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Moduli of vector bundles on surfaces: some basic results

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These are preliminary lecture notes, intended only for distribution to participants

Moduli of vector bundles on surfaces: some basic results.

We will address two basic questions concerning moduli spaces of vector-bundles on a projective surface (over $\mathbb{C}$).

1 – When is the moduli space (with fixed rank, determinant and 2^{nd} Chern class) generically smooth (and of the expected dimension)?

2 – When is the moduli space irreducible?

It turns out that the answer to both questions is affermative in "general", i.e if the expected dimension is sufficiently large. One should notice that these moduli spaces might behave wildly for low values of the expected dimension (they might have dimension greater than the expected one, or might be non-reduced or they might have many irreducible components).

<u>Remark</u> Why study these moduli spaces? By Donaldson's theory they contain subtle information on the C^∞-structure of the underlying 4-manifold. From a purely algebro-geometric viewpoint, consider a curve C: then $\mathrm{Pic}(C)$ plays a crucial rôle in the study

of G. If S is a surface, $\mathrm{Pic}(S)$ might not contain much information, e.g. if S is regular $\mathrm{Pic}(S)$ is a sublattice of $H^2(S;\mathbb{Z})$. On the other hand moduli spaces of vector-bundles of higher rank are non-empty of big dimension (as soon as the expected dimension is large enough).

Before stating our results we need to introduce some notation.

<u>Notation</u> S = projective smooth irreducible surface / $\mathbb{C}$

H = ample divisor on S

ξ = set of sheaf data on S = $(r_\xi, \det_\xi, c_2(\xi))$,

where (i) $r_\xi \in \mathbb{Z}$, $r_\xi \geq 2$,

(ii) $\det_\xi$ is a line bundle on S

(iii) $c_2(\xi) \in \mathbb{Z}$.

$$\mathcal{M}_\xi := \left\{ \begin{array}{l} F \text{ a rank-}r_\xi \text{ } H\text{-semistable torsion-free} \\ \text{sheaf on } S \text{ with } \det F \cong \det_\xi \text{ , } c_2(F) = c_2(\xi) \end{array} \right\} \Big/ \text{equivalence}$$

$$\Delta_\xi := c_2(\xi) - \frac{r_\xi - 1}{2 r_\xi} c_1(\xi)^2 \qquad \left(c_1(\xi) := c_1(\det \xi) \right).$$

Remark: By Bogomolov's theorem, if $\Delta_\xi < 0$ then $\mathcal{M}_\xi = \emptyset$. On the other hand it has been proved that if $\Delta_\xi \gg 0$ then $\mathcal{M}_\xi \neq \emptyset$.

Deformation theory. For F a torsion-free sheaf on S, and L a line bundle, set

$$h^i(F, F\otimes L)^0 := \dim \underbrace{Ext^i(F, F\otimes L)^0}_{\substack{\shortparallel \\ \text{kernel of the trace}: Ext^i(F,F\otimes L)\to H^i(L)}}$$

(If F is locally-free, $Ext^i(F, F\otimes L)^0 = H^i(ad F\otimes L)$).

(the 1^{st}-order deformations of F fixing the isomorphism class of $\det F$) $\cong Ext^1(F,F)^0$

By a theorem of Mukai-Artamkin the obstruction space for deformations of F can be taken to be

$$Ext^2(F,F)^0 \underset{\text{Serre duality}}{\cong} \left(Hom(F, F\otimes K)^0\right)^\vee.$$

As a consequence we have that, if $[F]\in \mathcal{M}_\xi$ and F is stable, then:

$$\dim_{[F]} \mathcal{M}_\xi \geq 2r_\xi \Delta_\xi - (r_\xi^2-1)\chi(\mathcal{O}_S) =: \text{exp. dim. of } \mathcal{M}_\xi$$

$$\dim T_{[F]}\mathcal{M}_\xi = 2r_\xi\Delta_\xi - (r_\xi^2-1)\chi(\mathcal{O}_S) + h^0(F, F\otimes K)^0.$$

From this we see that if every irreducible component of $\mathcal{M}_\xi$ contains a point $[F]$ with F stable and

$h^0(F, F\otimes K) = 0$, then $\mathcal{M}_{\mathcal{F}}$ is generically smooth of the expected dimension. So for a line-bundle L on S', set

$$W_{\mathcal{F}}^{L} := \{ [F] \in \mathcal{M}_{\mathcal{F}} \,/\, h^0(F, F\otimes L)^0 > 0 \}$$

Then ($\mathcal{M}_{\mathcal{F}}$ generically smooth of the expected dimension, in short "$\mathcal{M}_{\mathcal{F}}$ good")

$$\Uparrow$$

($\dim W_{\mathcal{F}}^{K} <$ exp. dim. $= 2r_{\mathcal{F}}\Delta_{\mathcal{F}} - (r_{\mathcal{F}}^2 - 1)\,\chi(\mathcal{O}_{S'})$, and the set of non-stable sheaves is nowhere dense)

Remark: the subsets $W_{\mathcal{F}}^{L}$ have geometric meaning also for different choices of L, e.g. if $L = K + C$, where $C \subset S$ is a curve.

Main results

Theorem 1 Let L be a line-bundle on S. There exist numbers $\alpha_2(r)$, $\alpha_1(r, S, H)$, $\alpha_0(r, S, H, L)$, with $\alpha_2(r) < 2r$, such that

$$\dim W_{\mathcal{F}}^{L} < \alpha_2(r_{\mathcal{F}})\Delta_{\mathcal{F}} + \alpha_1(r_{\mathcal{F}}, S, H)\sqrt{\Delta_{\mathcal{F}}} + \alpha_0(r_{\mathcal{F}}, S, H, L)$$

for all sheaf data $\mathcal{F}$.

Remark: The $\alpha_2, \alpha_1, \alpha_0$ are given by explicit formulas (involving $H^2, H\cdot K, K^2, L\cdot H, \chi(\mathcal{O}_S)$). Notice that (by the formulae on page 3) the theorem is non-vacuous because $\alpha_2(r) < 2r$.
Donaldson has proved (also Friedman – Zuo) a result as above for $r_{\mathcal{F}} = 2$. Donaldson's constant ~~this~~ α_2 equals 3, while we have $\alpha_2 = \frac{23}{6}$.
However the constants α_1, α_0 obtained by Donaldson have eluded computation.

Corollary There exists (an effective) $\Delta_0(r, S, H)$ such that if $\Delta_{\mathcal{F}} > \Delta_0(r_{\mathcal{F}}, S, H)$ then $M_{\mathcal{F}}$ is good.
(One also proves by dimension count that ~~this~~ it is also the closure of the moduli space of slope-stable vector-bundles.)

Remark: the above corollary has also been proved by Gieseker & Li, but their result is not effective. (They also prove a weaker version of Theorem 1).

Theorem 2 There exists $\Delta_1(r, s, H)$ such that if $\Delta_{\mathcal{F}} > \Delta_1(r, s, H)$ then $\mathcal{M}_{\mathcal{F}}$ is irreducible. (But Δ_1 is not effective.)

Example: Assume K is ample, and set $H := K$. ~~Let $r_{\mathcal{F}} = 2$~~ Assume also that $K^2 \gg 0$ ($K^2 > 100$ will suffice). Let $r_{\mathcal{F}} = 2$.

If

$$\Delta_{\mathcal{F}} \geq 42 K^2 + 15\,\chi(\mathcal{O}_S)$$

then $\mathcal{M}_{\mathcal{F}}$ is good.

All of the above results follow from a theorem concerning the boundary of $\mathcal{M}_{\mathcal{F}}$. Let $X \subset \mathcal{M}_{\mathcal{F}}$

$\partial \cancel{\mathcal{M}_{\mathcal{F}}}$ = boundary of $\cancel{X} := \{[F] \in X \mid F$ is not locally-free$\}$.

We notice that if $[F] \in \partial\mathcal{M}_{\mathcal{F}}$, and F is stable, then

$$\operatorname{cod}_{[F]}(\partial\mathcal{M}_{\mathcal{F}}, \mathcal{M}_{\mathcal{F}}) \leq r_{\mathcal{F}} - 1.$$

(The expected codimension is equal to $(r_{\mathcal{F}} - 1)$. This is the actual codimension if, for example, F^{**} is slope-stable and $h^0(F^{**}, F^{**} \otimes K)^0 = 0$.)

Theorem 3 There exists $\beta(r, s, H)$ such that if $X \subset \mathcal{M}_\xi$ is closed and

$$\dim X > \alpha_2 \Delta_\xi + \alpha_1 \sqrt{\Delta_\xi} + \beta,$$

then $\partial X \neq \emptyset$.

Theorem(1) follows from theorem(3): One argues by contradiction, if $\dim W_\xi^L$ is large, $\partial W_\xi^L \neq \emptyset$, so we consider all $[F^{**}]$ for $[F] \in \partial W_\xi^L$ generic. One observes that F^{**} is stable, so $[F^{**}] \in \mathcal{M}_{\xi_1}$ for some ξ_1 with $\Delta_{\xi_1} < \Delta_\xi$. Furthermore

$$h^0(F, F \otimes L)^0 \leq h^0(F^{**}, F^{**} \otimes L)^0.$$

Iterating this procedure one arrives at a contradiction.

Theorem(2) follows from theorems (1)-(3) by an argument due to Gieseker & Li.

We will concentrate on the proof of theorem(3): this is the key result.

A method for proving that a closed subset $X \subset \mathcal{M}_F$ has non-empty boundary.

Let $C \subset S$ be a smooth connected curve (of genus g). Assume there exists $[F] \in X$ such that

$$F|_C \text{ is not stable.}$$

(For this method to work we must also assume that F is stable.)
Of course we can assume F is locally-free, otherwise there is nothing to prove. Let

$$0 \to \mathcal{L}_0 \to F|_C \to Q_0 \to 0 \qquad (*)$$

be destabilizing (with $\mathcal{L}_0$, Q_0 locally-free), so

$$\mu(\mathcal{L}_0) \geq \mu(Q_0)$$

Idea:

Given $(*)$ we will construct a family of sheaves on S obtained by taking two elementary modifications of F along C. One sheaf in this family will be isomorphic to F, and the boundary of this family will have some sort of "weak ampleness" property. This will allow us to prove that $\partial X \neq \emptyset$, provided $\operatorname{cod}(X, \mathcal{M}_F)$ is ~~relatively~~ not large.

Construction of the family of elementary modifications.

Let E be the elementary modification of F associated to $(*)$ on p. 8. Thus E is the ~~sheaf~~ (torsion-free) sheaf on S' fitting into the exact sequence

$$0 \to E \to F \to \iota_* Q_0 \to 0, \qquad (\star)$$

where $\iota : C \hookrightarrow S'$ is the inclusion. Restricting $(\star)$ to C one gets

$$0 \to Q_0 \otimes \mathcal{O}_C(-C) \to E|_C \to \mathcal{L}_0 \to 0. \qquad (+)$$

Let $Y_F := \mathrm{Quot}^{P}(E|_C)$, where

$$P := \text{Hilbert polynomial of } \mathcal{L}_0.$$

Let $\mathcal{L} :=$ tautological quotient of $\pi_C^*(E|_C)$, where $\pi_C : C \times Y_F \to C$ is the projection. Let $0 \in Y_F$ correspond to the quotient in $(+)$, and for $y \in Y_F$ set

$$\mathcal{L}_y := \mathcal{L}|_{C \times \{y\}}.$$

Let $\mathcal{G} :=$ elementary modification of $\pi_S^* E$ associated to the tautological quotient of $\pi_C^*(E|_C)$

Thus we have

$$0 \to \mathcal{G} \to \pi_S^* E \to \iota_* \mathcal{L} \to 0$$

(these are sheaves on $S \times Y_F$.)

For $y \in Y_F$ we have

$$0 \to \underset{\mathcal{G}|_{S \times \{y\}}}{\underset{\|}{\mathcal{G}_y}} \to E \to \iota_* \mathcal{L}_y \to 0$$

and $\mathcal{G}$ is a ~~flat~~ family of torsion-free sheaves on S parametrized by Y_F (i.e. it is Y_F-flat).

Set $\quad \mathcal{F} := \mathcal{G} \otimes \pi_S^* \mathcal{O}_S(C)$. Then

(i) $\mathcal{F}_0 \cong F$

(ii) $\mathcal{F}$ is a family of torsion-free sheaves ~~parametrize~~, on S <u>parametrized by Y_F</u>.

Now let's examine

$$\partial Y_F := \{ y \in Y_F \mid \mathcal{F}_y \text{ is not locally-free} \}$$

<u>Claim</u> $\partial Y_F = \{ y \in Y_F \mid \mathcal{L}_y \text{ is not locally-free} \}$.

<u>Proof</u> One has

$$0 \to \mathcal{L}_y \otimes \mathcal{O}_C(-C) \to \mathcal{F}_y|_C \to Q_y \otimes \mathcal{O}_C(-C) \to 0$$

(where Q_y is defined by $Q_y \otimes \mathcal{O}_C(-C) = \ker (E|_C \to \mathcal{L}_y)$)

thus $\mathcal{G}_y|_{C_1}$ is locally-free if and only if $\mathcal{L}_y$ is locally-free. Since F is locally-free, and since $F|_{(S-C_1)} = \mathcal{G}_y|_{(S-C_1)}$ the claim follows. q.e.d.

the following proposition shows that ∂Y_F has a sort of "weak ampleness" property

Proposition Let $\Sigma \subset Y_F$ be closed, and assume

$$\dim \Sigma > \frac{r_{\mathcal{F}}^2}{4}.$$

then $\Sigma \cap \partial Y_F \neq \emptyset$.

At this point we can explain our method for proving that $\partial X \neq \emptyset$. Assume that:

(I) $\mathcal{F}_y$ is semistable for all $y \in Y_F$

(II) $\dim X + \dim Y_F > \dim T_{[F]} \mathcal{M}_{\mathcal{F}} + \frac{r_{\mathcal{F}}^2}{4}$

then $\partial X \neq \emptyset$. In fact by (I), the sheaf $\mathcal{F}$ defines a morphism

$$\varphi : Y_F \to \mathcal{M}_{\mathcal{F}},$$

and by ⓘⓘ we have

$$\dim \varphi^{-1}X > \frac{r_F^2}{4} \qquad \left(\begin{array}{l}\varphi^{-1}X \neq \phi \text{ because} \\ \varphi(X) \ni [F] \in X\end{array}\right)$$

By the proposition on page 11 we conclude that $\varphi^{-1}X \cap \partial Y_F \neq \phi$, hence $\partial X \neq \phi$.

<u>Ensuring that (I) – (II) hold.</u>

For (I) it suffices that F is "very stable", more precisely we require that for every non-zero subsheaf

$$0 \to M \to F$$

we have $\mu(M) < \mu(F) - C_1 \cdot H$.

Now let's consider (II): here the ~~initial~~ initial requirement that $F|_{C_1}$ be not stable enters into action.

One computes

$$\dim_0 Y_F \geq \chi(\mathcal{Q}_0^* \otimes \mathcal{O}_{C_1}(C_1) \otimes \mathcal{L}_0) = r_{\mathcal{L}_0} r_{\mathcal{Q}_0} [C_1^2 + 1 - g + \mu(\mathcal{L}_0) - \mu(\mathcal{Q}_0)] \geq$$

$$\geq r_{\mathcal{L}_0} r_{\mathcal{Q}_0} [C_1^2 + 1 - g] = \frac{1}{2}(r_F - 1)[C_1^2 - C_1 \cdot K].$$

Hence if $C_1 \sim nH$, then $\dim Y_F \to +\infty$ as $n \to +\infty$.

At this point ~~we~~ we have two contrasting requirements on n. To satisfy (I) we can't have n too large, while (II) requires n big. If ~~$\mathrm{cod}(X, M_\xi)$~~ $\dim X$ satisfies the hypotheses of theorem (3) one can find an appropriate n such that both (I) and (II) are satisfied.

~~How do we produce $[F] \in X$ such that $F/_C$ is not stable?~~

<u>Proof of Theorem (3)</u>

Of course we will apply the method described above. However we will not proceed directly, but rather we argue by contradiction.

We let $G \cap H$ as above, and we assume that g (= genus of G) satisfies

$$\dim X > (r_\xi^2 - 1)(g-1) \qquad (*)$$

~~One proves that~~ Let $X \subset M_\xi$ be closed, satisfying $(*)$.

<u>Proposition</u> If $\partial X = \phi$, then there exists $[F] \in X$ such that $F/_G$ is not stable.

<u>Proof</u> By contradiction. If $F|_C$ is stable for all $[F] \in X$, then restriction to C defines a morphism

$$\rho : X \longrightarrow \overline{\mathcal{M}}_C(r_{\mathcal{F}}, \det{}_{\mathcal{F}}|_C)$$

$$\left(\begin{array}{l}\text{moduli of semistable vector-bundles} \\ \text{on } C \text{ of rank} = r_{\mathcal{F}},\ \text{determinant} \cong \det{}_{\mathcal{F}}|_C\end{array}\right.$$

Since the RHS of $(*)$ on p. 13 is the dimension of $\overline{\mathcal{M}}_C(r_{\mathcal{F}}, \det{}_{\mathcal{F}}|_C)$, we conclude that

$$(\rho^* \Theta)^{\dim X} = 0, \qquad (\star)$$

where Θ = theta-divisor. But $\rho^*\Theta$ is identified with the restriction to X of (positive multiple of) a certain determinant line-bundle ~~on $\mathcal{M}_{\mathcal{F}}$~~ $\mathcal{L}_H$ on $\mathcal{M}_{\mathcal{F}}$, and one knows that

$$(\mathcal{L}_H|_X)^{\dim X} > 0.$$

This contradicts $(\star)$. <u>q.e.d.</u>

End of proof of theorem (3): one chooses n such that (for $C \wedge nH$) both $(*)$ of p 13 and the "method for proving $\partial X \neq \emptyset$" apply. Now assume $\partial X = \emptyset$. By the above proposition there exists $[F] \in X$ such that $F|_C$ is not stable. By the "method.." we conclude $\partial X \neq \emptyset$. This is absurd, and <u>concludes</u> the proof.