular representation on $R[G]$. Then composing with p, we get a G-homomorphism

$$\psi : V \longrightarrow R[G/T]$$

such that $(\psi \otimes k)(x_0)$ is a $(G \otimes k)$ invariant point of ~~$R[G/T]$ and $(\psi \otimes k)(x_0) \neq 0$.~~ Then the proof goes along the same lines as in Chap. I (proof of Theorem 1) and we get a G-homomorphism

$$\varphi : V \rightarrow W_{mp} \otimes W_{mp} \quad (m \gg 0),$$

(W_{mp} - sections of the line bundle on G/B associated to $m\rho$) such that $(\varphi \otimes k)(x_0) \neq 0$. ~~If Char k = 0,~~ From the fact that ~~W_{mp}~~

$$(W_{mp} \otimes k) \xrightarrow[\sim]{(G \otimes k)\text{ isom}} (W_{mp}^{*} \otimes k)$$

(m arbitrary if Char $k = 0$ and $m = p^{\alpha} - 1$ if Char $k = p$), we deduce that

$$W_{mp} \xrightarrow[\sim]{G\text{-isom}} W_{mp}^{*},$$

A being a discrete valuation with residue field k. Then as in Chap. I, the "det" function does the required job, q.e.d.

§3. Quotient spaces

Let there be given a linear action of a reductive group G over $S = \operatorname{Spec} R$ on the affine space $\mathbb{A}^n_S$ and let X be a closed G-stable subscheme of $\mathbb{A}^n_S$. Let $X = \operatorname{Spec} B$.

Definition 2: A geometric point $x \in X(k)$ (k algebraically closed) is semi-stable if the closure (in $X \otimes k$) of the $G \otimes k$ orbit through x does not contain (0). The geometric point x is stable if the $G \otimes k$ orbit through x is closed and its dimension $= \dim (G \otimes k)$ (note that x stable $\Rightarrow x$ semi-stable if $\dim (G \otimes k) \geq 1$).

Proposition 6: (1) Let X, G be as in Def. 2 and $x \in X(k)$ a semi-stable point with ~~X~~ $\mathbb{A}^n_S = \operatorname{Spec} S(V^*)$, V^* being a G module, free of finite rank over R. Then $\exists\ F \in S(V^*)^G$ homogeneous of deg > 0 such that $F(x) \neq 0$.

(2) $\exists$ a well-determined open G-stable subscheme X^{ss} of X whose geometric points are precisely the semi-stable points of X. In fact, $X - X^{ss}$ ~~is~~ is defined by the ideal in B generated by the homogeneous elements of deg > 0 in B^G (in fact, the ideal in B generated by the image of

the canonical homomorphism $S(V^*)_+^G \longrightarrow B_+^G (\hookrightarrow B^G)$, defines $X - X^{ss}$ set theoretically. Here $S(V^*)_+$ denotes the ideal in $S(V^*)$ generated by homogeneous elements of degree >0).

~~Proof: It suffices to prove~~

(3) Let $x_1, x_2 \in X(k)$ such that if $O(x_i)$ denote the $G \otimes k$ orbits in $X \otimes k$ through x_i, we have

$$\overline{O(x_1)} \cap \overline{O(x_2)} = \phi$$

Then $\exists\ F \in S(V^*)^G$ such that $F(x_1)=1$ and $F(x_2)=0$.

Proof: It suffices to prove (3). The proof proceeds along the same lines as in Remark 1(a), Chap. I (except there is a minor technical point). Of course we ~~t~~ can take $X = \operatorname{Spec} S(V^*)$ for proving (3). Consider the morphism $f : X \longrightarrow Y$ with $Y = \operatorname{Spec} S(W)$, $S(W) \hookrightarrow S(V^*)$ as in Remark 1(a). However, we cannot say that Y is again an affine space. To get over this difficulty, we can assume that G is split over R (by the techniques already employed) ~~so~~ and in fact that G is split and $R = \mathbb{Z}$ (using Prop. 4 etc.). Then in this case, W is in fact free (and finitely ~~over~~ generated over $\mathbb{Z}$) so that we can write $W = W_1^*$, W_1 dual of W. Then we see Y is again ~~an~~ an affine space, the geometric point x_1 ~~of any~~ maps to (0) in Y (by f) and x_2 maps to a non-zero invariant in Y and by Th. 1, we are through, q.e.d.

Let now V be a ~~free~~ $G-R$ module (G reductive over R) such that V is free of rank $(n+1)$ over R. We take the canonical ~~(linear)~~ action of G on $\mathbb{P}(V) = \operatorname{Proj} S(V^*)$ (projective space with the usual notation, not Grothendieck's), called linear action of G on $\mathbb{P}(V)$. Let X be a closed G-stable subscheme of $\mathbb{P}^n_S$ and say $X = \operatorname{Proj.} B$, where B is a graded R-algebra and quotient of $S(V^*)$. We denote by $\hat{X}$ the cone over ~~of~~ X i.e. $\hat{X} = \operatorname{Spec} B$. We have a canonical action of G on X. As in Def. 8 (Chap I), we could refer to this as a linear action of G on $(X, \mathbb{P}(V))$ and consider it more intrinsically as an action of G ~~which~~ on X, ~~whi~~ which lifts to an action on the very ample line bundle on ~~$\mathbb{P}(V)$~~ X coming the ample tautological line bundle on $\mathbb{P}(V)$.

Definition 3: With the notations, as above, a geometric point $x \in X(k)$ is semi-stable (resp. stable) if for some $\hat{x}$ over $\hat{X}(k) - (0)$ ((0)-vertex of $\hat{X}$), $\hat{x}$ is semi-stable (resp. stable) for the action of G on the affine scheme $\hat{X}$. (One sees that this definition is independent of the choice of $\hat{x}$ over x).

<u>Proposition 7</u>: With the notations as in Def. 3 above, we have the following:

(1) $\exists$ a well-determined G-stable open subscheme X^{ss} of X such that the geometric points of X^{ss} are precisely the semi-stable points of X. Besides, if $T \to S$ is a morphism, we have

$$(X \times_S T)^{ss} = X^{ss} \times_S T$$

where the LHS denotes the subscheme of semi-stable points for the action $G \times_S T$

(2) Given a finite number x_i of semi-stable points of X, $\exists$ a homogeneous $F \in S(V^*)^G$ of deg > 0 such that $F(x_i) \neq 0 \;\; \forall i$.

(3) Given $x_1, x_2 \in X^{ss}(k)$ (k algebraically closed), the following are equivalent:

(a) $\overline{O(x_1)} \cap \overline{O(x_2)} = \phi$ (orbit closures in $X^{ss} \otimes k$)

(b) $\exists \; F \in S(V^*)^G$ homogeneous of degree > 0 such that $F(x_1) \neq 0$ and $F(x_2) = 0$ (note that this property is equivalent to saying that if $\hat{x}_1, \hat{x}_2 \in \hat{X}(k)$ lie over x_1, x_2 (respectively), then $\overline{O(\hat{x}_1)} \cap \overline{O(\hat{x}_2)} = \phi$ (orbit closures taken in $\hat{X} \otimes k$).

(c) Let $f \in S(V^*)^G$ be homogeneous of deg >0 such that $f(x_1) \neq 0$ and $f(x_2) = 0$ and X_f the G-stable affine open subscheme of X of points x such that $f(x) \neq 0$. Then $\overline{O(x_1)} \cap \overline{O(x_2)} = \phi$, orbit closures taken in $X_f \otimes k$.

Proof: The proof follows on the same lines as in Chap. I. and we leave them.

Thus we define stability and semi-stability for geo-metric points. Now to carry over the results on quotient spaces, the crucial thing that one has to prove is that if $Y = \text{Proj}\, S(V^*)^G$ or $\text{Proj}\, B^G$ (or $X = \text{Spec}\, B^G$.), then Y is of finite type over R i.e. B^G is a finitely generated R-algebra. ~~Also we have to show (for this~~ Also we have to show that $Y(k)$ identifies with $X^{ss}(k)$ modulo "the orbit closure equivalence relation" (this is not difficult, as we shall see below). To prove the finite generation of the ring of invariants, we shall outline a method which in the case of a base field seems different from the one given by Nagata. Our method shows that even if we work over a base field, it is good to use "base change" i.e. work over a more general base. Our methods seem more natural.

Remark 1: We have been so far working with affine base schemes for the sake of simplicity. If the base scheme S is not necessarily affine, we ~~so~~ should work with ~~G $\mathcal{O}_S$ such~~ a reductive group scheme over S and a ~~with~~ with G-$\mathcal{O}_S$ modules V which are locally free of finite rank over ~~S~~ $\mathcal{O}_S$. We take the canonical action of G on the projective bundle $\mathbb{P}(V)$ and define semi-stable and stable points for geometric points etc.

Now the crucial property is the following (rather preceding it):

Proposition 8: We take the notations as in Def. 3 above. Then "X^{ss} mod G is proper" i.e. if

$$f : X^{ss} \longrightarrow Z \quad (Z \text{ separated of finite type over } S = \operatorname{Spec} R)$$

is a dominant, G-invariant S-morphism (G-invariance means that f is a G-morphism for the trivial action of G on Z), then f is surjective (it follows that if Z is quasi-projective or can be embedded in something proper, then Z is projective or proper over S).

~~Remark~~ (Note that the above property should be a posteriori true if we expect the results of Chap. I to carry through, for if $R = k$ (field), then f factors through $Y = \operatorname{Proj} B^G$ since $X^{ss} \to Y$ is a categorical quotient and Y being projective, it follows that f is surjective).

<u>Proof</u> We shall now outline the proof which is quite simple intuitively. The first idea is that if we base change by $Z \to S$, we are reduced to the case ~~when~~ of proving the proposition when $Z = S$ i.e. when f is the structure morphism. Let us first assume that this reduction. Then surjectivity means ~~m~~ that if the generic fibre of $X^{ss} \to S$ (say S is irreducible) ~~is~~ is non-empty, then the <u>closed fibre</u> is also <u>non-empty</u>. Now to prove this assertion, by the usual techniques we can also suppose that R is a discrete valuation ring. Let K be the quotient field of R and k the residue field. Since ~~$X^{ss}(\bar{K})$ ($\bar{K}$ algebraic closure of K), we see (by Th 1) that~~
~~$\exists\ F \in (B \otimes_R K)^{G \otimes K}$ homogeneous of $\deg > 0$ such that~~
$X^{ss}(\bar{K}) \neq \phi$ ($\bar{K}$ algebraic closure of K), we see that
$\exists\ F \in (B \otimes_R K)^{G \otimes K}$ such that $F \not\equiv 0$ and F is homogeneous of $\deg > 0$. Now if π is the local uniformiser of R, multiplying F by a power of π, we can suppose that $F \in B^G$. Now $F \in B_d^G \hookrightarrow B_d$ (elements of deg d of the graded ring B, $d > 0$). Now B_d is a free R-module of finite rank over R. Let $\bar{F}$ denote the canonical image of F in

$$\overline{B} = B \otimes_R k = \oplus (B_d \otimes_R k) = \oplus \overline{B}_d$$

$$\overline{B}_d = B_d \otimes k.$$

Again multiplying F by a suitable power of π, we can ~~suppose~~ that $\overline{F} \not\equiv 0$ i.e. we have produced a non-trivial invariant ($\overline{F} \in \overline{B}_d^G$, $d > 0$) in $\overline{B}$ so that $X^{ss}(\bar{k}) \neq \phi$.

Thus it remains only to prove the assertion that we can reduce to the case $Z = S$. For this consider the graph morphism of f

$$\Gamma_f : X^{ss} \longrightarrow X^{ss} \times_S Z$$

Let H be the group scheme over Z, ~~acting on H~~

$$H = G \times_S Z$$

acting on $X^{ss} \times_S Z$ by base change of the action of G on X^{ss}. We claim that we have a canonical action of H on X^{ss} (considered as a scheme over Z by f) such that Γ_f is an H-morphism. This is intuitively obvious and easily proved. It is in fact a general assertion regarding G-invariant morphisms. We leave this.

We have then canonical morphisms

$$X^{ss} \xhookrightarrow{\Gamma_f} X^{ss} \times_S Z \xhookrightarrow{i} X \times_S Z \xhookrightarrow{j} \mathbb{P}^n_S \times_S Z$$

where Γ_f and j are closed immersions. We have

$$(X \times_S Z)^{ss} = (X^{ss} \times_S Z). \text{ (by Prop. 7, (1))}$$

Let W be the closure of X^{ss} in $X \times_S Z$ (closed subscheme structure extending Γ_f). Suppose that the action of G on X^{ss} extends to W so that W is a closed G-stable subscheme of $X \times_S Z$. Then we see easily that

$$W^{ss} = X^{ss}$$

~~(for $W^{ss} = W \cap (X^{ss} \times_S Z)$~~

This reduces to the case $Z = S$. Now extension of the action of G to W can be achieved, if necessary, by going W_{red}, q.e.d.

<u>Proposition 9</u>: With the notations ~~and~~ preceding Def. 3, we have the following: We can find a filtration P_i of B^G by graded subalgebras of <u>finite type</u> over R

$$\bigcup_i P_i = B^G, \quad P_i \subset P_{i+1}$$

such that if $Y_i = \operatorname{Proj} P_i$ and

$$\Psi_i' : X \to Y_i, \quad \cancel{\theta_i : Y_i \to Y_{i+1}}, \quad \theta_i : Y_{i+1} \to Y_i$$

the canonical rational morphisms induced, respectively, by the inclusions (of graded algebras)

$$P_i \subset B, \quad P_i \subset P_{i+1}, \text{ we have}$$

(a) $\forall i$, Ψ_i' is a morphism in X^{ss}, set $\Psi_i = \Psi_i' / X^{ss}$.

Further all θ_i are morphisms

(b) $\forall i$, Ψ_i induces a bijection of $Y_i(k)$ (k algebraically closed) with $X^{ss}(k)$ modulo the equivalence relation

$$x_1 \sim x_2 \iff \overline{O(x_1)} \cap \overline{O(x_2)} \neq \phi$$
(orbit closures in $X^{ss} \otimes k$)

(c) B^G is integral over P_i ($\forall i$); in fact P_{i+1} is integral over P_i ($\forall i$).

(d) If Z is a closed G-stable subscheme of Z^{ss}, $\Psi_i(Z)$ is closed in Y_i; further, if Z_1, Z_2 are disjoint closed G-stable subschemes of X^{ss}, we have

$$\Psi_i(Z_1) \cap \Psi_i(Z_2) = \phi, \ \forall i.$$

<u>Proof:</u> The first point is to note that $\exists$ a <u>finite</u> <u>number</u>, $F_1, \dots, F_k \in B^G_+$ and homogeneous such that given any $x \in X^{ss}(k)$ (k-algebraically closed) $\exists$ some F_i such that $F_i(x) \neq 0$; besides given any $x_1, x_2 \in X^{ss}(k)$ which are <u>not</u> equivalent under the relation in (b), $\exists$ some F_j such that $F_j(x_1) \neq 0$ and $F_j(x_2) = 0$. To see this, let I be the graded ideal in $B \otimes_R B$ generated by elements of the form

$$(f \otimes 1 - 1 \otimes f), \ f \in B^G_+, \text{ homogeneous.}$$

and Γ the closed subscheme of $X \times_S X$ defined by I. Then by Prop. 7, we see that $(x_1, x_2) \in (X \times_S X)(k)$ is <u>not</u> in $\Gamma(k)$ if and only if

(i) either one of them, say $x_1 \in X^{ss}(k)$ and $x_2 \in (X - X^{ss})(k)$; or

(ii) both $x_1, x_2 \in X^{ss}(k)$, but
$$\overline{O(x_1)} \cap \overline{O(x_2)} = \phi \text{ (orbit closures in } X^{ss} \otimes k)$$

Since $B \otimes_R B$ is Noetherian (B an algebra of finite type over R, R Noetherian), $\exists$ $F_i \in B^G_+$, homogeneous $(1 \leq i \leq n)$ such that $(F_i \otimes 1 - 1 \otimes F_i)$ generate the ideal I. It follows then that if (x_1, x_2) satisfying (i) or (ii) above, $\exists$ some F_i such that
$$F_i(x_1) \neq 0, \quad F_i(x_2) = 0$$
Thus we have found the required $F_1, \ldots, F_n$.

Let P_1 be the graded subalgebra of B^G generated by $F_1, \ldots, F_n$. Choose now any filtration of B^G by finitely generated R-algebras P_i, all containing P_1
$$\bigcup_{i \geq 1} P_i = B^G; \quad P_1 \subset P_2 \subset P_3 \ldots \subset P_i \subset P_{i+1}$$
Let $Y_i = \operatorname{Proj}. P_i$. Then Y_i' is the

canonical rational morphism $\psi_i' : X \longrightarrow Y_i$, we see that ψ_i' is a morphism in X^{ss}. Let $\psi_i = \psi_i' | X^{ss}$. Then by Prop. 8, ψ_i is surjective [ψ_i being dominant $(P_i \hookrightarrow B)$, $P_i \subset B^G$ and G-invariant]. It is now clear that the property (b) of the Proposition follows, as well as: ~~that~~

$$\begin{tikzcd} X^{ss} \arrow[d, "\psi_i"'] \arrow[dr, "\psi_{i+1}"] & \\ Y_i & Y_{i+1} \arrow[l, "\theta_i"'] \end{tikzcd}$$

θ_i is a <u>bijective</u> morphism

Since Y_i are projective, the bijective morphism θ_i is in fact a finite morphism. It follows that P_{i+1} is integral over P_i etc. q.e.d.

<u>Corollary 1</u>: Assume that R is of finite type over <u>universally Japanese</u> ring and B is an <u>integral domain</u>. Then B^G is an R-algebra of finite type. (A ring A is said to be universally Japanese, if it is a Noetherian domain such that if A' is any domain which is an A-algebra of finite type, the integral closure of A' in a finite extension of the quotient field of A' is an A' module of finite type). In particular $B^G =$ some P_i. $(Y = \operatorname{Proj} P_i$. We denote by $\psi = \psi_i$ the canonical morphism $\psi : X^{ss} \to Y$.

<u>Proof</u>: Now P_1 is a <u>domain</u> and being of finite type over R is also universally Japanese. Let L_1 be the quotient field of P_1, L_2 that of B^G and L_3 that of B $(L_1 \subset L_2 \subset L_3)$.

Since B is of finite type over R, it follows that L_3 is of finite type over L_1 and hence L_2 is of finite type over L_1. Since B^G is integral over P_1, it follows that L_2 is algebraic over L_1, so that it follows that L_2 is a finite extension of L_1. Hence the integral closure Q of P_1 in ~~B^G_λ~~ L_2 is a P_1 module of finite type and ~~&~~ since $B^G \subset Q$, B^G is a P_1-module of finite type. This implies that B^G is an R-algebra of finite type.

<u>Corollary 2:</u> Let $f \in B^G$ be homogeneous of deg > 0. and B an integral domain with R of finite type over a universally ring. Then the canonical morphism (of affine schemes)

$$\Psi_f : X_f \longrightarrow Y_f \quad (Y = \mathrm{Proj}\, B^G)$$

is surjective, Y_f is of finite type over ~~P~~ S and Ψ_f has properties (b) and (d) of the Proposition.

<u>Proof:</u> This is an ~~ad~~ immediate consequence of Cor 1.

<u>Corollary 3:</u> Let G act on $\mathbb{A}^n_S = \mathrm{Spec}\, S(V^*)$, where V is a G-R module, free of rank n over R. Let X_1 be a ~~G~~ closed G-stable subscheme of $\mathbb{A}^n_S$ with $X_1 = \mathrm{Spec}\, B_1$ (B_1 not necessarily graded). Suppose that B_1 is a <u>domain</u> and R is of ~~finite~~ finite type over a

a universally Japanese ring, then B_1^G is an R-algebra of finite type and the canonical morphism $\psi_1 : X_1 \to Y_1 \,(= \operatorname{Spec} B_1^G)$ is surjective and it has the properties (b) and (c) of the Proposition.

Proof: Now consider the G-module $W = V \oplus R$, where G acts trivially on R. Let f be the element in W^* ~~such that f~~ given by $f : W \to R$ such that $f(V) = 0$ and $f|R = $ Identity. Then $f \in (W^*)^G$ and we have

$$\mathbb{P}(W)_f = \mathbb{A}^n_S = \operatorname{Spec} S(V^*)$$

and then we reduce easily to Cor. 2, q.e.d.

Theorem 2: Let G be a reductive group scheme over $S = \operatorname{Spec} R$, R being of finite type over a universally Japanese ring. We are given a linear action of G on $\mathbb{A}^n_S$ and a closed G-stable subscheme $X = \operatorname{Spec} B$ of $\mathbb{A}^n_S$. Let M be a G-B module of finite type over B (i.e. a coherent G-$\mathcal{O}_X$ module). Then we have

(i) $Y = \operatorname{Spec} B^G$ is of finite type over S i.e. B^G is an R-algebra of finite type

(ii) M^G is a B^G module of finite type

(iii) the canonical morphism $\varphi : X \to Y$

induced by $B^G \hookrightarrow B$ is surjective and it has the properties (b) and (d) of Prop. 9.

<u>Proof:</u> We have only to prove (i) and (ii). ~~For this~~ Since we can consider M canonically as an $S(V^*)$ module, it suffices to consider the case $B = S(V^*)$. In this case (i.e. $B = S(V^*)$) we see that proving (i) and (ii) are equivalent to proving that M^G is a Noetherian B^G module, for in particular, B^G is a Noetherian module over itself and being graded, it follows that it is of finite type over R.

The proof is by a "devissage" argument (or Noetherian induction). We have seen (Cor. 3, Prop. 9) that if C is a quotient of $B = S(V^*)$ and is a domain C^G is an R-algebra of finite type.

Let $\mathcal{M}$ be the category of G-B modules M of finite type over B such that M^G is Noetherian over B^G. Then we see easily the following:

(i) $M \in \mathcal{M}$ and N a G-B submodule of M, then $N \in \mathcal{M}$

(ii) if $0 \to M_1 \to M_2 \to M_3 \to 0$ is an exact sequence of G-B, then we have

$$M_1, M_3 \in \mathcal{M} \implies M_2 \in \mathcal{M}$$

(for $0 \to M_1^G \to M_2^G \to M_3^G$ is exact).

We have only to show that

$\mathcal{M}$ = category of all G-B modules of finite type over B

Let then M be a G-B module of finite type over B. For proving the above equality, we can assume the (Noetherian) inductive hypothesis, that if N is any G-B module of ~~finite type~~ over B such that

$$\operatorname{Supp} N \text{ (Support of } N) \subsetneqq \operatorname{Supp} M$$
(support in the sense of B modules)

then $N \in \mathcal{M}$. Let $I = \operatorname{Ann} M$ (as a B-module). Then I is a G-stable ideal in B and $C = B/I$ is a G-R algebra and M is canonically a G-C module (Cor., Prop. 2)

We show now that $M \in \mathcal{M}$ if C is a <u>domain</u>. If $T(M)$ is the torsion submodule of M, we have

$$0 \to T(M) \to M \to M/T(M) \to 0, \text{ exact}$$

Now by the induction hypothesis $T(M) \in \mathcal{M}$; hence it suffices ~~to~~ to show that the torsion free C-module ~~is in M is in $\mathcal{M}$~~. $M/T(M)$ is in $\mathcal{M}$. Suppose then M is ~~f~~ torsion free. If $M^G = 0$, there is nothing to prove. Suppose then $M^G \neq 0$. Then the map

$$C \to M, \quad x \mapsto x \cdot m, \quad m \in M^G, m \neq 0$$

is an injective G-B homomorphism. Then

Consider
$$0 \to C \to M \to M/C \to 0$$

Now the rank or the ~~support~~ support of M/C drops and $C \in \mathcal{M}$. Hence by our ~~d~~ induction hypothesis $M \in \mathcal{M}$.

Then by a simple argument, we show that $M \in \mathcal{M}$ if C is <u>reduced</u>. For the general case, let
$$J = \mathrm{Ker}; C \to C_{red}$$
~~We have $J^s = 0$ and we have ~~a~~ a filtration~~

We have seen J^m are G-stable ideals in C (Cor., Prop. 2). This implies that $J^m M$ are G-C submodules of M, since $J^m M$ is the image under the G-R homomorphism (Prop. 2)
$$J^m \otimes_R M \longrightarrow M$$
We have then a filtration of G-C modules:
$$M \supset JM \supset J^2 M \supset \cdots \supset J^s M \quad (J^s M = 0 \text{ for an } s).$$
Now M/JM and JM/J^2M are modules over $C/J = C_{red}$ and hence they are in $\mathcal{M}$. Then it follows easily that $M \in \mathcal{M}$, q.e.d.

<u>Theorem 3:</u> Let G be a reductive group scheme ~~acting~~ over $S = \mathrm{Spec}\, R$ with R of finite type over a universally Japanese ring. Let V be a G-R module, free of ~~finite rank over R~~ $(n+1)$ over R. Let X be a

closed G-stable subscheme of $\mathbb{P}^n_S = \operatorname{Proj} S(V^*)$, with $X = \operatorname{Proj} B$ (B graded quotient of $S(V^*)$). Let $Y = \operatorname{Proj} B^G$ and $\varphi' : X \to Y$ the canonical rational morphism induced by $B^G \hookrightarrow B$. Then we have

(i) φ' is a morphism in X^{ss}, denote $\varphi = \varphi'$.

(ii) φ is surjective

(iii) Y is of finite type over S.

(iv) the morphism $\varphi : X^{ss} \to Y$ satisfies the properties (b) and (d) of Prop. 9.

Proof: By Th. 2, Y is of finite type over S. The other properties ~~now~~ follow by our now familiar arguments.

Remark 2: It can be shown easily that the morphism φ in Th. 2 and Th. 3 are categorical quotients. However it need not be a universal categorical quotient i.e. say for the morphism $\varphi : X \to Y$, $\forall$ base change $Y' \to Y$, $X \times_Y Y' \to Y'$ need not be a categorical quotient. If R contains a field of characteristic zero, φ is a universal categorical quotient, however even if R = field of char > 0, φ need not be a universal categorical quotient.