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On reductive algebraic semigroups

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ON REDUCTIVE ALGEBRAIC SEMIGROUPS

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An (affine) algebraic semigroup is an affine algebraic variety S with an associative multiplication
$$\mu : S \times S \to S,$$
which is a morphism of algebraic varieties. A zero of a semigroup S is such an element 0 (if it exists) that $0s = s0 = 0$ for any $s \in S$.

Any (affine) algebraic group is an algebraic semigroup. An important example of an algebraic semigroup which is not a group, is the semigroup $\operatorname{End} V$ of endomorphism of a (finite-dimensional) vector space V. Moreover, if $\dim V = n$, then for any $r = 1, \dots, n$
$$\operatorname{End}_r V = \{A \in \operatorname{End} V : \operatorname{rk} A \le r\}$$
is an algebraic semigroup with zero (but without unit, unless $r = n$).

It is well-known that any algebraic group is isomorphic to a (Zarisky) closed subgroup of the group $\operatorname{Aut} V = GL(V)$ of automorphisms of a suitable vector space V. A slight modification of the proof of this theorem allows us to prove that any algebraic semigroup S is isomorphic to a closed subsemigroup of $\operatorname{End} V$ for a suitable V. Moreover, if S has a unit, one may assume that it corresponds to the identity map of V under this isomorphism. (See [3] or [7] for details.) In this situation, an element of S is invertible if and only if it corresponds to an element of $GL(V)$. It follows that the group $G(S)$ of invertible elements (the unit group) of S is open in S and is an algebraic group. In particular, if S is a group, it is an algebraic group.

In what follows we assume that the base field k is algebraically closed of characteristic 0 and the variety S is irreducible. An algebraic semigroup S is called (geometrically) normal, if the variety S is normal.

For semigroups with units (monoids), we shall assume that their homomorphisms take the unit to the unit. Note that if $\varphi : S \to S'$ is a dominant homomorphism of algebraic semigroups with units, then $\varphi(G(S))$ is an open subgroup in $G(S')$ and hence $\varphi(G(S)) = G(S')$.

An algebraic semigroup S with unit is called reductive, if the group $G(S)$ is reductive. One can show (see [7], [8] and Proposition 1 below) that $G(S)$ cannot be semisimple, unless S is a group.

Reductive algebraic semigroups were studied by Putcha [2], [3] and Renner [4]–[7]. In particular, Renner classified the reductive semigroups S satisfying the following conditions:

(R1) the center of $G(S)$ is one-dimensional;

(R2) S has a zero;

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(R3) S is normal.

An example of a reductive semigroup, satisfying the conditions (R1) and (R2), is

$$S = \overline{k^* G_0} \subset \operatorname{End} V,$$

where $G_0 \subset GL(V)$ is a connected semisimple linear group. In this example, $G(S) = k^* G_0$. In particular, if $G_0 = SL(V)$, then $S = \operatorname{End} V$.

Roughly speaking, Renner's result reduces to the assertion, that such a semigroup S is uniquely determined by $G(S)$ and the closure of a maximal torus T of $G(S)$, which may be any affine embedding of T, equivariant with respect to the action of T on itself by multiplications and to the Weyl group. Apparently, the condition (R1) is not really essential for Renner's method. However, in this article we propose another approuch to the classiication problem.

The commutative reductive semigroups were studied by Neeb [18].

Now we state the results of the article. Their proofs are given in §1 – §9.

1. Let S be a reductive semigroup and $G = G(S)$. We define an action of $G \times G$ on S by

$$(g_1, g_2) \circ s = g_1 s g_2^{-1}.$$

The algebra $k[S]$ is a $(G \times G)$-invariant subalgebra of $k[G]$.

Let T be a Cartan subgroup of G and B a Borel subgroup containing T. We denote by $\mathfrak{X}$ the character group of T and by $\mathfrak{X}_+$ the semigroup of dominant characters with respect to B.

It is well-known that

$$k[G] = \bigoplus_{\Lambda \in \mathfrak{X}_+} k[G]_\Lambda, \tag{1}$$

where $k[G]_\Lambda$ denotes the linear space of the matrix entries of the irreducible linear representation $R^{(\Lambda)}$ of G with highest weight Λ. The summands of (1) are minimal $(G \times G)$-invariant subspaces, and the corresponding irreducible representations of $G \times G$ are mutually non-isomorphic. It follows that any $(G \times G)$-invariant subspace of $k[G]$ is a sum of some of $k[G]_\Lambda$. In particular,

$$k[S] = \bigoplus_{\Lambda \in \mathfrak{L}} k[G]_\Lambda, \tag{2}$$

where $\mathfrak{L} = \mathfrak{L}(S)$ is a subset of $\mathfrak{X}_+$.

The multiplication

$$\mu : G \times G \to G$$

in the group G defines, and is defined by, the algebra homomorphism

$$\mu^* : k[G] \to k[G] \otimes k[G],$$

which is called the comultiplication in the algebra $k[G]$. It is given by the following formula: if $f_{ij}^{(\Lambda)}$ denotes the (i,j)-th matrix entry of $R^{(\Lambda)}$, then

$$\mu^* f_{ij}^{(\Lambda)} = \sum_k f_{ik}^{(\Lambda)} \otimes f_{kj}^{(\Lambda)}. \tag{3}$$

Obviously, the comultiplication in $k[S]$ is just the restriction of that in $k[G]$.

Thus distinguishing S among all algebraic semigroups, containing G as the unit group, reduces to indicating $\mathfrak{L}$. We shall say that $\mathfrak{L}$ defines S, and denote $S = S(\mathfrak{L})$.

For any $\Lambda, M \in \mathfrak{X}_+$, we denote by $\mathfrak{X}(\Lambda, M)$ the set of the highest weights of irreducible components of the representation $R^{(\Lambda)}R^{(M)}$. It is known that $\mathfrak{X}(\Lambda, M) \ni \Lambda + M$. We have

$$k[G]_\Lambda k[G]_M = \bigoplus_{N\in\mathfrak{X}(\Lambda,M)} k[G]_N. \tag{4}$$

It follows that $\mathfrak{L}(S) = \mathfrak{L}$ satisfies the condition

$$\Lambda, M \in \mathfrak{L} \Rightarrow \mathfrak{X}(\Lambda, M) \subset \mathfrak{L}. \tag{5}$$

In particular, $\mathfrak{L}$ is a subsemigroup (containing 0) of $\mathfrak{X}_+$.

We call a subsemigroup $\mathfrak{L} \subset \mathfrak{X}_+$ perfect if it contains 0 and satisfies the condition (5).

Theorem 1. *A subset $\mathfrak{L} \subset \mathfrak{X}_+$ defines an algebraic semigroup, containing G as the unit group, if and only if it is a perfect finitely generated subsemigroup, generating the group $\mathfrak{X}$.*

2. It we require that S be normal, a more explicit description of $\mathfrak{L}(S)$ is available.

Let $\mathfrak{g}$ and $\mathfrak{t}$ be the tangent algebras of G and T, respectively. Identifying characters of T with their differentials[1], we put

$$\mathfrak{t}(\mathbb{Q}) = \{h \in \mathfrak{t} : \Lambda(h) \in \mathbb{Q} \quad \forall\Lambda \in \mathfrak{X}\},$$

so the dual space $\mathfrak{t}(\mathbb{Q})^*$ is identified with $\mathfrak{X} \otimes \mathbb{Q}$. Let $\alpha_1, \ldots, \alpha_n$ be the simple roots of G and $h_1, \ldots, h_n$ the corresponding dual roots. The Weyl chamber $C \subset \mathfrak{t}(\mathbb{Q})^*$ is defined by

$$C = \{\Lambda \in \mathfrak{t}(\mathbb{Q})^* : \Lambda(h_i) \geq 0 \qquad (i = 1, \ldots, n)\}.$$

The group G and the torus T decompose into the almost direct products

$$G = ZG_0, \quad T = ZT_0,$$

where Z is the connected center and G_0 the commutator group of G, and $T_0 = T \bigcap G_0$ a Cartan subgroup of G_0.

Let $\mathfrak{z}, \mathfrak{g}_0$, and $\mathfrak{t}_0$ denote the tangent algebras of Z, G_0, and T_0, respectively. Then

$$\mathfrak{g} = \mathfrak{z} \otimes \mathfrak{g}_0, \quad \mathfrak{t} = \mathfrak{z} \otimes \mathfrak{t}_0.$$

It $\mathfrak{z}(\mathbb{Q}) = \mathfrak{z} \cap \mathfrak{t}(\mathbb{Q}), \mathfrak{t}_0(\mathbb{Q}) = \mathfrak{t}_0 \cap \mathfrak{t}(\mathbb{Q})$, then

$$\mathfrak{t}(\mathbb{Q})^* = \mathfrak{z}(\mathbb{Q})^* \otimes \mathfrak{t}_0(\mathbb{Q})^*, \tag{6}$$

$$C = \mathfrak{z}(\mathbb{Q})^* + C_0,$$

where $C_0 \subset \mathfrak{t}_0(\mathbb{Q})^*$ is the Weyl chamber of G_0.

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[1] Since the addition in the group $\mathfrak{X}$ corresponds to the multiplication in the algebra $k[T]$, characters of T, when considered as elements of this algebra, are denoted as exponentials.

Theorem 2. *A subset $\mathfrak{L} \subset \mathfrak{X}_+$ defines a normal algebraic semigroup, containing G as the unit group, if and only if $\mathfrak{L} = \mathfrak{X}_+ \bigcap K$, where K is a closed convex polyhedral cone in $\mathfrak{t}(\mathbb{Q})^*$ satisfying the conditions*

(1) $K \ni -\alpha_1, \dots, -\alpha_n$;
(2) *the cone $K \bigcap C$ generates $\mathfrak{t}(\mathbb{Q})^*$. The semigroup $S(\mathfrak{L})$ has a zero if and only if*
(3) *the cone $D = K \bigcap \mathfrak{z}(\mathbb{Q})^*$ is pointed;*
(4) $K \bigcap C_0 = \{0\}$.

We emphasize that any subset $\mathfrak{L} \subset \mathfrak{X}_+$ satisfying the conditions (1) and (2) of the theorem, automatically satisfies the conditions of Theorem 1.

*Remarks.*1. The projection of K on $\mathfrak{t}_0(\mathbb{Q})^*$ is a convex cone containing an interior point of C_0 and the negative simple roots. Hence it is the whole space $\mathfrak{t}_0(\mathbb{Q})^*$.

2. Since $C_0 \subset \operatorname{conv}\{\alpha_1, \dots, \alpha_n\}$, the projection of $K \bigcap C$ on $\mathfrak{z}(\mathbb{Q})^*$ is contained in K (and coincides with D). It is a generating convex cone in $\mathfrak{z}(\mathbb{Q})^*$.

3. We may (cand will) assume that the cone K is the greatest one among the convex cones having the same intersection with C. This means that any hyperplane bounding K bounds $K \bigcap C$. Under this condition, the cone K is uniquely determined by the semigroup.

Corollary. *Any normal reductive semigroup decomposes into an almost direct product of a reductive group and a (normal) reductive semigroup with zero.*

(For non-normal semigroups this is not true.)

"An almost direct product" means a quotient of the direct product with respect to a finite central subgroup. Note that if Γ is a finite central subgroup of an algebraic semigroup S, then the quotient semigroup S/Γ turns to be an algebraic semigroup, being supplied with a structure of an affine algebraic variety as the invariant-theoretic quotient $S/\!\!/\Gamma$. (Since Γ is finite, the fibers of the canonical morphism $S \to S/\!\!/\Gamma$ are exactly the Γ-orbits: see, for example, [9].) We shall say that S is a covering semigroup of S/Γ.

3 Let S be a reductive semigroup with $G(S) = G$. Since the Borel subgroup of $G \times G$ has an open orbit in G, S is a spherical $(G \times G)$-variety and hence ontains only finitely many $(G \times G)$-orbits [11]. (More immediately, this follows from their description given below.)

Consider now the $(G_0 \times G_0)$-action on S. Let

$$A = A(S) = S/\!\!/(G_0 \times G_0) \tag{7}$$

be the invariant-theoretic quotient of S with respect to this action. By definition, A is the spectrum of the subalgebra

$$k[S]^{G_0 \times G_0} \subset k[S],$$

consisting of the $(G_0 \times G_0)$-invariant polynomial functions on S.

Denote by $\mathfrak{X}_Z$ the subgroup of $\mathfrak{X}$, consisting of the characters, vanishing on $\mathfrak{t}_0$. (These are the (highest) weights of the one-dimensional representations of G.) Then

$$k[A] = k[S]^{G_0 \times G_0} = \bigoplus_{\Lambda \in \mathfrak{L}_Z} k[G]_\Lambda, \tag{8}$$

where $\mathfrak{L}_Z = \mathfrak{L} \bigcap \mathfrak{X}_Z$. The embedding $k[A] \subset k[S]$ defines the canonical morphism

$$\pi : S \to A \tag{9}$$

According to a general theorem of the invariant theory (see, for example, [9]), π is surjective. It S is normal, A is also normal.

Since the subalgebra $k[A] \subset k[S]$ is $(G \times G)$-invariant, the action of the group $G \times G$ on S induces its action on A in such a way that the morphism π is equivariant. Obviously, the latter action reduces to an action of the torus

$$G/G_0 = Z/Z_0,$$

where $Z_0 = Z \bigcap G_0$.

Moreover, it follows from (3) that

$$\mu^* k[A] \subset k[A] \otimes k[A].$$

Thereby A is endowed with a structure of a (commutative) algebraic semigroup in such a way that the morphism π is a semigroup homomorphism. The image of the unit of S is a unit of A. It S has a zero, its image is a zero of A.

Definition 1. The algebraic semigroup $A = A(S)$, together with the homomorphism $\pi : S \to A$, is called the abelization of S.

According to the theory of toric varieties [13], the $G(A)$-orbits in A are in a one-to-one correspondence with the (closed) faces of the cone $D = \mathbb{Q}_+ \mathfrak{L}_Z$ in such a way that the ideal of (the closure of) an orbit is spanned by those subspaces $k[A]_\chi = k[G]_\chi$, for which χ does not belong to the corresponding face. This correspondence is monotone in the following sense: for two orbits O_1, O_2, corresponding to faces F_1, F_2, we have $O_1 \subset \overline{O_2}$ if and only if $F_1 \subset F_2$. The orbit O, corresponding to a face F, contains a (unique) idempotent e_F, defined by

$$\chi(e_F) = \begin{cases} 1, & \chi \in F, \\ 0, & \chi \notin F. \end{cases} \tag{10}$$

In an analogous way, the $(G \times G)$-orbits in S are in a monotone one-to-one correspondence with faces of the cone $\mathbb{Q}_+ Z$, but in general not with all of them. The ideal of the orbit, corresponding to a face F, is spanned by those subspaces $k[G]_\Lambda$, for which $\Lambda \notin F$. In particular, there are only finitely many $(G \times G)$-orbits.

Denote by $\bar{Z}$ the closure of Z in S.

Theorem 3. *Let S be a reductive semigroup. Then*

1) $\pi^{-1}(e) = G_0$;
2) $\pi(\bar{Z}) = A$;
3) *the closed $(G_0 \times G_0)$-orbits are exactly those meeting $\bar{Z}$. Moreover, if S is normal, then*
4) π *induces an isomorphism $\bar{Z}/Z_0 \simeq A$;*
5) *the closure of any $(G_0 \times G_0)$-orbit is normal.*

It ıollows from 1) that

$$G(A) = G/G_0 = Z/Z_0$$

and the restriction of π on G is the canonical homomorphism $G \to G/G_0$. Moreover,

$$\pi^{-1}(G(A)) = G.$$

4. It is of special interest to distinguish the cases when the morphism π is flat. In these cases, the fibers of π are equidimensional and, if S has a zero, the triple (S, A, π) can be considered as a multi-parameter contraction of the $(G_0 \times G_0)$-action on G_0 to that on $\pi^{-1}(0)$. We shall see that, under some restrictions, the result of this contraction does not depend on S. The action $G_0 \times G_0$ on $\pi^{-1}(0)$ is special which means that the stabilizer of any point contains a maximal unimpotent subgroup of $G_0 \times G_0$.

A canonical (one-parameter) contraction of any action of a reductive group on an affine variety to a special one was considered by Popov [10] (see also [11]). In the case of the action of $G_0 \times G_0$ on G_0, the result of our contraction is just the same.

Definition 2. A normal reductive semigroup S is called flat, if the morphism π is flat and its fibers are reduced (as schemas) and irreducible.

The morphism π is flat if and only if $k[S]$ is a free $k[A]$-module (Proposition 3). Even in this case, the fibers of π need not be reduced : see an example in 4.2.

According to the decomposition (6), we represent an element of $\mathfrak{t}(\mathbb{Q})^*$ as a pair (χ, λ), where $\chi \in \mathfrak{z}(\mathbb{Q})^*, \lambda \in \mathfrak{t}_0(\mathbb{Q})^*$.

Theorem 4. *Let $S = S(\mathfrak{L})$ be a normal reuctive semigroup. In the notation of Theorem 2, the semigroup S is flat if and only of there are such a convex polyhedral cone $D \subset \mathfrak{z}(\mathbb{Q})^*$ and a homomorphism $\theta : Z \to T_0$ that*

$$\theta|_{Z_0} = id \tag{11}$$

and the cone $K = K(S)$ has the form

$$K = \{(\chi, \lambda) \in \mathfrak{t}(\mathbb{Q})^* : \chi - \theta^*(\lambda) \in D\}. \tag{12}$$

For such a cone K, the conditions of Theorem 2 look as follows:

(1) $\theta^*(\alpha_i) \in D \quad (i = 1, \ldots, n)$;
(2) the cone D generates $\mathfrak{z}(\mathbb{Q})^*$;
(3) the cone D is pointed;
(4) ${\theta^*}^{-1}(D) \bigcap (-C_0) = \{0\}$.

It S has a zero, the fiber $\pi^{-1}(0)$ is an ideal of S. As an algebraic semigroup with a $(G_0 \times G_0)$-action, it depends only on G_0, provided S is flat (see 4.4). Like the asymptotic cone of a hyperboloid, it reflects on the behaviour of G_0 at infinity. We call it the asymptotic semigroup of G_0 and denote by $\operatorname{As} G_0$. A separate paper [12] is devoted to a more detailed investigation of it.

In general, if S is flat, all the fibers of π are spherical $(G_0 \times G_0)$-varieties and $(G_0 \times G_0)$-orbits are just the intersections of $(G \times G)$-orbits with the fibers (Proposition 5). (For a reductive group L, an irreducible L-variety X is called spherical, if the Borel subgroup of L has an open orbit in X. In this case, L has only finitely many orbits in X [11].)

5. Any homomorphism

$$\varphi : S' \to S$$

of reductive algebraic semigroups gives rise to a homomorphism of their abelizations:

$$\varphi_{\mathrm{ab}} : A' \to A$$

in such a way that the diagram

$$\begin{array}{ccc} S' & \xrightarrow{\ \varphi\ } & S \\ {\scriptstyle \pi'}\downarrow & & \downarrow{\scriptstyle \pi} \\ A' & \xrightarrow[\ \varphi_{\mathrm{ab}}\ ]{} & A \end{array}$$

is commutative.

Consider the fibered product

$$\hat{S} = A' \times_A S = \{(a', s) \in A' \times S : \varphi_{\mathrm{ab}}(a') = \pi(s)\}.$$

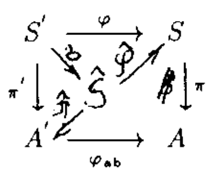
It is a closed subsemigroup of $A' \times S$ and the canonical projections

$$\hat{\pi} : \hat{S} \to A', \quad \hat{\varphi} : \hat{S} \to S$$

are semigroup homomorphisms. There is a (unique) homomorphism

$$\sigma : S' \to \hat{S}$$

such that the diagram



is commutative.

Definition 3. The homomorphism φ is called excellent if σ is an isomorphism.

Note that the semigroup $\hat{S}$ is reductive and φ maps isomorphically the commutator group of $G(\hat{S})$ onto that of $G(S)$. So if the homomorphism φ is excellent, it maps isomorfically the commutator group of $G(S')$ onto that of $G(S)$. Moreover, if S is flat, so is S' (Proposition 6).

A standard consideration with commutative diagrams shows that the product of excellent homomorphisms is also excellent.

For a fixed connected semisimple group G_0, denote by $\mathcal{FS}(G_0)$ the class of all the flat reductive semigroups, whose commutator group of the unit group is isomorphic to G_0. It turns out that there is a distinguished semigroup $S \in \mathcal{FS}(G_0)$, which is universal in a sense. In the statement of the following theorem, we identify the commutator group of $G(S)$ with G_0.

Theorem 5. *There is a semigroup with zero $S \in \mathcal{FS}(G_0)$ satisfying the following condition:*

(*) For any semigroup $S' \in \mathcal{FS}(G_0)$ and any isomorphism φ_0 of the commutator group G_0' of $G(S')$ onto G_0, there is an excellent homomorphism

$$\varphi : S' \to S,$$

whose restriction to G_0' coincides with φ_0. Moreover, if S' has a zero, such a homomorphism is unique.

It is clear that such a semigroup S is unique up to isomorphism. We call it the enveloping semigroup of G_0 and denote by $\operatorname{Env} G_0$.

In terms of Theorem 4, the semigroup $S = \operatorname{Env} G_0$ is described as follows:

(1) Z_0 is the whole center of G_0;

(2) θ is an isomorphism;

(3) the cone D is generated by the forms $\theta^*(\alpha_i), i = 1, \ldots, n$.

For any $\lambda \in \mathfrak{t}_0(\mathbb{Q})^*$, we shall denote $\theta^*(\lambda)$ by $\bar{\lambda}$.

6. Now we describe the $(G \times G)$-orbit structure of $S = \operatorname{Env} G_0$.

The faces of the cone D are enumerated by the subsets of $\Omega = \{1, \ldots, n\}$ in such a way that to a subset I, there corresponds the face D_I spanned (as a convex cone) by $\alpha_i, i \in I$. We denote by O_I the Z-orbit ($= G(A)$-orbit) in A, corresponding to D_I.

The cone $K \bigcap C$ is linearly, and hence combinatorially, isomorphic to the direct product $D \times C_0$ (so it is a simplicial cone). The cone C_0 is spanned by the fundamental weights $\omega_1, \ldots, \omega_n$ of $\mathfrak{g}_0$. For $J \subset \Omega$ let us denote by C_J its face spanned by $\omega_j, j \in J$. In this notation, the faces of $K \bigcap C$ are

$$F_{I,J} = \{(\chi, \lambda) \in \mathfrak{t}(\mathbb{Q})^* : \chi - \bar{\lambda} \in D_I, \lambda \in C_J\}, \tag{13}$$

where $I, J \subset \Omega$.

Let Σ be the Dynkin diagram of $\mathfrak{g}$ and $v_1, \ldots, v_n$ its vertices enumerated in accordance with the enumeration of the simple roots. For $I \subset \Omega$, we denote by Σ_I the subdiagram of Σ, constituted by the vertices $v_i, i \in I$. The subsets of I, corresponding to the connected components of Σ_I, will be called the connected components of I.

Definition 4. A pair (I, J) and the corresponding face $F_{I,J}$ of $K\bigcap C$ are called essential, if no connected component of the complement of J is entirely contained in I.

Theorem 6. *The $(G\times G)$-orbits in $S=\operatorname{Env} G_0$ are in a monotone one-to-one correspondence with the essential faces of the cone $K\bigcap C$.*

We denote by $O_{I,J}$ the orbit corresponding to an (essential) face $F_{I,J}$. Its ideal is spanned by the subspaces $k[G]_\Lambda$ with $\Lambda\notin F_{I,J}$. Clearly,

$$\pi(O_{I,J})=O_I. \tag{14}$$

In particular, if $I=\Omega$, then the only possibility for J is to be equal to Ω as well. This means that $\pi^{-1}(G(A))=G$, which also follows from Theorem 3. On the contrary, if $I=\varnothing$, then J may be an arbitrary subset of Ω, so $\pi^{-1}(0)=\operatorname{As} G_0$ decomposes into 2^n $(G\times G)$-orbits.

For any I, there is the least admissible J, namely, the union of the connected components of Ω, entirely contained in I. The corresponding orbit $O_{I,J}$ is the unique orbit which is closed in $\pi^{-1}(O_I)$. At the same time, there is the greatest admissible J, namely, the whole set Ω. The corresponding orbit is the unique orbit which is open in $\pi^{-1}(O_I)$.

7. Let us describe the stabilizers of the $(G\times G)$-action on S.

According to general results of Putcha [1],[2] for reductive semigroups, each $(G\times G)$-orbit $O_{I,J}$ contains an idempotent defined up to conjugacy. It can be chosen in $\bar{T}$ and, under this condition, it is defined up to the action of the Weyl group. We denote such an idempotent by $e_{I,J}$ and will describe its stabilizer. An interpretation of $e_{I,J}$ is given in 7.3.

Let B be the Borel subgroup of G and $\mathfrak{b}$ its tangent algebra. For any subset $M\subset\Omega$, we denote by $P(M)$ the parabolic subgroup of G, whose tangent algebra is generated by $\mathfrak{b}$ and the root vectors, corresponding to the roots $-\alpha_i, i\in M$ (so $B=P(\varnothing)$). We have

$$P(M)=U(M)R(M), \tag{15}$$

where $U(M)$ is the unipotent radical and $R(M)$ a maximal reductive subgroup of $P(M)$. We shall assume that $R(M)\supset T$. Under this condition, $R(M)$ is uniquely defined. We denote by $G(M)$ its commutator group.

Let $P_-(M)$ be the parabolic subgroup which is opposite to $P(M)$ and $U_-(M)$ its unipotent radical. Then

$$P_-(M)=U_-(M)R(M). \tag{16}$$

We denote by δ (resp. δ_-) the projection of $P(M)$ (resp. $P_-(M)$) onto $R(M)$ with respect to the decomposition (15) (resp. (16)).

We call two elements of Ω adjacent, if such are the corresponding vertices of the Dynkin diagram. For a subset $M\subset\Omega$ we denote by $C(M)$ its complement and by M° its "interior", consisting of its elements, which are not adjacent to any elements of $C(M)$.

Let now $O_{I,J}$ be a $(G\times G)$-orbit in $S=\operatorname{Env} G_0$. Put

$$M=(I\bigcap J^\circ)\bigcup C(J) \tag{17}$$

and define a torus $T_{I,J} \subset T$ by

$$T_{I,J} = \{t \in T : \Lambda(t) = 1 \text{for} \Lambda \in F_{I,J}\}. \tag{18}$$

Note that $T_{I,J}$ contains $T \bigcap G(C(J))$ and $G(C(J))T_{I,J}$ is a normal subgroup of $R(M)$.

Theorem 7. *Under a suitable choice of the idempotent $e_{I,J} \in O_{I,J} \bigcap \bar{T}$, its stabilizer $H_{I,J}$ is the subgroup of $P(M) \times P_-(M)$, consisting of the pairs (g, g_-), satisfying the condition*

$$\delta(g) \equiv \delta_-(g_-) (\operatorname{mod} G(C(J))T_{I,J}). \tag{19}$$

In other words, $H_{I,J}$ is the (semidirect) product of $U(M) \times U_-(M)$, the diagonal in $R(M) \times R(M)$, and the group $G(C(J))T_{I,J} \times \{e\}$.

In particular, $H_{I,J}$ is reductive if and only if $M = \Omega$, which means that $I \supset J = J^\circ$. This is just the case when $O_{I,J}$ is closed in $\pi^{-1}(O_{I,J})$. Another characterrization of this case is that $e_{I,J} \in \bar{Z}$.

On the contrary, for the orbit $O_{I,\Omega}$, which is open in $\pi^{-1}(O_I)$, we have

$$J = J^\circ = \Omega, \quad M = I \bigcap J^\circ = I,$$

so $H_{I,\Omega}$ is the product of $U(I) \times U_-(I)$, the diagonal in $R(I) \times R(I)$, and the torus $T_I \times \{e\}$, where

$$T_I = T_{I,\Omega} = \{z\theta(z)^{-1} : z \in Z, \bar{\alpha}_i(z) = 1 (i \in I)\}. \tag{20}$$

The idempotents $e_{I,J}$ chosen as in Theorem 7 subject the relations

$$e_{I_1,J_1} e_{I_2,J_2} = e_{I_1 \bigcap I_2, J_1 \bigcap J_2}. \tag{21}$$

8. The idempotents $e_{I,\Omega}$ are just those lying in the closure of the diagonal torus

$$T_\varnothing = \{z\theta(z)^{-1} : z \in Z\}, \tag{22}$$

so

$$G\bar{T}_\varnothing G = \bigcup_I O_{I,\Omega}. \tag{23}$$

Clearly, this is an open subvariety of S.

It is easy to see that $\bar{T}_\varnothing \simeq k^n$. This fact gives rise to the following theorem.

Theorem 8. *The variety*

$$S^{\text{pr}} = \bigcup_I O_{I,\Omega} \tag{24}$$

is smooth and there is a geometric quotient S^{pr}/Z which is a smooth projective variety.

The definition of a geometric quotient see, for example, in [9].

The variety $S^{\rm pr}/Z$ inherits the $(G_0 \times G_0)$-action and contains the adjoint group G_0/Z_0 as an open orbit of this action. One can show that it is nothing else than the "wonderful" equivariant completion of G_0/Z_0 constructed by DeConcini and Procesi [14].

The subset $S^{\rm pr} \subset S$ is not a subsemigroup, so there is no natural semigroup structure on $S^{\rm pr}/Z$. On the other hand, the set theoretic quotient S/Z is a semigroup but not an algebraic variety. It seems that, for classical groups G_0, the semigroup S/Z is close, if not identical, to that constructed by Neretin [15]. Neretin's results are not used in this work, but his ideology influenced me to a certain extent.

9. If the group G_0 acts on an affine variety X_0, we can be interested in the extension of this action to an action of the semigroup $S = \operatorname{Env} G_0$ on an affine variety E containing X_0.

More generally, we can consider G_0-equivariant morphisms

$$\varphi : X_0 \to E,$$

where E is an affine variety with an action of S on it. (We assume that the unit of S acts as the identity map.) Let us call such an S-variety E, together with the morphism φ, an enveloping S-variety of X_0, if for any pair (E', φ') of the same kind, there is a unique S-equivariant morphism $\psi : E \to E'$ such that the diagram

$$\begin{array}{ccc} X_0 & \xrightarrow{\varphi} & E \\ & {\scriptstyle\varphi'}\searrow \quad \swarrow{\scriptstyle\psi} & \\ & E' & \end{array}$$

is commutative. It is clear that an enveloping S-variety, if exists, is unique in a natural sense.

Theorem 9. *For any affine G_0-variety X_0, there exists an enveloping S-variety E. The corresponding morphism φ is an isomorphism of X_0 onto a closed subvariety of E.*

We shall denote the enveloping S-variety of X_0 by $\operatorname{Env} X_0$ and identify X_0 with its image in $\operatorname{Env} X_0$.

For any affine S-variety E, we can consider the invariant-theoretic quotient $E/\!\!/G_0$. It inherits the action of S, which reduces to an action of the abelization $A = S/\!\!/(G_0 \times G_0)$ of S.

For $E = \operatorname{Env} X_0$, we have

$$E/\!\!/G_0 = A \times X_0/\!\!/G_0. \tag{25}$$

Moreover, $X_0/\!\!/G_0$ is embedded into E as the subvariety of fixed points of S (see 9.6).

10. The basic results of this work were obtained during my visit to Institut des Hautes Études Scientifiques in August of 1993. A preliminary version of the

research was reported at the meeeting on "Invariant ordering in geometry and algebra" at Mathematisches Forschungsinstitut Oberwolfach in October of 1993 and at the international meeting organized by Sondervorschungsbereiche 343 "Diskrete Strukturen in der Mathematik" in Bielefeld in November of 1993. I thank all these institutions for their hospitality. I also thank Yu.A.Neretin for fruitful discussions.

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§1. Proof of Theorem 1.

1. In the subsequent proofs, we make use of the following results which are apparently due to Khadzhiev [16], Vust [20], and Popov [10]. (See [10] for details.)

Let A be a commutative associative algebra with unit, and let a reductive group R act on A by automorphisms. We assume that any elements of A is contained in a finite-dimensional R-invariant subspace and, for any such subspace V, the induced linear representation $R \to GL(V)$ is algebraic. Let U be a maximal unipotent subgroup of R and A^U the subalgebra of U-invariant elements of A.

Consider the following properties of an algebra:

(a) it is finitely generated;
(b) it has no nilpotent elements;
(c) it has no zero divisors;
(d) it is normal.

Theorem. *([16],[20],[10]). Let (P) be any of the properties (a)–(d). The algebra A has the property (P) if and only if the algebra A^U has this property.*

2. Let $\mathfrak{L}$ be a perfect subsemigroup of $\mathfrak{X}_+$. To prove Theorem 1, we are to find out, under which conditions the algebra

$$k[G]_{\mathfrak{L}} = \bigoplus_{\Lambda \in \mathfrak{L}} k[G]_\Lambda \tag{26}$$

is finitely generated and generates the field $k(G)$.

Consider the action of $G \times G$ on $k[G]$. Denote by U the unipotent radical of the Borel subgroup B of G and by U_- the unipotent radical of the opposite Borel subgroup B_-. Then $U_- \times U$ is a maximal unipotent subgroup of $G \times G$.

For each irreducible linear representation

$$R^\Lambda : G \to GL(V^{(\Lambda)})$$

we choose a basis in $V^{(\Lambda)}$, consisting of weight vectors, the highest vector being the first of them. Then the algebra $k[G]^{U_- \times U}$ is spanned by the functions

$$\delta^{(\Lambda)} = f_{11}^{(\Lambda)}.$$

Since the highest vector of $V^{(\Lambda+M)}$ is the tensor product of those of $V^{(\Lambda)}$ and $V^{(M)}$, we have

$$\delta^{(\Lambda+M)} = \delta^{(\Lambda)}\delta^{(M)},$$

so the algebra $k[G]^{u_- \times U}$ is isomorphic to the semigroup algebra of $\mathfrak{X}_+$.

In the same way, the algebra $k[G]_{\mathfrak{L}}^{U_- \times U}$ is isomorphic to the semigroup algebra of $\mathfrak{L}$. It follows that the algebra $k[G]_{\mathfrak{L}}$ is finitely generated if and only if such is the semigroup $\mathfrak{L}$.

3. Let Quot A denote the field of quotients of an algebra A.

We have Quot $k[G]_{\mathfrak{L}} = k(G)$ if and only if the functions of $k[G]_{\mathfrak{L}}$ separate points of G, i.e. if the intersection of the kernels of the representations $R^\Lambda, \Lambda \in \mathfrak{L}$, is trivial. Let us denote this intersection by G_1.

If $G_1 \neq \{e\}$, then $T_1 = G_1 \bigcap T \neq \{e\}$, so $\mathfrak{L}$ belongs to the proper subgroup

$$\mathfrak{X}_1 = \{\Lambda \in \mathfrak{X} : \Lambda|_{T_1} = 1\} \subset \mathfrak{X}.$$

To prove the converse, we need

Lemma. *Let $\Lambda \in \mathfrak{X}_+$ and $i \in \Omega$ be such that $(\Lambda, \alpha_i) > 0$. Then $2\Lambda - \alpha_i \in \mathfrak{X}(\Lambda, \Lambda)$.*

This follows from Proposition 9 below and the decomposition rule for products of irreducible representations of SL_2.

So, if $\Lambda \in \mathfrak{L}$ and $(\Lambda, \alpha_i) > 0$, then $2\Lambda - \alpha_i \in \mathfrak{L}$, and $\alpha_i \in \mathfrak{L} - \mathfrak{L}$. Moreover, if $(\alpha_i, \alpha_j) < 0$, then $(2\Lambda - \alpha_i, \alpha_j) > 0$. It follows that the set

$$\Omega_1 = \{i \in \Omega : \alpha_i \in \mathfrak{L} - \mathfrak{L}\}$$

is a union of connected components of Ω.

Let now $G_1 = \{e\}$. Then, for each connected component of Ω, there is such $\Lambda \in \mathfrak{L}$ that $(\Lambda, \alpha_i) > 0$ for some i of this component. Hence $\Omega_1 = \Omega$, so the group $\mathfrak{L} - \mathfrak{L}$ contains the root lattice $\mathfrak{R}$ of $\mathfrak{g}$.

The quotient group $\mathfrak{X}/\mathfrak{R}$ is naturally isomorphic to the character group of the center of G, and if $\mathfrak{L} - \mathfrak{L} \neq \mathfrak{X}$, there is such an element $z \neq e$ of the center, that $\Lambda(z) = 1$ for all $\Lambda \in \mathfrak{L}$, and hence $z \in G_1$. This contradicts our assumption.

§2. Proof of Theorem 2

1. Let first $\mathfrak{L} = \mathfrak{X}_+ \bigcap K$, where $K \subset \mathfrak{t}(\mathbb{Q})^*$ is a convex polyhedral cone satisfying the conditions 1) and 2) of the theorem.

It is known (and easy to show) that the intersection of a lattice in $\mathbb{Q}^m$ with a convex polyhedral cone is a finiteely generated semigroup. Hence the semigroup $\mathfrak{L} = \mathfrak{X} \bigcap (C \bigcap K)$ is finitely generated. Since the cone $C \bigcap K$ generates the space $\mathfrak{t}(\mathbb{Q})^*$, the semigroup $\mathfrak{L}$ generates the group $\mathfrak{X}$.

Since any $N \in \mathfrak{X}(\Lambda, M)$ has the from

$$N = \Lambda + M - \sum_i k_i \alpha_i, \quad k_i \geq 0,$$

the condition 1) of the theorem guarantees that the semigroup $\mathfrak{L}$ is perfect. So it defines an algebraic semigroup S with $G(S) = G$.

Let now S be an algebraic semigroup with $G(S) = G$ defined by a semigroup $\mathfrak{L}$.

In view of Theorem stated in 1.1, the algebra $k[S]$, defined by (2), is normal if and only if such is the algebra $k[S]^{U_- \times U}$. The last algebra is isomorphic to the semigroup algebra of $\mathfrak{L}$, which is normal if and only if

$$\mathfrak{L} = \mathfrak{X} \bigcap \mathbb{Q}_+ \mathfrak{L}$$

(see, for example, [13] or [17]).

The cone $\mathbb{Q}_+\mathfrak{L}$ is a convex polyhedral cone contained in the Weyl chamber C. Let $H_1, \ldots, H_s$ be its walls distinct from the walls of C and $H_1^+, \ldots, H_s^+$ the half-spaces bounded by $H_1, \ldots, H_s$ respectively and containing $\mathbb{Q}_+\mathfrak{L}$. We put

$$K = H_1^+ \bigcap \cdots \bigcap H_s^+.$$

so that

$$\mathfrak{L} = \mathfrak{X} \bigcap (K \bigcap C) = \mathfrak{X}_+ \bigcap K.$$

Let us prove that $K \ni -\alpha_1, \ldots, \alpha_n$.

Suppose $-\alpha_i \notin H_j^+$ for some i, j and take an interior point Λ of the face $H_j \bigcap (K \bigcap C)$ of the cone $K \bigcap C$. Multiplying Λ by an integer we may assume that $\Lambda \in \mathfrak{L}$. Then $2\Lambda - \alpha_i \in \mathfrak{X}(\Lambda, \Lambda)$ but $2\Lambda - \alpha_i \notin \mathfrak{L}$, which is a contradiction.

A maximal ideal of the algebra $k[S]$ defines a zero of S if and only if it is $(G \times G)$-invariant. It such an ideal exists, it must be equal to

$$k[S]_+ = \bigoplus_{\Lambda \in \mathfrak{L} \setminus \{0\}} k[G]_\Lambda.$$

The subspace $k[S]_+$ is really an ideal if and only if

$$0 \notin \mathfrak{X}(\Lambda, M) \quad \forall \Lambda, M \in \mathfrak{L} \setminus \{0\}. \tag{27}$$

It follows from the condition 1) of the theorem that the projection of $K \bigcap C$ on $\mathfrak{z}(\mathbb{Q})^*$ is contained in $K \bigcap C$. So if the condition 3) is satisfied, $0 \in \mathfrak{X}(\Lambda, M)$ implies $\Lambda, M \in C_0$; so, if the condition 4) is satisfied, $\Lambda = M = 0$.

Conversely, if the condition 3) is violated, there are such $\Lambda, M \in (\mathfrak{L} \setminus \{0\}) \bigcap \mathfrak{z}(\mathbb{Q})^*$ that $\Lambda + M = 0$. If the condition 4) is violated, there exists $\Lambda \in (\mathfrak{L} \setminus \{0\}) \bigcap C_0$. Let m be the dimension of the representation R^Λ. We have $\det R(g) = 1$ for $g \in G$, which implies that the m-th (tensor) power of R^Λ contains the trivial representation. It follows that the condition (27) is violated.

2. For example, consider the case $G = k^* \times SL_2$. We identify the space $\mathfrak{t}(\mathbb{Q})^*$ with $\mathbb{Q}^2$ in such a way that $\mathfrak{t}_0(\mathbb{Q})^*$ is identified with the x-axis, $\mathfrak{z}(\mathbb{Q})^*$ with the y-axis, the group $\mathfrak{X}$ with $\mathbb{Z}^2$, and the only simple root α with (2,0). Then the Weyl chamber C has the form

$$C = \{(x, y) \in \mathbb{Q}^2 : x \geq 0\}.$$

A normal algebraic semigroup S with $G(S) = G$ is define, in terms of Theorem 2, by a convex cone $K \subset \mathbb{Q}^2$ with the following properties: $K \ni (-1, 0)$ and K^0 meets the right half-plane C. Moreover, since only the intersection $K \bigcap C$ is essential,

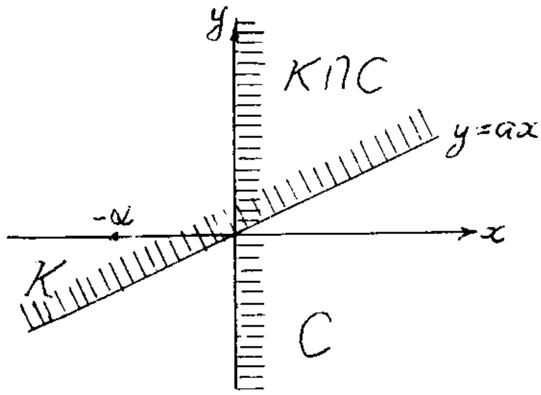

FIG.1

we may assume that each side of K meets C^0, so K is either the whole plane or a half-plane distinct from $-C$. Since the above identification of $\mathfrak{t}(\mathbb{Q})^*$ with $\mathbb{Q}^2$ is defined up to multiplying y by -1, we also may assume that $K \ni (0,1)$.

It K is the whole plane, we have $\mathfrak{L} = \mathfrak{X}_+$ and $S = G$. It K is the upper half-plane, $\mathfrak{L} = \mathbb{Z}_+^2$ and $S = k \times SL_2$.

In all the other cases, K has the form

$$K = \{(x,y) \in \mathbb{Q}^2 : y \geq ax\} \quad (a \in \mathbb{Q}, a > 0)$$

(see Fig.1), so the conditions 3) and 4) of Theorem 2 are satisfied and S is a semigroup with zero. For the further citation, we denote this semigroup by S_a.

3. Now we prove Corollary to Theorem 2.

The condition 1) of the theorem implies that if the cone K contains an interior point of some face of C_0, it contains the whole face. So $F = K \bigcap C_0$ is a face of C_0. Moreover, if $\alpha_i|_F \neq 0$ and $(\alpha_i, \alpha_j) < 0$, then $\alpha_j|_F \neq 0$, too. Hence F is spanned by the fundamental weights of some ideal of $\mathfrak{g}_0$, say, $\mathfrak{g}_0^{(1)}$, Let $\mathfrak{g}_0^{(2)}$ be the complementary ideal, $G_0^{(1)}$ and $G_0^{(2)}$ the connected subgroups of G_0, whose tangent algebras are $\mathfrak{g}_0^{(1)}$ and $\mathfrak{g}_0^{(2)}$, respectively.

Let further $Z^{(1)}$ and $Z^{(2)}$ be such almost complementary subtori of Z, that $\mathfrak{z}^{(2)}$, the tangent algebra of $Z^{(2)}$, is the annihilator of $K \bigcap (-K) \bigcap \mathfrak{z}(\mathbb{Q})^*$.

Put

$$G^{(1)} = Z^{(1)} G_0^{(1)}, \quad G^{(2)} = Z^{(2)} G_0^{(2)}.$$

Passing to a suitable covering semigroup, we may assume that

$$G = G^{(1)} \times G^{(2)}.$$

Then

$$\mathfrak{X} = \mathfrak{X}^{(1)} \oplus \mathfrak{X}^{(2)},$$

where $\mathfrak{X}^{(1)}$ and $\mathfrak{X}^{(2)}$ are the character groups of $T^1 = T \bigcap G^{(1)}$ and $T^2 = T \bigcap G^{(2)}$, respectively. With respect to this decomposition,

$$\mathfrak{L} = \mathfrak{X}_+^{(1)} \oplus (\mathfrak{X}_+^{(2)} \bigcap K^{(2)}),$$

where the cone $K^{(2)} = K \bigcap \mathfrak{z}^{(2)}(\mathbb{Q})^*$ satisfies the conditions 3) and 4) of the theorem. It follows that

$$S = G^{(1)} \times S^{(2)},$$

where $S^{(2)}$ is an algebraic semigroup with $G(S^{(2)}) = G^{(2)}$ defined by the semigroup

$$\mathfrak{L}^{(2)} = \mathfrak{X}_+^{(2)} \bigcap K^{(2)}.$$

According to the theorem, $S^{(2)}$ has a zero.

§3. Proof of Theorem 3

1. The first assertion of Theorem 3 is implied by the following known fact [8].

Proposition 1. *Any reductive algebraic semigroup, whose unit group is semisimple, is a group.*

Indeed, the proposition implies that $\bar{G}_0 = G_0$. It follows that G_0 is the unique closed $(G_0 \times G_0)$-orbit in the fiber $\pi^{-1}(e)$. All other orbits in $\pi^{-1}(e)$, if they exist, must be higher-dimensional. But each coset $gG_0 (g \in G)$ is a $(G_0 \times G_0)$-orbit, and, consequently, these orbits have the highest demention. Hence $\pi^{-1}(e) = G_0$.

For the sake of completeness, we give a proof of Proposition 1 below. It is based on the following

Proposition 2. *Let $H \subset GL(V)$ be a connected semisimple algebraic linear group. Then any irreducible linear representation of H is realized in a suitable tensor power of V.*

Proof. We have $H \subset SL(V)$, so any irreducible linear representation of H is contained in the restriction to G of some irreducible linear representation of $SL(V)$. But any irreducible linear representation of $SL(V)$ is realized in a suitable tensor power of V. □

Proof of Proposition 1. Let S be an algebraic semigroup, whose unit group $G(S) = G$ is semisimple, and $\mathfrak{L}$ the corresponding subsemigroup of $\mathfrak{X}_+$ (see the formula (2)). Since $\mathfrak{L}$ generates the group $\mathfrak{X}$, it contains such characters $\Lambda_1, \ldots, \Lambda_m$ that the representation

$$R = R^{(\Lambda_1)} + \cdots + R^{(\Lambda_m)}$$

of the group G is faithful. By Proposition 2, any irreducible representation of G is contained in some product of $R^{(\Lambda_1)}, \ldots, R^{(\Lambda_m)}$. Hence $\mathfrak{L} = \mathfrak{X}_+$ and $S = G$. □

2. Since $G_0 \bar{Z} \supset G$, the restriction homomorphism

$$\rho : k[A] = k[S]^{G_0 \times G_0} \to k[\bar{Z}]$$

is injective. We shall prove that $k[\bar{Z}]$ is integral over $\rho(k[A])$, which will imply that the corresponding morphism

$$\rho^* = \pi|_{\bar{Z}} : \bar{Z} \to A$$

is surjective.

The algebra $k[\bar{Z}] \subset k[Z]$ is spanned by the characters $e^{\Lambda}|_Z, \Lambda \in \mathfrak{L}$, of Z. For $\Lambda \in \mathfrak{L}$, let $m = \dim R^{(\Lambda)}$. Then the m-th exterior power of $R^{(\Lambda)}$ is a one-dimensional representation of G. Its (highest) weight $M \in \mathfrak{L}_Z$ is such that $e^M|_Z = (e^{\Lambda})^m|_Z$, so $e^{\Lambda}|_Z$ is integral over $\rho(k[A])$.

3. Since $\rho(k[A]) \subset k[\bar{Z}]^{Z_0}$, we have a commutative diagram

$$\begin{array}{ccc} \bar{Z} & \xrightarrow{\pi} & A \\ & p\searrow \quad \swarrow\sigma & \\ & \bar{Z}/Z_0 & \end{array}$$

where p is the canonical homomorphism. Moreover, since the fibers of $\pi|_Z$ are just the cosets of Z_0, σ is a birational morphism. Since $\pi|_{\bar{Z}}$ is surjective, σ is also surjective. But any surjective birational morphism onto a normal variety is an isomorphism. This implies the fourth assertion of the theorem.

4. The $(G_0 \times G_0)$-orbit of an element $u \in \bar{Z}$ is $G_0 u$. Let us prove that it is closed. Since Zu contains an idempotent, we may assume that u is an idempotent. In this case, the morphism

$$G_0 \to G_0 u, \quad g \mapsto gu,$$

is a semigroup homomorphism. It defines an isomorphism

$$G_0/N \simeq G_0 u,$$

where $N = \{g \in G_0 : gu = u\}$. Hence $\overline{G_0 u}$ is an algebraic semigroup, whose unit group is semisimple. In virtue of Proposition 1, this implies that $\overline{G_0 u} = G_0 u$.

Let now O be any closed $(G_0 \times G_0)$-orbit. Since $\pi(\bar{Z}) = A$, the fiber $\pi^{-1}(\pi(0))$ meets $\bar{Z}$. But any fiber contains only one closed orbit. Hence O meets $\bar{Z}$. This proves the third assertion of the theorem.

5. Let S be normal and O be the $(G \times G)$-orbit in S, corresponding to the face F of the cone $K \bigcap C$ (in the notation of Theorem 2). We know (see 1.1) that the normality of $k[\bar{O}]$ is equivalent to the normality of $k[\bar{O}]^{U_- \times U}$. In 1.2, we saw that the algebra $k[S]^{U_- \times U}$ is naturally isomorphic to the semigroup agebra of $\mathfrak{L}$. Correspondingly, the algebra $k[\bar{O}]^{U_- \times U}$ is isomorphic to the semigroup algebra of

$$\mathfrak{L}_F = \mathfrak{L} \bigcap F = \mathfrak{X} \bigcap F$$

and hence normal (see 2.1).

§4 Proof of Theorem 4

1. Let $S = S(\mathfrak{L})$ be a normal reductive semigroup with $G(S) = G$ and A its abelization.

We introduce a preorder on $\mathfrak{L}$:

$$\Lambda_1 \geq \Lambda_2, \quad \text{if } \Lambda_1 - \Lambda_2 \in \mathfrak{L}_Z.$$

It $\Lambda_1 \geq \Lambda_2$ and $\Lambda_2 \geq \Lambda_1$, we shall call Λ_1 and Λ_2 equivalent and write $\Lambda_1 \sim \Lambda_2$. More explicitly, $\Lambda_1 \sim \Lambda_2$, if $\Lambda_1 - \Lambda_2 \in \mathfrak{M}_0$, where

$$\mathfrak{M}_0 = \mathfrak{L}_Z \bigcap (-\mathfrak{L}_Z)$$

is the greatest subgroup contained in $\mathfrak{L}_Z$ (and in $\mathfrak{L}$). An element $M \in \mathfrak{L}$ will be called minimal, of $\Lambda \leq M$ implies $\Lambda \sim M$. Let $\mathfrak{M}$ denote the set of all minimal elements of $\mathfrak{L}$. It is evident that

$$\mathfrak{L} = \mathfrak{M} + \mathfrak{L}_Z.$$

Proposition 3. *The following conditions are equivalent:*

1) $k[S]$ *is a flat* $k[A]$*-module;*
2) $k[S]$ *is a free* $k[A]$*-module;*
3) *if* $M_1 + \chi_1 = M_2 + \chi_2 \quad (M_1, M_2 \in \mathfrak{M}, \quad \chi_1, \chi_2 \in \mathfrak{L}_Z)$, *then* $M_1 \sim M_2$ *(and* $\chi_1 \sim \chi_2$*).*
4) $k[S]$ *decomposes as a vector space into the tensor product*

$$k[S] = k[A] \otimes k[G]_{\mathfrak{M}_1}, \tag{28}$$

where $\mathfrak{M}_1$ *is a set of representatives of the cosets of* $\mathfrak{M}_0$ *in* $\mathfrak{M}$ *and*

$$k[G]_{\mathfrak{M}_1} = \bigoplus_{M \in \mathfrak{M}_1} k[G]_M. \tag{29}$$

In the case, when S has a zero, $\mathfrak{M}_0 = \{0\}$ and $\mathfrak{M}_1 = \mathfrak{M}$.

Proof. Obviously, 3) $\Rightarrow$ 4) $\Rightarrow$ 2) $\Rightarrow$ 1), so we are only to prove the implication 1) $\Rightarrow$ 3).

Let $k[S]$ be a flat $k[A]$-module. It is easy to see that the subalgebra $k[S]^{U_- \times U}$ (see 1.2) is a direct summand of $k[S]$ as a $k[A]$-module. Consequently, it is also a flat $k[A]$-module.

For a semigroup $\mathfrak{S}$, we denote by $k\mathfrak{S}$ its semigroup algebra over k. We saw in 1.2 that $k[S]^{U_- \times U} \simeq k\mathfrak{L}$. Under this isomorphism, the subalgebra $k[A]$ corresponds to $k\mathfrak{L}_Z$. Thus $k\mathfrak{L}$ is a flat $k\mathfrak{L}_Z$-module. It follows that for any ideal $\mathfrak{J}$ of $\mathfrak{L}_Z$ the natural homomorphism

$$k\mathfrak{J} \otimes_{k\mathfrak{L}_Z} k\mathfrak{L} \to k\mathfrak{L} \tag{30}$$

is injective.

Let us call two pairs $(\Lambda_1, \chi_1), (\Lambda_2, \chi_2) \in \mathfrak{L} \times \mathfrak{J}$ adjacent, if $\Lambda_1 + \chi_1 = \Lambda_2 + \chi_2$ and $\Lambda_1 \geq \Lambda_2$ or $\Lambda_2 \geq \Lambda_1$. Extending this relation by transitivity, we obtain some equivalence relation on $\mathfrak{L} \times \mathfrak{J}$, which we shall call the $\mathfrak{J}$-equivalence. The injectivity of the homomorphism (30) means that any two pairs $(\Lambda_1, \chi_1), (\Lambda_2, \chi_2) \in \mathfrak{L} \times \mathfrak{J}$ such that $\Lambda_1 + \chi_1 = \Lambda_2 + \chi_2$ are $\mathfrak{J}$-equivalent.

Now we prove that if $M_1, M_2 \in \mathfrak{M}$ are such that $M_1 - M_2 \in \mathfrak{L}_Z - \mathfrak{L}_Z$, then $M_1 \sim M_2$, which is equivalent to the condition 3) of the proposition.

Consider the ideal

$$\mathfrak{J} = (M_1 - M_2 + \mathfrak{L}_Z) \bigcap \mathfrak{L}_Z.$$

Let χ_2 be a minimal element of $\mathfrak{I}$ and $\chi_1 \in \mathfrak{L}_Z$ such that

$$M_1 + \chi_1 = M_2 + \chi_2.$$

Since M_2 is minimal in $\mathfrak{L}$ and χ_2 is minimal in $\mathfrak{I}$, the only pairs adjacent to (M_2, χ_2) are $(M_2 + \chi, \chi_2 - \chi)$, where $\chi \in \mathfrak{M}_0$. It follows that $M_1 \sim M_2$. □

2. Assume now that $k[S]$ is a flat $k[A]$-module, that is the morphism $\pi : S \to A$ is flat.

We keep the notation $\mathfrak{M}$ for the set of minimal elements of $\mathfrak{L}$.

Proposition 4. *The fiber of π are reduced and irreducible if and if $\mathfrak{M}$ is a subsemigroup of $\mathfrak{L}$.*

Proof. Let e_0 be the idempotent of A defined by

$$\chi(e_0) = \begin{cases} 1 & \text{for} \quad \chi \in \mathfrak{M}_0, \\ 0 & \text{for} \quad \chi \in \mathfrak{L}_Z \setminus \mathfrak{M}_0. \end{cases} \qquad (31)$$

It is easy to see that any neighbourhood of e_0 contains representativees of all $G(A)$-orbits in A. Since the set of points $a \in A$ for which the fiber $\pi^{-1}(a)$ is reduced and irreducible, is open [19] and $G(A)$-invariant, we may restrict ourselves to the investigation of the fiber $\pi^{-1}(e_0)$.

Let $\mathfrak{p}_0$ denote the ideal of $k[S]$ generated by the maximal ideal of $k[A]$ corresponding to e_0. Obviously, it is spanned by the subspaces $k[G]_\Lambda$ with $\Lambda \in \mathfrak{L} \setminus \mathfrak{M}$ and $(e^\chi - 1)k[G]_M$ with $M \in \mathfrak{M}$ and $\chi \in \mathfrak{M}_0$, so the subspace (28) is complementary to $\mathfrak{p}_0$.

"The fiber $\pi^{-1}(e_0)$ is reduced and irreducible" means that the quotient algebr $k[S]/\mathfrak{p}_0$ has no zero divisors. It $M_1, M_2 \in \mathfrak{M}$, but $M_1 + M_2 \notin \mathfrak{M}$, then

$$k[G]_{M_1}, k[G]_{M_2} \not\subset \mathfrak{p}_0, \text{but} \quad k[G]_{M_1+M_2} \subset \mathfrak{p}_0,$$

so the above condition is not fulilled.

Let now $\mathfrak{M}$ be a subsemigroup. Then the algebra $(k[S]/\mathfrak{p}_0)^{U_- \times U}$ is isomorphic to the semigroup algebra of $\mathfrak{M}/\mathfrak{M}_0$ and consequently has no zero divisors. According to the theorem statedin 1.1, $k[S]/\mathfrak{p}_0$ has no zero divisors as well.

Example. For the semigroup $S = S_a$, defined in 1.5, the morphism π is always flat and its fibers are always irreducible, but they are reduced if and only if $a \in \mathbb{N}$. The case $a = \frac{1}{2}$ is depicted in Fig.2. The elements of $\mathfrak{L}$ are represented by dots, the minimal ones being distinguished by small circle. Those of them lying above the line $y = \frac{1}{2}x$ correspond to nilpotent elments of the algebra $k[S]/I_0$.

3. Let us represent the above results in terms of Theorem 4.

We denote by $\mathfrak{X}(T_0)$ (resp. $\mathfrak{X}(Z)$) the character group of T_0 (resp. Z) and by $\mathfrak{X}_+(T_0)$ the semigroup of dominant characters of T_0.

If the cone K has the form (12), then

$$\mathfrak{M}_0 = \mathfrak{X}_Z \bigcap (D \bigcap (-D)), \qquad (32)$$

$$\mathfrak{M} = \{(\theta^*(\lambda), \lambda) : \lambda \in \mathfrak{X}(T_0)\} + \mathfrak{M}_0. \qquad (33)$$

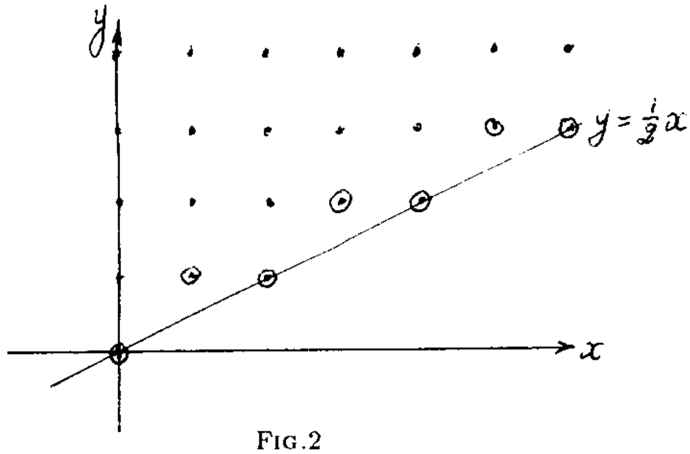

FIG.2

and the conditions of Proposition 3 and 4 are satisfied.

Conversely, if these conditions are satisfied, then the projection on $\mathfrak{t}_0(\mathbb{Q})^*$ defines a semigroup epimorphism

$$p : \mathfrak{M} \to \mathfrak{X}_+(T_0)$$

such that $p(M_1) = p(M_2)$ if and only if $M_1 \sim M_2$. It can be extended to a group epimorphism

$$p : \mathfrak{M} - \mathfrak{M} \to \mathfrak{X}(T_0)$$

with the same property. Let

$$q : \mathfrak{X}(T_0) \to \mathfrak{M} - \mathfrak{M}$$

be such a homomorphism that

$$pq = id.$$

We have

$$q(\lambda) = (r(\lambda), \lambda) \in \mathfrak{M} - \mathfrak{M} \subset \mathfrak{X},$$

where

$$r : \mathfrak{X}(T_0) \to \mathfrak{X}(Z)$$

is a group homomorphism such that the restrictions of λ and $r(\lambda)$ to Z_0 coincide for any $\lambda \in \mathfrak{X}(T_0)$. Hence $r = \theta^*$, where

$$\theta : Z \to T_0$$

is a homomorphism satisfying (11).

We have

$$\mathfrak{L} = \{(\chi, \lambda) \in \mathfrak{X}(Z) \times \mathfrak{X}_+(T_0) : \chi - \theta^*(\lambda) \in \mathfrak{L}_Z\}.$$

Hence the cone K has the form (12) with

$$D = K \bigcap \mathfrak{z}(\mathbb{Q})^* = \mathbb{Q}\mathfrak{L}_Z.$$

Remarks, 1. The cone D and, if S has a zero, the homomorphism θ are determined uniquely by S.

2. As a set of representatives of the cosets of $\mathfrak{M}_0$ in $\mathfrak{M}$ (see Proposition 3), we can choose

$$\mathfrak{M}_1 = \{(\theta^*(\lambda), \lambda) : \lambda \in \mathfrak{X}(T_0)\}. \tag{34}$$

4. The algebra $k[G_0]$ decomposes (as a vector space) into the direct sum

$$k[G_0] = \bigoplus_{\lambda \in \mathfrak{X}_+(T_0)} k[G_0]_\lambda, \tag{35}$$

where $K[G_0]_\lambda$ is the linear span of the matrix entries of the irreducible linear representation $R^{(\lambda)}$ of G_0 with highest weight λ. The multiplication in $k[G_0]$ has the form

$$fg = \sum_\nu p_\nu(f, g) \quad (f \in k[G_0]_\lambda, g \in k[G_0]_\mu, p_\nu(f, g) \in k[G_0]_\nu), \tag{36}$$

where ν runs over the highest weights of the irreducible components of $R^{(\lambda)}R^{(\mu)}$, The comultiplication has the form

$$\mu^*(f_{ij}^{(\lambda)}) = \sum_k f_{ik}^{(\lambda)} \otimes f_{kj}^{(\lambda)}, \tag{37}$$

where $f_{ij}^{(\lambda)}$ denotes the (i, j)-th matrix entry of $R^{(\lambda)}$.

It S has a zero, then $e_0 = 0$, $\mathfrak{M}_0 = \{0\}$ and the algebra

$$k[\pi^{-1}(0)] = k[S]/I_0$$

is naturally identified as a$(G_0 \times G_0)$-module with the subspace

$$k[S]_{\mathfrak{M}} = \bigoplus_{M \in \mathfrak{M}} k[G]_M.$$

On the other hand, in virtue of (33) we can identify $k[G_0]$ with $k[S]_{\mathfrak{M}}$ by means of the mapping

$$f \mapsto e^{\theta^*\lambda} f \quad (f \in k[G_0]_\lambda).$$

Thereby $k[\pi^{-1}(0)]$ is identified with $k[G_0]$.

To the multiplication in $k[\pi^{-1}(0)]$, there corresponds the $*$-multiplication in $k[G_0]$, defined by

$$f * g = p_{\lambda+\mu}(f, g) \quad (f \in k[G_0]_\lambda, g \in k[G_0]_\mu).$$

To the comultiplication in $k[\pi^{-1}(0)]$, there corresponds the comultiplication (37) in $k[G_0]$.

So we see that the semigroup $\pi^{-1}(0)$, together with the action of $G_0 \times G_0$, depends only on G_0, provided S is a flat semigroup with zero.

5. For a flat reductive semigroup S, the structure of $(G\times G)$-orbits in S is tightly connected with the structure of $(G_0\times G_0)$-orbits in the fiber of the morphism π.

Proposition 5. *Let S be a flat reductive semigroup. Then any fiber $\pi^{-1}(a)(a\in A)$ of the morphism π is a spherical $(G_0\times G_0)$-variety and*

$$GsG\bigcap\pi^{-1}(a)=G_0sG_0$$

for any $s\in\pi^{-1}(a)$.

Proof. For a reductive group L, an irreducible affine L-variety X is spherical if and only if $k[X]$ is a multiplicity free L-module [23].

Since the morphism π is flat and its fibers are reduced, all the $(G_0\times G_0)$-modules $k[\pi^{-1}(a)], a\in A$, are isomorphic. Since $k[\pi^{-1}(e)]=k[G_0]$ is multiplicity free, such are all of them.

Thus, any fiber $\pi^{-1}(a)\quad(a\in A)$ is a spherical $(G_0\times G_0)$-variety and, in particular, contains only finitely many $(G_0\times G_0)$-orbits. Denote by Z_a the stabilizer of a in Z. Obviously,

$$GsG\bigcap\pi^{-1}(a)=Z_aG_0sG_0$$

for any $s\in\pi^{-1}(a)$. The group Z_a, acting in $\pi^{-1}(a)$, can only permute $(G_0\times G_0)$-orbits. We have to prove that in fact it leaves each of them invariant. It will follow from the connectedness of Z_a/Z_0, which is proved below.

We have $\mathfrak{X}(Z/Z_0)=\mathfrak{X}_Z$ and $\mathfrak{X}_Z\bigcap D=\mathfrak{L}_Z$. There is such face F of the cone D that, for $\chi\in\mathfrak{L}_Z$,

$$\chi(a)\begin{cases}\neq 0, & \chi\in F,\\ =0, & \chi\notin F.\end{cases}$$

The subgroup $Z_a\subset Z$ is defined by the equations

$$\chi(z)=1,\quad \chi\in\mathfrak{L}_Z\bigcap F.$$

Since $\mathfrak{L}_Z\bigcap F=\mathfrak{X}_Z\bigcap F$, the subgroup of $\mathfrak{X}_Z$, generated by $\mathfrak{L}_Z\bigcap F$, is primitive (i.e. the quotient group is torsion-free). This means that the group Z_a/Z_0 is connected. □

§5. Proof of Theorem 5

1. Let S and S' be reductive semigroups and

$$\varphi: S'\to S$$

a homomorphism. We denote all the objects associated to S' by the same letters, as those associated to S, but equipped with a prime.

The homomorphism φ is excellent if and only if

$$k[S']\simeq k[A']\otimes_{k[A]}k[S], \tag{38}$$

the structure of a $k[A]$-module on $k[A']$ being defined by means of the homomorphism

$$\varphi^*_{\mathrm{ab}} : k[A] \to k[A']$$

and the isomorphism being realized by means of the map

$$\pi'^* \times \varphi^* : k[A'] \times k[S] \to k[S'].$$

Note that φ^*_{ab} is nothing else than the restriction of φ^* to $k[A]$ and π'^* is the identity embedding of $k[A']$ into $k[S']$.

Proposition 6. *Let the semigroup S be flat and the homomorphism φ be excellent. Then the semigroup S' is also flat.*

Proof. Since $k[S]$ is a flat $k[A]$-module, if follows from (38) that $k[S']$ is a flat $k[A']$-module [21].

Let now $\mathfrak{m}'$ be a maximal ideal of $k[A']$ and $\mathfrak{m}$ its pullback in $k[A]$. Then

$$k[S']/\mathfrak{m}'k[S'] \simeq (k[A']/\mathfrak{m}') \otimes_{k[A]/\mathfrak{m}} (k[S]/\mathfrak{m}k[S]) = k[S]/\mathfrak{m}k[S],$$

so $k[S']/\mathfrak{m}'k[S']$ has no zero divisors. □

Proposition 7. *Let the semigroups S and S' be flat. The homomorphism φ is excellent if and only if $\varphi^*(\mathfrak{M}_1)$ is a set of representatives of the cosets of $\mathfrak{M}'_0$ in $\mathfrak{M}'$.*

Proof. In virtue of (28), the right-hand side of (38) can be represented in the form

$$k[A'] \otimes_{k[A]} k[S] = k[A'] \otimes k[G]_{\mathfrak{M}_1}.$$

So the homomorphism φ is excellent if and only if

$$k[S'] = k[A'] \otimes k[G']_{\varphi^*(\mathfrak{M}_1)},$$

which is equivalent to the property stated in the proposition. □

2. For a connected semisimple group G_0, we construct a reductive semigroup $S = \mathrm{Env}\, G_0$ as described in the introduction. Obviously, it has a zero. We are to prove the property (*).

Let S' be another reductive semigroup from the class $\mathcal{FS}(G_0)$ and $\varphi_0 : G'_0 \to G_0$ an isomorphism. For simplicity of the notation, let us identify G'_0 with G_0 by means of this isomorphism.

Since θ is an isomorphism, there is a unique homomorphism $\varphi : Z' \to Z$ such that the diagram

$$\begin{array}{ccc} Z' & \xrightarrow{\varphi} & Z \\ & {\scriptstyle\theta'}\searrow \quad \swarrow{\scriptstyle\theta} & \\ & T_0 & \end{array} \tag{39}$$

is commutative. Since $Z_0' \subset Z_0$,

$$\varphi|_{Z_0'} = id.$$

It follows that, being combined with the identity map of G_0, φ gives rise to a homomorphism $G' \to G$. We also denote it by φ.

For any $\lambda \in \mathfrak{t}_0(\mathbb{Q})^*$ we have

$$\varphi^*(\tilde{\lambda}) = \varphi^*\theta^*(\lambda) = \theta'^*(\lambda).$$

Since the cone $D \subset \mathfrak{z}(\mathbb{Q})^*$ is generated by $\tilde{\alpha}_1, \ldots, \tilde{\alpha}_n$, while the cone $D' \subset \mathfrak{z}'(\mathbb{Q})^*$ contains $\theta'^*(\alpha_1), \ldots, \theta'^*(\alpha_n)$, we obtain

$$\varphi^*(D) \subset D', \quad \varphi^*(K) \subset K'.$$

Hence

$$\varphi^*(\mathfrak{L}) \subset \mathfrak{L}'; \tag{40}$$

and, moreover,

$$\varphi^*(\mathfrak{L}_Z) \subset \mathfrak{L}_Z', \quad \varphi^*(\mathfrak{M}) = \mathfrak{M}_1' \subset \mathfrak{M}', \tag{41}$$

where

$$\mathfrak{M}_1' = \{(\theta'^*(\lambda), \lambda) : \lambda \in \mathfrak{X}(T_0)\}.$$

It follows from (40) that, for any $\Lambda \in \mathfrak{L}$,

$$\varphi^*(k[G]_\Lambda) = k[G']_{\varphi^*(\Lambda)} \subset k[S']$$

This means that φ is extended to a homomorphism $S' \to S$. We still denote it by φ.

According to Proposition 7, the property (41) implies that the homomorphism φ is exellent.

Conversely, if the semigroup S' has a zero, any excellent homomorphism $\varphi : S' \to S$, which is the identity map on $G_0' = G_0$, must satisfy the condition $\varphi^*(\mathfrak{M}) = \mathfrak{M}'$. This condition is equivalent to commutativity of the diagram (39) and hence defines φ uniquely.

§6. Proof of Theorem 6

1. We need some known facts from the representation theory. For convenience of the reader, we give their proofs (cf. [22]).

For any $I \subset \Omega$ and $\Lambda \in \mathfrak{X}_+$, we use the following notation:

Π_I – the linear span of $\{\alpha_i : i \in I\}$ in $\mathfrak{t}(\mathbb{Q})^*$,

G_I – the connected (reductive) algebraic subgroup of G, whose tangent algebra is spanned by $\mathfrak{t}$ and the root subspaces, corresponding to the roots lying in Π_I,

$R_I^{(\Lambda)}$ – the irreducible linear representation of G_I with highest weight Λ,

$V_I^{(\Lambda)}$ – the subspace of $V^{(\Lambda)}$ spanned by the weight subspaces, corresponding to the weights lying in $\Lambda + \Pi_I$.

Evidently, $V_I^{(\Lambda)}$ is G_I-invariant.

Proposition 8. *The representation of G_I in $V_I^{(\Lambda)}$ is irreducible (and hence isomorphic to $R_I^{(\Lambda)}$).*

Proof. Every weight M of $R^{(\Lambda)}$ is (uniquely) represented in the form

$$M = \Lambda - \sum_i k_i \alpha_i \qquad (k_i \in \mathbb{Z}_+).$$

We have $M \in \Lambda + \Pi_I$ if and only if $k_j = 0$ for $j \notin I$. So if $M \in \Lambda + \Pi_I$ and $j \notin I$, then $M + \alpha_j$ is not a weight of $R^{(\Lambda)}$. It follows that every highest vector for G_I in $V_I^{(\Lambda)}$ is a highest vector for G. Consequently, such a vector is unique up to proportionality. □

Let now $R : G \to GL(V)$ be a (not necessarily irreducible) linear representation of G and $\bar{\Lambda} \in \mathfrak{t}(\mathbb{Q})^*$ be such that any weight M of R is represented in the form

$$M = \bar{\Lambda} - \sum_i k_i \alpha_i \qquad (k_i \in \mathbb{Z}_+).$$

Denote by V_I the subspace of V spanned by the weight subspaces corresponding to the weights lying in $\bar{\Lambda} + \Pi_I$.

Proposition 9. *The representation of G_I in V_I is isomorphic to $\sum_\Lambda m_\Lambda R_I^{(\Lambda)}$ where Λ runs over $\mathfrak{X}_+ \bigcap (\bar{\Lambda} + \Pi_I)$ and m_Λ denote the multiplicity of $R^{(\Lambda)}$ in R.*

Proof. In the same way, as in the preceding proof, we can see that any highest vector for G_I in V_I is a highest vector for G. □

Proposition 10. *For $\Lambda, M \in \mathfrak{X}_+$ and*

$$N = \Lambda + M - \sum_{i \in I} k_i \alpha_i \in \mathfrak{X}_+ \qquad (k_i \in \mathbb{Z}_+)$$

the multiplicity of $R^{(N)}$ in $R^{(\Lambda)} R^{(M)}$ is equal to that of $R_I^{(N)}$ in $R_I^{(\Lambda)} R_I^{(M)}$.

Proof. Apply Proposition 9 to $R = R^{(\Lambda)} R^{(M)}$ taking $\bar{\Lambda} = \Lambda + M$. note that in this case the representation of G_I in V_I is isomorphic to $R_I^{(\Lambda)} R_I^{(M)}$. □

2. Let G_0 be a connected semisimple group and $S = \operatorname{Env} G_0$.
Any prime $(G \times G)$-invariant ideal $\mathfrak{p}$ of $k[S]$ has the form

$$\mathfrak{p} = \bigoplus_{\Lambda \in \mathfrak{J}} k[G]_\Lambda, \tag{42}$$

where $\mathfrak{J}$ is such an ideal of the semigroup $\mathfrak{L}$ that its complement $\mathfrak{L} \setminus \mathfrak{J}$ is a subsemigroup. In its turn, any such ideal $\mathfrak{J}$ of $\mathfrak{L}$ has the form

$$\mathfrak{J} = \mathfrak{L} \setminus F \tag{43}$$

where F is a (closed) face of $\mathbb{Q}\mathfrak{L} = K \bigcap C$ [13].

Conversely, let $\mathfrak{I}$ be an ideal of $\mathfrak{L}$ of the form (43). Assume that the subspace $\mathfrak{p}$ defined by (42) is an ideal of $k[S]$. Then

$$(k[S]/\mathfrak{p})^{U_-\times U} \simeq k(\mathfrak{L}\setminus\mathfrak{I}),$$

and, in virtue of Theorem A, $k[S]/\mathfrak{p}$ has no zero divisors, i.e. $\mathfrak{p}$ is a prime ideal.

3. Let us now find out, for which faces F of $K\bigcap C$ the subspace $\mathfrak{p}$ defined by (42) is an ideal of $k[S]$, that is

$$\Lambda\in\mathfrak{L}, M\in\mathfrak{I}\Rightarrow\mathfrak{X}(\Lambda,M)\subset\mathfrak{I}. \tag{44}$$

Let $F=F_{I,J}$ in the notation of the introduction.

Assume that a connected component $\tilde{I}$ of the complement $C(J)$ of J is entirely contained in I. Let

$$\alpha=\sum_{i\in\tilde{I}}m_i\alpha_i=\sum_{i=1}^{n}l_i\omega_i$$

be the highest root of $G_{\tilde{I}}$. We have $m_i>0$ for $i\in\tilde{I}$, while

$$l_i\begin{cases}\geq 0 & \text{for } i\in\tilde{I},\\ \leq 0 & \text{for } i\in J,\\ =0 & \text{for } i\notin\tilde{I}\bigcup J;\end{cases}$$

moreover, there is such $i\in\tilde{I}$ that $l_i>0$. So

$$\lambda=\sum_{i\in\tilde{I}}l_i\omega_i\in C_0\setminus C_J,$$

$$\Lambda=(\bar{\lambda},\lambda)\in(K\bigcap C)\setminus F.$$

Take such integer $k>0$ that $k\Lambda\in\mathfrak{X}$. Then

$$k\Lambda\in\mathfrak{I}.$$

Consider the representation $R^{(k\Lambda)}$ of G. Its restriction to $(G_{\tilde{I}},G_{\tilde{I}})$ is the irreducible representation with highest weight $k\alpha$. It is self-dual, which implies that its square contains the trivial representation. Hence

$$(R_{\tilde{I}}^{(k\Lambda)})^2\supset R_{\tilde{I}}^{(2k(\Lambda-\alpha))}$$

and, by Proposition 10,

$$(R^{(k\Lambda)})^2\supset R^{(2k(\Lambda-\alpha))}.$$

However

$$2k(\Lambda-\alpha)=2k(\bar{\lambda},-\sum_{j\in J}l_j\omega_j)\in F,$$

so

$$2k(\Lambda-\alpha)\notin\mathfrak{I}.$$

Consequently, $\mathfrak{p}$ is not an ideal of $k[S]$.

4. Conversely, let no connected component of $C(J)$ be entirely contained in I. Let

$$\Lambda = (\chi, \lambda) \in \mathfrak{L}, \quad M = (\psi, \mu) \in \mathfrak{J}.$$

Then any $N \in \mathfrak{X}(\Lambda, M)$ has the form

$$N = (\chi + \psi, \nu),$$

where

$$\nu = \lambda + \mu - \sum_{i=1}^{n} k_i \alpha_i \qquad (k_i \geq 0).$$

We have either $\psi - \bar{\mu} \notin D_I$, or $\mu \notin C_J$. In the first case

$$\chi + \psi - \bar{\nu} = (\chi - \bar{\lambda}) + (\psi - \bar{\mu}) + \sum_{i=1}^{n} k_i \alpha_i \notin D_I,$$

so $N \in \mathfrak{J}$. The same result is obtained if $k_s > 0$ for some $s \notin I$.

Let now $\mu \notin C_J$, so $\lambda + \mu \notin C_J$ as well, and

$$\nu = \lambda + \mu - \sum_{i \in I} k_i \alpha_i \qquad (k_i \geq 0). \tag{45}$$

Suppose $\nu \in C_J$ and let

$$I' = \{i \in I : k_i > 0\}.$$

If j is adjacent to some $i \in I'$, then it follows from (45) that $\nu(h_j) > 0$ and hence $j \in J$. But then our assumption about I and J implies that $I' \subset J$. So

$$\nu(h_s) = (\lambda + \mu)(h_s) \quad \text{for} \quad s \notin J,$$

which makes it impossible for ν to belong to C_J.

§7. Proof of Theorem 7

1. Let S be a reductive semigroup with $G(S) = G$, defined by a subsemigroup $\mathfrak{L} \subset \mathfrak{X}_+$. For any $\Lambda \in \mathfrak{L}$, the representation $R^{(\Lambda)}$ of G is extended to a representation of S, which will be denote in the same way. The sum of all these representations will be denoted by $\mathcal{R}$. Obviously, $\mathcal{R}$ is a faithful (infinite-dimensional) representation of S.

For any $s \in S$, $\mathcal{R}(s)$ can be represented as the set $\{R^{(\Lambda)}(s) : \Lambda \in \mathfrak{L}\}$, where $R^{(\Lambda)}(s) \in \operatorname{End} V^{(\Lambda)}$.

Proposirion 11. *A set $\{\mathcal{A}^{(\Lambda)} : \Lambda \in \mathfrak{L}\}$, where $\mathcal{A}^{(\Lambda)} \in \operatorname{End} V^{(\Lambda)}$, belongs to $\mathcal{R}(S)$ if and only if for any $\Lambda, M, N \in \mathfrak{L}$ and any G-equivariant linear map*

$$\varphi : V^{(\Lambda)} \otimes V^{(M)} \to V^{(N)} \tag{46}$$

the diagram

$$\begin{array}{ccc} V^{(\Lambda)} \otimes V^{(M)} & \xrightarrow{\ \varphi\ } & V^{(N)} \\ {\scriptstyle \mathcal{A}^{(\Lambda)}\otimes\mathcal{A}^{(M)}}\Big\downarrow & & \Big\downarrow{\scriptstyle \mathcal{A}^{(N)}} \\ V^{(\Lambda)} \otimes V^{(M)} & \xrightarrow{\ \varphi\ } & V^{(N)} \end{array} \tag{47}$$

is commutative.

Proof. The maps (46) contain all the information about the decomposition rule of the tensor products of the G-modules $V^{(\Lambda)}, \Lambda \in \mathfrak{L}$, and thereby about the multiplication law of the matrix entries of the representations $R^{(\Lambda)}, \Lambda \in \mathfrak{L}$. The commutativity of the diagrams (47) means that the matrix entries of $\mathcal{A}^{(\Lambda)}$'s are multiplied in the same way as the matrix entries of the representations $R^{(\Lambda)}$. □

Remark. One can easily get a finite-dimensional faithful representation of S. Namely, if $\Lambda_1, \dots, \Lambda_m$ generate the semigroup $\mathfrak{L}$, then $R = R^{(\Lambda_1)} + \dots + R^{(\Lambda_m)}$ is such a representation. Moreover, $R(S)$ is a closed subsemigroup of $\operatorname{End} V^{(\Lambda_1)} \times \dots \times \operatorname{End} V^{(\Lambda_m)}$. However, to describe $R(S)$ explicitly is a difficult task.

2. Let now G_0 be a connected semisimple group, $S = \operatorname{Env} G_0$, and A the abelization of S.

The characters $\bar{\alpha}_i = \theta^* \alpha_i (i = 1, \dots, n)$ of Z, considered as elements of the algebra $k[A] \subset k[S]$, will be denoted by π_i. We have

$$k[A] = k[\pi_1, \dots, \pi_n],$$

so $A = k^n$ and the homomorphism $\pi : S \to A$ is given by

$$\pi(s) = (\pi_1(s), \dots, \pi_n(s)).$$

For any $I \subset \Omega$, let e_I be the corresponding idempotent of A, defined by

$$\pi_i(e_I) = \begin{cases} 1, & i \in I, \\ 0, & i \in I. \end{cases} \tag{48}$$

We will describe the subsemigroup

$$S_I = \pi^{-1}(e_I) \subset S \tag{49}$$

in terms of the representation $\mathcal{R}$.

For $\lambda \in \mathfrak{X}_+(T_0)$, let

$$R^{(\lambda)} : G_0 \to GL(V^{(\lambda)}) \tag{50}$$

be the irreducible representation of G_0 with highest weight λ.

Any $\Lambda = (\chi, \lambda) \in \mathfrak{L}$ is represented in the form

$$\Lambda = (\tilde{\lambda} + \sum_i k_i \bar{\alpha}_i, \lambda) \qquad (k_i \in \mathbb{Z}_+). \tag{51}$$

The space of the representation $R^{(\Lambda)}$ can be identified with $V^{(\lambda)}$ and, if we put

$$\tilde{R}^{(\lambda)} = R^{(\tilde{\lambda}, \lambda)}, \tag{52}$$

then

$$R^{(\Lambda)} = (\prod_i \pi_i^{k_i})\bar{R}^{(\lambda)}.$$

In particular, for $s \in S_I$

$$R^{(\Lambda)}(s) = \begin{cases} \bar{R}^{(\lambda)}(s), & \text{if } \chi - \bar{\lambda} \in D_I, \\ 0, & \text{otherwise.} \end{cases} \tag{53}$$

It follows that

$$\bar{\mathcal{R}} = \bigoplus_{\lambda \in \mathfrak{X}_+(T_0)} \bar{R}^{(\lambda)} \tag{54}$$

is a faithful representation of S_I.

Proposition 11 together with (53) implies the following description of $\bar{\mathcal{R}}(S_I)$.

Proposition 12. *A set $\{\mathcal{A}^{(\lambda)} : \lambda \in \mathfrak{X}_+(T_0)\}$, where $\mathcal{A}^{(\lambda)} \in \operatorname{End} V^{(\lambda)}$, belongs to $\bar{\mathcal{R}}(S_I)$ if and only if for any $\lambda, \mu, \nu \in \mathfrak{X}_+(T_0)$ and any G_0-equivariant linear map*

$$\varphi : V_0^{(\lambda)} \otimes V_0^{(\mu)} \to V_0^{(\nu)}, \tag{55}$$

one has.

$$\varphi \circ (\mathcal{A}^{(\lambda)} \otimes \mathcal{A}^{(\mu)}) = \begin{cases} \mathcal{A}^{(\nu)} \circ \varphi, & \text{if } \nu \in \lambda + \mu + \Pi_I, \\ 0, & \text{otherwise.} \end{cases} \tag{56}$$

(Here Π_I denotes the linear span of $\{\alpha_i : i \in I\}$ in $\mathfrak{t}_0(\mathbb{Q})^*$.)

3. For $I \subset \Omega$ and $\lambda \in \mathfrak{X}_+(T_0)$ let $V_I^{(\lambda)}$ denote the subspace of $V^{(\lambda)}$ spanned by the weight subspaces, corresponding to the weights lying in the plane $\lambda + \Pi_I$, and $\mathcal{P}_I^{(\lambda)}$ the (unique) T_0-equivariant projection on $V_I^{(\lambda)}$.

Let now $I, J \subset \Omega$ constitute an essential pair.

Lemma 1. *It $\lambda \in \mathfrak{X}_+(T_0)$ and $\lambda - \sum_{i\in I} k_i\alpha_i \in C_J$ $(k_i \in \mathbb{Z}_+)$, then*

1) $\lambda \in C_J$;
2) $k_i = 0$ *for* $i \notin J^\circ$.

(See the notation in 0.6 and 0.7.)

Proof. Let $I' = \{i \in I : k_i \neq 0\}$. Suppose $i_0 \in I' \setminus J^\circ$. Let m be such an element of $C(J)$ that $\alpha_{i_0}(h_m) < 0$, and M the connected component of $C(J)$ containing m. Since $I' \not\supset M$, we may assume that $m \notin I'$. Then

$$(\lambda - \sum_i k_i\alpha_i)(h_m) = \lambda(h_m) - \sum_i k_i\alpha_i(h_m) > 0,$$

which is a contradiction. Hence $I' \subset J^\circ$, and for any $m \in C(J)$

$$\lambda(h_m) = (\lambda - \sum k_i\alpha_i)(h_m) = 0,$$

so $\lambda \in C_J$. $\square$

Define for any $\lambda \in \mathfrak{X}_+(T_0)$

$$\mathcal{P}_{I,J}^{(\lambda)} = \begin{cases} \mathcal{P}_{I\bigcap J^\circ}^{(\lambda)}, & \text{if } \lambda \in C_J, \\ 0, & \text{otherwise}, \end{cases} \tag{57}$$

and prove that the set $\{\mathcal{P}_{I,J}^{(\lambda)}\}$ satisfies the condition of Proposition 12.

Consider any non-trivial G_0-equivariant linear map (55). Note that $\mathcal{P}_{I,J}^{(\lambda)} \otimes \mathcal{P}_{I,J}^{(\mu)}$ is either 0, or, if $\lambda, \mu \in C_J$, the T_0-equivariant projection of $V^{(\lambda)} \otimes V^{(\mu)}$ on the sum of the weight subspaces, corresponding to the weights, lying in the plane $\lambda + \mu + \Pi_{I\bigcap J^\circ}$.

It $\nu \notin \lambda + \mu + \Pi_I$, then $R^{(\nu)}$ has no weights in the plane $\lambda + \mu + \Pi_I$, so

$$\varphi \circ (\mathcal{P}_{I,J}^{(\lambda)} \otimes \mathcal{P}_{I,J}^{(\mu)}) = 0.$$

Let now $\nu \in \lambda + \mu + \Pi_I$. Under this condition, we have to prove that

$$\varphi \circ (\mathcal{P}_{I,J}^{(\lambda)} \otimes \mathcal{P}_{I,J}^{(\mu)}) = \mathcal{P}_{I,J}^{(\nu)} \circ \varphi, \tag{58}$$

It $\lambda + \mu \notin C_J$, then by Lemma 1 $\nu \notin C_J$ and both sides of (58) vanish.

It $\lambda + \mu \in C_J$, but $\nu \notin C_J$, then $\nu \notin \lambda + \mu + \Pi_{I\bigcap J^\circ}$ and hence $R^{(\nu)}$ has no weights in this plane. It this case, both sides of (58) still vanish.

Finally, if $\lambda, \mu, \nu \in C_J$, then by Lemma 1 $\nu \in \lambda + \mu + \Pi_{I\bigcap J^\circ}$, so

$$\nu + \Pi_{I\bigcap J^\circ} = \lambda + \mu + \Pi_{I\bigcap J^\circ}.$$

It follows that

$$\varphi \circ (\mathcal{P}_{I\bigcap J^\circ}^{(\lambda)} \otimes \mathcal{P}_{I\bigcap J^\circ}^{(\mu)}) = \mathcal{P}_{I\bigcap J^\circ}^{(\nu)} \circ \varphi,$$

which is just the equality (58) in this case.

Thus, there is such an idempotent $e_{I,J} \in S_I$ that

$$\bar{\mathcal{R}}(e_{I,J}) = \{\mathcal{P}_{I,J}^{(\lambda)}\}. \tag{59}$$

The definition of $\mathcal{P}_{I,J}^{(\lambda)}$ implies that $e_{I,J} \in O_{I,J}$.

4. In this subsection we prove that $e_{I,J} \in \bar{T}$. This also can be considered as an independent proof of the existence of an element $e_{I,J} \in S_I$ satisfying (59).

Lemma 2.. *Let Δ be an indecomposable root system and Π its base (whose elements will be called simple roots). For any $\alpha \in \Pi$ there exists a positive linear combination of simple roots whose scalar products with all of them but α are negative.*

Proof. Let $\Pi_1, \dots, \Pi_s$ be the indecomposable components of $\Pi \setminus \{\alpha\}$ and α_i the root of Π_i which is adjacent to α. Proceeding by induction on $\operatorname{rk} \delta$, we may assume that for each i, there exists a positive linear combination β_i of the roots of Π_i, whose scalar products with all of them but α_i are negative. The sum $\sum_{i=1}^{s} \beta_i + c\alpha$ meets the requirement for sufficiently large positive c. □

Lemma 3. *For any essential pair (I, J), there exists such an element $h \in \mathfrak{t}(\mathbb{Q})$ that*

$$\bar{\alpha}_i(h) = \begin{cases} 0, & i \in I, \\ a_i > 0, & i \notin I, \end{cases} \tag{60}$$

$$(\bar{\omega}_j + \omega_j)(h) = \begin{cases} 0, & j \in J, \\ b_j > 0, & j \notin J, \end{cases} \tag{61}$$

$$\alpha_i(h) = \begin{cases} 0, & i \in I \bigcap J^\circ, \\ c_i < 0, & i \notin I \bigcap J^\circ. \end{cases} \tag{62}$$

Proof. For any positive rational $a_i (i \notin I)$ and b_j $(j \notin J)$ there exists a unique $h \in \mathfrak{t}(\mathbb{Q})$ satisfying (60) and (61). We have to show that a_i and b_j can be chosen in such a way that (62) be also satisfied.

For any connected component M of $C(J)$ chose $m \in M \setminus I$ and, making use of Lemma 2, take a positive linear combination h_M of the corresponding dual roots satisfying the condition $\alpha_i(h_M) < 0$ for all $i \in M \setminus \{m\}$. The sum $h_0 = \sum_M h_M$ satisfies the conditions

$$\omega_j(h_0) = \begin{cases} 0, & j \in J, \\ b_j > 0, & j \notin J, \end{cases} \tag{63}$$

$$\alpha_i(h_0) = \begin{cases} 0, & i \in J^\circ, \\ d_i < 0, & i \in I \setminus J^\circ. \end{cases} \tag{64}$$

Since $\bar{\alpha}_i + \alpha_i$ are expressed in terms of $\bar{\omega}_j + \omega_j$ in the same way as α_i are expressed in terms of ω_j, any $h \in \mathfrak{t}(\mathbb{Q})$ satisfying (61), automatically satisfies the conditions

$$(\bar{\alpha}_i + \alpha_i)(h) = \begin{cases} 0, & i \in J^\circ, \\ d_i < 0, & i \in I \setminus J^\circ. \end{cases} \tag{65}$$

Since

$$\alpha_i(h) = (\bar{\alpha}_i + \alpha_i)(h) - \bar{\alpha}_i(h),$$

(65) and (60) implies (62), provided a_i are sufficiently large. □

Proposition 13. *It $h \in \mathfrak{t}(\mathbb{Q})$ satisfies the conditions of Lemma 3, then*

$$\lim_{t \to -\infty} \exp th = e_{I,J}. \tag{66}$$

Proof. We shall prove (66), if we prove that all the eigenvalues of $d\mathcal{R}(h)$ are non-negative and

$$\operatorname{Ker} d\mathcal{R}(h) = \operatorname{Im} \mathcal{R}(e_{I,J}). \tag{67}$$

Let $\Lambda \in \mathfrak{L}$ has the form (51). Then all the weights of $R^{(\Lambda)}$ has the form

$$M = \sum_i k_i \bar{\alpha}_i + (\bar{\lambda} + \lambda) - \sum_i l_i \alpha_i \qquad (k_i, l_i \geq 0) \tag{68}$$

and it follows from (60)–(62) that $M(h) \geq 0$. Moreover, $M(h) = 0$ if and only if

1) $k_i = 0$ for $i \notin I$;
2) $\lambda \in C_J$;
3) $l_i = 0$ for $i \notin I \bigcap J^\circ$.

This gives (67). □

5. Now we find the stabilizer of $e_{I,J}$ in $G\times G$. Obviously, $g_1 e_{I,J} g_2^{-1} = e_{I,J} (g_1, g_2 \in G)$ if and only if

$$R^{(\Lambda)}(g_1)\mathcal{P}^{(\lambda)}_{I\bigcap J^\circ}R^{(\Lambda)}(g_2)^{-1} = \mathcal{P}^{(\lambda)}_{I\bigcap J^\circ} \tag{69}$$

for any $\Lambda = (\chi, \lambda) \in \mathfrak{X}\bigcap F_{I,J}$.

Denote by $U^{(\lambda)}_{I\bigcap J^\circ}$ the T_0-invariant complementary subspace of $V^{(\lambda)}_{I\bigcap J^\circ}$ in $V^{(\lambda)}$. The condition (69) is equivalent to the following three ones:

(S1) $V^{(\lambda)}_{I\bigcap J^\circ}$ is invariant under $R^{(\Lambda)}(g_1)$;

(S2) $U^{(\lambda)}_{I\bigcap J^\circ}$ is invariant under $R^{(\Lambda)}(g_2)$;

(S3) if we identify in the natural way the spaces $V^{(\lambda)}_{I\bigcap J^\circ}$ and $V^{(\lambda)}/U^{(\lambda)}_{I\bigcap J^\circ}$, then their endomorphisms, induced by $R^{(\Lambda)}(g_1)$ and $R^{(\Lambda)}(g_2)$ respectively, coincide.

The condition (S1) is satisfied if and only if $g_1 \in P(M)$, where $M = (I\bigcap J^\circ)\bigcup C(J)$. The kernel of the representation of $P(M)$ in $V^{(\lambda)}_{I\bigcap J^\circ}$ is $U(M)G(C(J))T_{I,J}$ (see the notation in 0.7).

In an analogous way, the condition (S2) is satisfied if and only if $g_2 \in P_-(M)$. The kernel of the representation of P_- in $V^{(\lambda)}_{I\bigcap J^\circ} = V^{(\lambda)}/U^{(\lambda)}_{I\bigcap J^\circ}$, is $U_-(M)G(C(J))T_{I,J}$.

This gives Theorem 7.

6. The formula (21) follows from the formula

$$\mathcal{P}^{(\lambda)}_{M_1}\mathcal{P}^{(\lambda)}_{M_2} = \mathcal{P}^{(\lambda)}_{M_1\bigcap M_2} \quad (M_1, M_2 \subset \Omega, \quad \lambda \in \mathfrak{X}_+(T_0))$$

and the obvious fact, that the interior of $J_1 \bigcap J_2$ is $J_1^\circ \bigcap J_2^\circ$.

§8. Proof of Theorem 8

1. Define a character $\tilde{\alpha}_i$ of the torus $T_\varnothing$ by

$$\tilde{\alpha}_i(z\theta(z)^{-1}) = \tilde{\alpha}_i(z) = \alpha_i(\theta(z)). \tag{70}$$

Obviously, the group $\mathfrak{X}(T_\varnothing)$ is freely generated by $\tilde{\alpha}_1, \dots, \tilde{\alpha}_n$. The algebra $k[\bar{T}_\varnothing]$ is spanned by the restrictions to $T_\varnothing$ of the weights of the representations $R^{(\Lambda)}, \Lambda \in \mathfrak{L}$. Since any such weight is represented in the form (68) and

$$(\bar{\lambda} + \lambda)(z\theta(z)^{-1}) = 1 \quad (\lambda \in \mathfrak{X}_+(T_0), z \in Z),$$

we obtain

$$k[\bar{T}_\varnothing] = k[e^{\tilde{\alpha}_1}, \dots, e^{\tilde{\alpha}_n}] \tag{71}$$

It follows that the semigroup $\bar{T}_\varnothing$ is isomorphic to k^n. It contains 2^n idempotents, enumerated by the subsets of $\Omega = \{1, \dots, n\}$. To a subset $I \subset \Omega$, there corresponds an idempotent $\tilde{e}_I \in \bar{T}_\varnothing \subset \bar{T}$ with the following properties:

$$\pi_i(\tilde{e}_I) = \tilde{\alpha}_i(\tilde{e}_i) = \begin{cases} 1, & i \in I, \\ 0, & i \notin I, \end{cases}$$

$$\bar{R}^{(\lambda)}(\tilde{e}_I) = \mathcal{P}_I^{(\lambda)} \quad (\lambda \in \mathfrak{X}_+(T_0))$$

(see the notation in §7). Comparing this with results of 7.3, we obtain

$$\tilde{e}_I = e_{I,\Omega}.$$

It follows that

$$G\bar{T}_\varnothing G = S^{\mathrm{pr}}. \tag{72}$$

It is easy to see that

$$T = Z \times T_\varnothing.$$

2. The following proposition describes "the big cell" of S.

Proposition 14. *The map*

$$\varphi : U_- \times Z \times \bar{T}_\varnothing \times U \to S, \quad (u_-, z, t, u) \mapsto u_- ztu,$$

is an open embedding. Its image is contained in S^{pr} and contains representatives of all $(G \times G)$-orbits in S^{pr}.

Proof. Two last assertions follow from (72). To prove the first one, we have only to check that φ is dominant and injective, because every injective dominant morphism of normal irreducible algebraic varieties is an open embedding.

Since $ZT_\varnothing = T$ and the big cell

$$\mathrm{BC}(G) = U_- TU$$

is dense in G, φ is dominant.

To prove the injectivity, one has to show that

$$u_- zt_1 e_{I,\Omega} u = t_2 e_{I,\Omega} \quad (u_- \in U_-, u \in U, z \in Z, t_1, t_2 \in T_\varnothing) \tag{73}$$

implies

$$u_- = u = z = e. \tag{74}$$

One can rewrite (73) as follows:

$$(t_2^{-1} u_- t_2) zte_{I,\Omega} u = e_{I,\Omega}, \tag{75}$$

where $t = t_1 t_2^{-1} \in T_\varnothing$. According to Theorem 7 the stabilizer of $e_{I,\Omega}$ in $G \times G$ is the semidirect product of $U(I) \times U_-(I)$, the diagonal in $R(I) \times R(I)$, and the torus $T_I \times \{e\}$. Note that $T_I \subset T_\varnothing$. Therefore (75) implies that

$$u_-, u \in R(I), \quad (t_2^{-1} u_- t_2) ztu \in T_I,$$

which, in its turn, implies (74). $\square$

Corollary. *The variety S^{pr} is smooth.*

The affine chart $U_- Z\bar{T}_{\mathfrak{a}} U$ will be called the big cell of S^{pr} and denoted by $\mathrm{BC}(S^{\mathrm{pr}})$.

3. Studying S^{pr}/Z can be reduced to the case, when G_0 is simply connected. Namely, let $\tilde{G}_0$ be the simply connected covering group of G_0 and $\tilde{S} = \mathrm{Env}\,\tilde{G}_0$. Let $\tilde{G} = G(\tilde{S})$ and $\tilde{Z}$ be the connected center of $\tilde{G}$. Then $S = \tilde{S}/\Gamma$, where Γ is a finite central subgroup of $\tilde{G}_0$ (contained in $\tilde{Z}$). It is easy to see that $S^{\mathrm{pr}} = \tilde{S}^{\mathrm{pr}}/\Gamma$. It there exists a geometrical quotient $\tilde{S}^{\mathrm{pr}}/\tilde{Z} = X$, it can be considered as a geometric quotient S^{pr}/Z via the commutative diagram

$$\begin{array}{ccc} \tilde{S}^{\mathrm{pr}} & \xrightarrow{\varphi} & S^{\mathrm{pr}} \\ & \searrow \quad \swarrow & \\ & X & \end{array}$$

In what follows we assume that G_0 is simply connected.

4. Let $\omega_1, \dots, \omega_n$ be the fundamental weights of $\mathfrak{g}_0$. Put

$$V^{(i)} = V^{(\omega_i)}, \quad R^{(i)} = \bar{R}^{(\omega_i)} (= R^{(\bar{\omega}_i, \omega_i)}), \quad \delta^{(i)} = \delta^{(\bar{\omega}_i, \omega_i)}$$

(see the notation in 7.2 and 1.2) and

$$V = V^{(1)} \oplus \dots \oplus V^{(n)}, \quad R = R^{(1)} + \dots + R^{(n)}.$$

Put further

$$\mathrm{End}' V^{(i)} = (\mathrm{End}\, V^{(i)}) \setminus \{0\},$$
$$\mathrm{End}' V = \mathrm{End}' V^{(1)} \times \dots \times \mathrm{End}' V^{(n)}.$$

It is clear that

$$S^{\mathrm{pr}} = R^{-1}(\mathrm{End}' V). \tag{76}$$

Proposition 15. *The representation R maps isomorphically the variety S^{pr} onto a closed subvariety of* $\mathrm{End}' V$.

The first step in proving the proposition is studying the representation of the big cell.

In each of the spaces $V^{(1)}, \dots, V^{(n)}$ we choose a basis as in 1.2 and put

$$\mathrm{End}'' V^{(i)} = \{\mathcal{A} \in \mathrm{End}\, V^{(i)} : a_{11} \neq 0\},$$

where (a_{kl}) is the matrix of $\mathcal{A}$ in the chosen basis, and

$$\mathrm{End}'' V = \mathrm{End}'' V^{(1)} \times \dots \times \mathrm{End}'' V^{(n)}.$$

Proposition 16. $\mathrm{BC}(S^{\mathrm{pr}}) = R^{-1}(\mathrm{End}''(V))$.

In other words, the complement of $\mathrm{BC}(S^{\mathrm{pr}})$ in S is the union of the divisors defined by the equations

$$\delta^{(i)} = 0 \quad (i = 1, \dots, n). \tag{77}$$

As we shall see, these divisors are prime.

Proof. Since $\mathrm{BC}(S^{\mathrm{pr}})$ is an affine variety, its complement in S is a divisors, say, D. A straightforward verification shows that $\delta^{(i)}(s) \neq 0$ for $s \in \mathrm{BC}(S^{\mathrm{pr}})$. This means that D contains all the divisor (77).

On the other hand, we have

$$\mathrm{BC}(S^{\mathrm{pr}}) \bigcap G = \mathrm{BC}(G).$$

It is well-known that the complement of $\mathrm{BC}(G)$ in G is the union of n prime divisors defined in G by the equations (77) (the center components of the highest weights do not matter here). Consequently, G is the union of the closures of these divisors and, may be, some divisor, which does not meet G.

Obviously, any prime divisor beyond G is $(G \times G)$-invariant and hence the closure of a $(G \times G)$-orbit. Since a pair of the form (Ω, J) is not essential for $J \neq \Omega$, the only $(G \times G)$-orbits of codimension 1 are $O_{I,\Omega}$, where $I = \Omega - \{i\}$. But they all are represented in $\mathrm{BC}(S^{\mathrm{pr}})$. Hence the divisors (77) are prime and exhaust D. □

Proposition 17. *The representation R maps isomorphically the variety $\mathrm{BC}(S^{\mathrm{pr}})$ onto a closed subvariety of $\mathrm{End}''\, V$.*

Proof. In algebraic terms, the assertion means that the algebra $k[\mathrm{BC}(S^{\mathrm{pr}})]$ is generated by the matrix entries of $R^{(1)}, \dots, R^{(n)}$ and the function $(\delta^{(1)})^{-1}, \dots, (\delta^{(n)})^{-1}$. In virtue of Proposition 16,

$$k[\mathrm{BC}(S^{\mathrm{pr}})] = k[S][(\delta^{(1)})^{-1}, \dots, (\delta^{(n)})^{-1}].$$

Since the algebra $k[S]$ is generated by the matrix entries of $R^{(1)}, \dots, R^{(n)}$ and the functions $\pi_1, \dots, \pi_n$ (see 7.2), we have only to check, that the latter functions can be expressed as polynomials in the matrix entries of $R^{(1)}, \dots, R^{(n)}$ and $(\delta^{(1)})^{-1}, \dots, (\delta^{(n)})^{-1}$.

For any i, the square of the representation $R^{(i)}$ contains the representation

$$R^{(2\bar{\omega}_i, 2\omega_i - \alpha_i)} = \pi_i R^{(2\bar{\omega}_i - \bar{\alpha}_i, 2\omega_i - \alpha_i)} = \pi_i \bar{R}^{(2\omega_i - \alpha_i)}$$

(see Proposition 10). In particular, the function $\pi_i \delta^{(2\bar{\omega}_i - \bar{\alpha}_i, 2\omega_i - \alpha_i)}$ can be expressed as a sum of products of matrix entries of $R^{(i)}$. On the other hand, if $2\omega_i - \alpha_i = \sum_j k_j \omega_j \quad (k_j \in \mathbb{Z}_+)$, then

$$\delta^{(2\bar{\omega}_1 - \bar{\alpha}_1, 2\omega_1 - \alpha_1)} = \prod_j (\delta^{(j)})^{k_j}.$$

Hence π_i can be represented in the desired form. □

Proof of Proposition 15. Since $S^{\rm pr} = G \cdot {\rm BC}(S^{\rm pr}) \cdot G$, it follows from Proposition 16 and 17 that R maps isomorphically $S^{\rm pr}$ onto a closed subvariety of $R(G) \cdot {\rm End}'' V \cdot R(G)$. We will prove that

$$R(G) \cdot {\rm End}'' V \cdot R(G) = {\rm End}' V, \tag{78}$$

which will imply the proposition.

It we identify in the canonical way ${\rm End}\, V^{(i)}$ with $V^{(i)} \otimes (V^{(i)})^*$, the matrix entry a_{11} as a linear form on ${\rm End}\, V^{(i)}$ is identified with some non-zero element $u \in V^{(i)*} \otimes V^{(i)}$. Since the representation $R^{(i)*} R^{(i)}$ of $G \times G$ is irreducible, the $(G \times G)$-orbit of u spans the space $V^{(i)*} \otimes V^{(i)}$. This means that for any $\mathcal{A} \in {\rm End}' V^{(i)}$, there exist such $g_1, g_2 \in G$ that

$$\langle \mathcal{A}, (R^{(i)*}(g_1) \otimes R^{(i)}(g_2))u \rangle = \langle R^{(i)}(g_1) \mathcal{A} R^{(i)}(g_2)^{-1}, u \rangle \neq 0.$$

Thus

$$R(G) \cdot {\rm End}'' V^{(i)} \cdot R(G) = {\rm End}' V^{(i)}, \tag{79}$$

For any set $(\mathcal{A}_1, \ldots, \mathcal{A}_n) \in {\rm End}' V$, let

$$M^{(i)} = \{(g_1, g_2) \in G \times G : R^{(i)}(g_1) \mathcal{A}_i R^{(i)}(g_2)^{-1} \in {\rm End}'' V^{(i)}\}.$$

It is clear that $M^{(i)}$ is open in $G \times G$ and it follows from (79) that $M^{(i)}$ is not empty. Hence $\bigcap_i M^{(i)} \neq \varnothing$, which means that

$$(\mathcal{A}_1, \ldots, \mathcal{A}_n) \in R(G) \cdot {\rm End}'' V \cdot R(G). \quad \square$$

5. For $z \in Z$, we have

$$R^{(i)}(z) = \bar{\omega}_i(z) \mathcal{E},$$

where $\mathcal{E}$ denotes the identity operator. It follows that the action of Z on $S^{\rm pr}$ is induced via the representation R by the action of $(k^*)^n$ on ${\rm End}' V$, defined by

$$(t_1, \ldots, t_n) \circ (\mathcal{A}_1, \ldots, \mathcal{A}_n) = (t_1 \mathcal{A}_1, \ldots, t_n \mathcal{A}_n).$$

The latter action has the standard geometric quotient

$$p : {\rm End}' V \to P({\rm End}\, V^{(1)}) \times \cdots \times P({\rm End}\, V^{(n)}),$$

where $P(U)$ denotes the projective space, associated with the vector space U. The restriction of p to $R(S^{\rm pr})$ defines a geometric quotient of $S^{\rm pr}$, which is a closed subvariety of $P({\rm End}\, V^{(1)}) \times \cdots \times P({\rm End}\, V^{(n)})$ and hence a projective variety. Moreover, since $R(S^{\rm pr})$ is a smooth variety, such is $p(R(S^{\rm pr}))$. Thus, Theorem 8 is proved.

6. For a centerless connected semisimple group H, the wonderful $(H \times H)$-eqquivariant embedding of H, constructed by DeConcini and Procesi [14], can be characterized by the following properties:

1) it is complete;
2) it is simple, i.e. contains only one closed orbit;
3) it is toroidal, i.e. the closed orbit is not contained in the closure of the complement of the big cell in H.

The embedding $\mathrm{Ad}(G_0) = G_0/Z_0 \subset S^{\mathrm{pr}}/Z$ is $(G_0 \times G_0)$-equivariant and projective. It follows from Proposition 14 that S^{pr}/Z decomposes into 2^n orbits, ordered as the subsets of Ω, so only one of them is closed. It is not contained in the closure of the complement of the big cell in G_0/Z_0, because this complement lies beyond $\mathrm{BC}(S^{\mathrm{pr}})$, while all the orbits meet $\mathrm{BC}(S^{\mathrm{pr}})$. Thus S^{pr}/Z coincides with the wonderful embedding of $\mathrm{Ad}(G_0)$.

§9 Proof of Theorem 9

1. Let G be a connected reductive group, and let us use the notation of 0.1 for objects, associated to G.

Let G act on an affine variety X. For each $\Lambda \in \mathfrak{X}_+$, denote by $k[X]_\Lambda$ the isotypic component of the G-module $k[X]$, corresponding to the irreducible representation, dual to $R^{(\Lambda)}$. Then

$$k[X] = \bigoplus_{\Lambda \in \mathfrak{X}_+} k[X]_\Lambda \tag{80}$$

and one can choose such a basis $\{\varphi_{ip}\}$ of $k[X]_\Lambda$, that

$$\varphi_{ip}(gx) = \sum_j f_{ij}^{(\Lambda)}(g)\varphi_{jp}(x) \quad (g \in G, x \in X). \tag{81}$$

The space $k[X]_\Lambda$ need not be finite-dimensional, but it is known (see, for example, [9]) that it is a finitely generated module over $k[X]^G = k[X]_0$.

Obviously,

$$k[X]_\Lambda k[X]_M \subset \bigoplus_{N \in \mathfrak{X}(\Lambda, M)} k[X]_N. \tag{82}$$

Considering the products of the highest vectors, we see that, if X is irreducible, the set

$$\mathfrak{L}(X) = \{\Lambda \in \mathfrak{X}_+ : k[X]_\Lambda \neq 0\} \tag{83}$$

is a subsemigroup of $\mathfrak{X}_+$.

2. Let now S be an algebraic semigroup with $G(S) = G$, defined by a (perfect) subsemigroup $\mathfrak{L} \subset \mathfrak{X}_+$. The G-action on X is extended to an S-action if and only if $\mathfrak{L}(X) \subset \mathfrak{L}$, the extension being given by the formulas (81), in which $g \in G$ is replaced by $s \in S$.

In the general case, the subspace

$$k[X]_{\mathfrak{L}} = \bigoplus_{\Lambda \in \mathfrak{L}} k[X]_\Lambda$$

is a subalgebra of $k[X]$. If it is finitely generated, we can consider the variety

$$E = \operatorname{Spec} k[X]_{\mathfrak{L}}. \tag{84}$$

The semigroup S acts in a naatural way on E, and the morphism

$$\varphi : X \to E,$$

defined by the embedding $k[X]_{\mathcal{L}} \subset k[X]$, is G-equivariant.

Moreover, it is easy to see that for each affine S-variety E' and G-equivariant morphism

$$\varphi' : X \to E'$$

there is a unique S-equivariant morphism $\psi : E \to E'$ such that the diagram

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & E \\ & {\scriptstyle\varphi'}\searrow \quad \swarrow{\scriptstyle\psi} & \\ & E' & \end{array}$$

is commutative.

3. If the commutator group G_0 of G acts on an affine variety X_0, then the formula

$$(z_1, g) \circ (z_2, x) = (z_1 z_2, gx)$$

defines an action of $Z \times G_0$ on $Z \times X_0$ and thereby an action of $G = (Z \times G_0)/Z_0$ on

$$X = (Z \times X_0)/Z_0,$$

where $Z_0 = Z \bigcap G_0$ is assumed to be embedded in $Z \times G_0$ by means of the map

$$z_0 \mapsto (z_0, z_0^{-1}).$$

The formula

$$\varphi(x) = (e, x) \bmod Z$$

defines a G_0-equivariant closed embedding $\varphi : X_0 \to X$. Obviously, $X = Z\varphi(X_0)$.

Let now X' be an affine G-variety and $\varphi' : X_0 \to X'$ a G_0-equivariant morphism. Then there is a unique G_0-equivariant morphism $\psi; X \to X'$ such that the diagram

$$\begin{array}{ccc} X_0 & \xrightarrow{\varphi} & X \\ & {\scriptstyle\varphi'}\searrow \quad \swarrow{\scriptstyle\psi} & \\ & X' & \end{array}$$

is commutative. It is defined by

$$\psi(z\varphi(x)) = z\varphi'(x).$$

4. The combination of the preceding constructions permits us to obtain in a canonical way an affine S-variety E, starting from an affine G_0-variety X_0, provided the algebra $k[X]_{\mathcal{L}}$ is finitely generated, Moreover, there is a canonical G_0-equivariant morphism

$$\varphi : X_0 \to E$$

and, for any affine S-variety E' and G_0-equivariant morphism $\varphi' : X_0 \to E'$, there is a unique S-equivariant morphism $\psi : E \to E'$ such that the diagram

$$\begin{array}{ccc} X_0 & \xrightarrow{\varphi} & E \\ & {\scriptstyle \varphi'} \searrow \quad \swarrow {\scriptstyle \psi} & \\ & E' & \end{array}$$

is commutative.

The first part of Theorem 9 will follow, of we prove that in the case $S = \operatorname{Env} G_0$ the algebra $k[X]_{\mathfrak{L}}$ is always finitely generated. We will prove a more general result.

Proposition 18. *If the semigroup S is flat, then the algebra $k[X]_{\mathfrak{L}}$ is finitely generated.*

Proof. We make use of the theorem stated in 1.1. Since

$$k[X_0]^U_\lambda k[X_0]^U_\mu \subset k[X_0]^U_{\lambda+\mu},$$

the subspace

$$\bigoplus_{\lambda \in \mathfrak{X}_+(T_0)} k[X]^U_{(\theta^*\lambda, \lambda)}$$

is a subalgebra, isomorphic to $k[X_0]^U$. Since every $\Lambda \in \mathfrak{L}$ is uniquely represented in the form

$$\Lambda = \chi + (\theta^*\lambda, \lambda) \quad (\chi \in \mathfrak{L}_Z, \lambda \in \mathfrak{X}_+(T_0))$$

and

$$\bigoplus_{\chi \in \mathfrak{L}_Z} k[X]_\chi = k[A],$$

we obtain

$$k[X]^U_{\mathfrak{L}} \simeq k[A] \otimes k[X_0]^U.$$

It follows that the algebra $k[X]^U_{\mathfrak{L}}$, and hence the algebra $k[X]_{\mathfrak{L}}$, is finitely generated. □

5. The second part of Theorem 9 is also valid in a more general situation.

Proposition 19. *If the semigroup S is normal, then the morphism $\varphi : X_0 \to E$ is a closed embedding.*

Proof. In algebraic terms, the assertion means that the corresponding algebra homomorphism

$$\varphi^* : k[E] = k[X]_{\mathfrak{L}} \to k[X_0]$$

is surjective. It is clear that

$$\varphi^* k[X]_{(\chi,\lambda)} = k[X_0]_\lambda,$$

so we are to prove that, for any $\lambda \in \mathfrak{L}(X)$, there exists such $\chi \in \mathfrak{z}(\mathbb{Q})^*$ that $(\chi, \lambda) \in \mathfrak{L}$. Actually, we shall prove this for any $\lambda \in \mathfrak{X}_+(T_0)$.

In the notation of Theorem 2, there is such $\chi_0 \in \mathfrak{z}(\mathbb{Q})^*$ that $(\chi_0, \lambda) \in K$ (see Remark 1 to the theorem). Then

$$(\chi_0, \lambda) + D \subset K.$$

Since the cone D generates $\mathfrak{z}(\mathbb{Q})^*$ (see Remark 2 to the theorem), we can find such $\chi \in \chi_0 + D$ that $(\chi, \lambda) \in \mathfrak{X}$. Then $(\chi, \lambda) \in \mathfrak{L}$. □

Remark. I guess that the proposition is true for any S.

6. For any S, we have in the preceding notation

$$k[E] = k[X]_{\mathfrak{L}} = (k[Z] \otimes k[X_0])_{\mathfrak{L}}$$

and hence

$$k[E]^{G_0} = (k[Z] \otimes k[X_0]^{G_0})_{\mathfrak{L}} = k[Z]_{\mathfrak{L}} \otimes k[X_0]^{G_0} = k[A] \otimes k[X_0]^{G_0},$$

or, in geometric terms,

$$E/\!\!/G_0 = A \times (X_0/\!\!/G_0). \tag{85}$$

It S contains a zero, then (27) holds and hence

$$k[E]_+ = \bigoplus_{\Lambda \in \mathfrak{L}\setminus\{0\}} k[E]_\Lambda$$

is an ideal of E. The quotient algebra $k[E]/k[E_+]$ is naturally isomorphic to

$$k[E]_0 = k[E]^G = k[X_0]^{G_0}.$$

This defines a closed embedding of $X_0/\!\!/G_0$ into E, which is a section of the decomposition (85). Its image is nothing else than the set of fixed points of S.

§10. An example

We denote by L_m the semigroup of all the matrices of order m.

It is easy to see that

$$\operatorname{Env} SL_2 = L_2.$$

However, $\operatorname{Env} SL_3$ is more complicated than L_3. It can be described in terms of its faithful linear representation

$$R^{(1)} + R^{(2)} + \pi_1 + \pi_2$$

(see the notation in 8.4 and 7.2).

The restriction of $R^{(1)}$ to the group SL_3 is its tautological representation, while the restriction of $R^{(2)}$ is the dual one or, wich is the same, the exterior square of $R^{(1)}$. So, if

$$R^{(1)}(g) = A_1, \quad R^{(2)}(g) = A_2 \quad (g \in SL_3),$$

then

$$A_1A_2^\top = A_1^\top A_2 = E, \tag{87}$$
$$\wedge^2 A_1 = A_2, \quad \wedge^2 A_2 = A_1, \tag{88}$$

where $\wedge^2 A$ denotes the matrix constituted by the algebraic ccomplements of the entries of A.

The center Z of $G(\operatorname{Env} SL_3)$ is represented by quadruples

$$(\lambda_1 E, \lambda_2 E, \lambda_1^2\lambda_2^{-1}, \lambda_2^2\lambda_1^{-1}) \quad (\lambda_1, \lambda_2 \in k^*) \tag{89}$$

It follows from (87)–(89) that any quadruple

$$(A_1, A_2, t_1, t_2) \in \operatorname{Env} SL_3$$

satisfies the relations

$$A_1A_2^\top = A_1^\top A_2 = t_1t_2E, \tag{90}$$
$$\wedge^2 A_1 = t_1A_2, \quad \wedge^2 A_2 = t_2A_1. \tag{91}$$

One can show that these relations define $\operatorname{Env} SL_3$.

The orbital decomposition of $\operatorname{Env} SL_3$ is given by the following table.

I	t_1	t_2	J	rk A_1	rk A_2	dim $O_{I,J}$
$\{1,2\}$	$\neq 0$	$\neq 0$	$\{1,2\}$	3	3	10
$\{1\}$	$\neq 0$	0	$\{1,2\}$	2	1	9
			$\{1\}$	1	0	6
			$\varnothing$	0	0	1
$\{2\}$	0	$\neq 0$	$\{1,2\}$	1	2	9
			$\{2\}$	0	1	6
			$\varnothing$	0	0	1
$\varnothing$	0	0	$\{1,2\}$	1	1	8
			$\{1\}$	1	0	5
			$\{2\}$	0	1	5
			$\varnothing$	0	0	0

REFERENCES

1. M. S. Putcha, *Green's relation on a connected algebraic monoid*, Linear and Multilinear Algebra **12** (1982), 205–214.
2. ———, *Reductive groups and regular semigroups*, Semigroup Forum **30** (1984), 253–261.
3. ———, *Linear algebraic monoids*, London Math. Soc. Lecture Notes, Vol 133 , Cambridge University Press, 1988.
4. L. Renner, *Reductive monoids are von Neumann regular*, J. Algebra **93** (1985), 237–245.
5. ———, *Classification of semisimple algebraic monoids*, Trans. Amer. Math Soc. **292** (1985), 193–223.
6. ———, *Classification of semisimple algebraic varieties*, J. Algebra **122** (1989), 275–287.
7. ———, *Algebraic varieties and semigroups*, in The analytical and topological theory of semigroup (K. H. Hofmann, J. D. Lawson, J. S. Pym, eds.), Walter de Gruyter, 1990, pp. 81–101.
8. W. C. Waterhouse, *The unit groups of affine algebraic monoids*, Proc. Amer. Math. Soc. **85** (1982), 506–508.
9. V. L. Popov, E. B. Vinberg, *Invariant theory*, in Encyclopaedia of Math. Sci., vol. 55, Springer-Verlag, 1994.
10. V. L. Popov, *A contraction of actions of reductive algebraic groups*, Mat. Sbornik **130** (1986), 310–334.
11. E. B. Vinberg, *Complexity of actions of reductive groups*, Funkt. Anal. Appl. **20** (1986), 1–13.
12. ———, *The asymptotic semigroup of a semisimple Lie group*, in Invariant ordering in geometry and algebra (K. H. Hofmann, J. D. Lawson, E. B. Vinberg, eds.), Walter de Gruyter (to appear).
13. G. Kempf, F. Knudsen, D. Mumford, B. Saint-Donat, *Toroidal embeddings, I*, Lecture Notes Math, Vol. 339, Springer-Verlag, 1973.
14. C. DeConcini, C. Procesi, *Complete symmetric varieties*, in Lecture Notes in Math., vol. 996, 1983, pp. 1–44.
15. Yu. A Neretin, *Universal completions of the complex classical groups*, Funkt. Anal. Appl. **26** (1992), 30–44.
16. D. Khadzhiev, *Some questions on the theory of vector invariants*, Mat. Sbornik **72** (1967), 420–435.
17. E. B. Vinberg, V. L. Popov, *On a class of quasihomogeneous affine varieties*, Izv. Akad. Nauk SSSR, Ser. mat. **36** (1972), 749–763.
18. K.-H. Neeb, *Toric varieties and algebraic monoids*, Seminar Sophus Lie **2** (1992), 159–188.
19. A. Grothendieck, J. Dieudonné,, *Eléments de Géométrie Algébrique, IV*, Publ. Math. IHES , vol. 28, 1966.
20. Th. Vust, *Sur la théorie des invariants des groupes classiques*, Ann. Inst. Fourier **26** (1976), 1–31.
21. N. Bourbaki, *Algèbre commutative, Ch. 1–2*, Hermann, Paris, 1961.
22. M. Krämer, *Some tips on the decomposition of tensor product representations of compact connected Lie groups*, Rep. Math. Physics **13** (1978), 295–304.
23. E. B. Vinberg, B. N. Kimel'feld, *Homogeneous domains on flag manifolds and spherical subgroups of semisimple Lie groups*, Funkt. Anal. Appl. **12** (1978), 12–19.