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The problem of describing invariants of a system of d vectors of a vector space V with respect to a group of linear transformations $H \subset GL(V)$ is the classical one. If V has a structure of algebra, not necessarily associative, and H acts by its automorphisms, there is the construction of invariants by means of the traces of products of the operators of left and right multiplications. I discuss here "how many" invariants, polynomial or rational, can be obtained in this way, and "how many" actions are covered by means of this method of constructing invariants.

Fix as a ground field an algebraically closed field k of characteristic zero.

The following standard notation and terminology is used below:

- k^* is the multiplicative group of k,
- $\mathrm{Frac}\, R$ is the field of fractions of a commutative integral domain R,
- $\mathrm{Mat}_n(C)$ is the ring of $n\times n$-matrices over a ring C,
- $E_n = \mathrm{diag}(1, \dots, 1) \in \mathrm{Mat}_n(C)$,
- $K\cdot S$ denotes the linear span of a subset S of a vector space over a field K,

• $P(U)$ is the projective space associated to a vector space U,
• $k[X]$ denotes the algebra of regular functions on an algebraic variety X,
• $k(X)$ denotes the field of rational functions on an irreducible algebraic variety X,
• M^G is the fixed point set of an action of a group G on a set M,
• G^0 denoted the connected component of an algebraic group G containing unit element,
• Lie G is the Lie algebra of G,
• $R(\lambda)$ denotes the simple G-module with the highest weight λ of a connected semisimple algebraic group G,
• $\omega_1, \ldots, \omega_\ell$ are the fundamental weights of G (the Bourbaki numeration, [Bo], is used),
• id_M - identity transformation of a set M.

We understand under "algebra" an algebra which is not necessarily associative, and under "ideal" its twosided ideal. Simple algebra is an algebra with nonzero multiplication and without proper ideals

If not specially pointed out, vector spaces (algebras, linear mappings) are assumed to be vector spaces ... over k

1. History of the problem

1.1. The source of our considerations lies in the following old problem:

Describe the algebra and the field of invariants of a system of d linear operators in an n-dimensional vector space.

1.2. Using matrix language it is reformulated in the form accepted in modern Invariant Theory as follows:

Let $A = \mathrm{Mat}_n(k)$. The group $G = GL_n(k)$ acts on A by conjugation. Consider the diagonal action of G on the vectors space $V_{d,n} = A^d = A \oplus \ldots \oplus A$ (d summands). The problem is to describe the algebra $k[V_{d,n}]^G$ and the field $k(V_{d,n})^G$.

1.3. The algebra $k[V_{d,n}]^G$ is finitely generated because of Hilber's theorem on invariants. Since the quotient of G by the inefficiency kernel of the action on $V_{d,n}$ is the group $PGL_n(k)$ which has no nontrivial characters, $k(V_{d,n})^G = \operatorname{Frac} k[V_{d,n}]^G$, cf. [PV], n. 3.2.

1.4. One obtains a number of nonconstant invariants by means of the following construction.

Let $i_1, \ldots, i_t$ be a sequence of numbers, not necessarily different, ~~taken~~ from the set $\{1, \ldots, d\}$. Since the traces of the conjugated matrices are equal, the function

$$V_{d,n} \to k \quad , \quad (a_1, \ldots, a_d) \mapsto \operatorname{tr} a_{i_1} \ldots a_{i_t} , \qquad (1.4.1)$$

is constant on G-orbits. It is a polynomial function, i.e. an element of $k[V_{d,n}]^G$. One can write it down in coordinates as follows.

Let $x_{ij}^{(s)} \in k[V_{d,n}]$ be (i,j)-th standard coordinate function on the s-th summand of $V_{d,n}$, i.e.

$$x_{ij}^{(s)}(v) = a_{ij}^{(s)} , \quad v = (a_1, \ldots, a_d) \in V_{d,n} , \quad a_\ell = (a_{pq}^{(\ell)}) \in A$$

Then $k[V] = k[\ldots, x_{ij}^{(s)}, \ldots]$. Consider the "generic matrix" X_s of the s-th summand of $V_{d,n}$, i.e.

$$X_s = (x_{ij}^{(s)}) \in \operatorname{Mat}_n(k[V_{d,n}]) , \quad s = 1, \ldots, d .$$

Then the invariant (1.4.1) has the appearance

$$tr\, X_{i_1} \dots X_{i_t} \qquad (1.4.2)$$

1.5. M. Artin conjectured in [Ar] that

All invariants of the type (1.4.2) generate the algebra of invariants $k[V_{d,n}]^G$

If $n=2$, this was proved by the classical researchers, cf. [GY]. If $n=3$, the proof was given in [Sib], [Sm]. Moreover, in both of the cases the (finite) minimal systems of generators of $k[V_{d,n}]^G$ were found.
If $d=1$ and n is arbitrary, the statement follows from the classical theorem that the coefficients of the characteristic polynomial of the matrix $X=X_1$ are (algebraically independent) system of generators of $k[A]^G$ (one can express these coefficients via $tr\,X$, $tr\,X^2$, ..., $tr\,X^n$ by means of Newton's formulas).
In general case M. Artin's conjecture was proved in [Pr1] in the following more precise form*):

Theorem 1 The algebra of invariants $k[V_{d,n}]^G$ is generated by all the monomials (1.4.2) with $t \leq 2^n-1$.

1.6. In [Ra] and [Pr1] it is proved also that all the relations between the invariants (1.4.1) are obtained (in a certain precisely defined sense) from the Hamilton-Cayley theorem.
The algebra $k[V_{d,n}]^G$ is generated by algebraically inde-

*) As a matter of fact one can find the proof of M. Artin's conjecture also in [Gu].

pendent elements iff either $d=1$ or $(d,n)=(2,2)$, cf [S1], [AG]. In general case its structure seems to be rather complicated, cf. [[BT]].

1.7. However, this does not exclude the hope that the structure of the field $k(V_{d,n})^G = \mathrm{Frac}\, k[V_{d,n}]^G$ is simple, more precisely that this field is rational or at least stably rational (over k).

No counterexamples to the conjecture on rationality (stable rationality) of $k(V_{d,n})^G$ are at this writing known *)

Moreover, it is proved that $k(V_{d,n})^G$ is rational, if $n=2$ ([Pr 2]), $n=3$ ([Fo 1]) and $n=4$ ([Fo 2]), and stably rational, if $n=5$ and 7 ([BLB]). It is also proved that stable rationality of $k(V_{s,n})^G$ and $k(V_{t,n})^G$ implies stable rationality of $k(V_{st,n})^G$ provided that s and t are relatively prime, [Ka], [Sc].

It is proved in [Pr 2] that $k(V_{d,n})^G$ for $d \geq 2$ is rational over $k(V_{2,n})^G$. Therefore the key case in the problem of (stable) rationality of $k(V_{d,n})^G$ is the case $d=2$.

1.8. Certainly, one has to consider this problem in the context of the general known open problem:

Is it true that the field of invariants of any connected linear algebraic group is rational?

*) One can find in [Ros] the arguments which are used to show that $k(V_{d,n})^G$ is not rational (as a matter of fact, even stably). However these arguments are wrong (nonsurjectivity of $R_{n,F}$ does not follow from the existence of the generic algebras which are not crossed products, and in fact conflicts with Merkurjev–Suslin theory.)

(according to [Sa], the assumption on the connectedness of G is essential).

2. Applications

2.1. One can show, [Pr3], [LB1], that if $k(V_{2,n})^G$ is stably rational, then for any field $K \supset k$ and any central simple associative K-algebra of dimension n^2 over K its class in the Brauer group $Br\, K$ is a product of the classes of cyclic algebras. Therefore, stable rationality of the fields $k(V_{2,n})^G$ for all n implies the Merkurjev-Suslin theorem for the fields K containing k (i.e. that $Br\, K$ is generated by the classes of cyclic algebras).

This explains the old interest of the ring-theorists in the problem of (stable) rationality of the fields $k(V_{d,n})^G$.

2.2. There is the interpretation of the fields $k(V_{d,n})^G$ as the fields of rational functions on the moduli varieties of the appropriate geometrical objects, which explains the interest of the algebraic-geometers in this problem.

Namely, one can show, [LB1], that $k(V_{2,n})^G$ is the field of rational functions on the moduli variety of stable vector bundles of rank n over $\mathbb{P}^2$ with the Chern numbers $(0, n)$.

There exists another interpretation, cf. [LB1], [VdB] and also [Vn]: $k(V_{2,n})^G$ is the field of rational functions on the variety of pairs $(Y, \mathcal{L})$, where Y is a smooth plane projective curve of degree n, and $\mathcal{L}$ is a divisor of degree $n(n-1)/2$ on Y.

See the survey of the other applications, including the application to PDE, in [LB2], [LBT], [LBS].

3. Generalization

3.1. The construction of invariants described in 1.4, is based only on the fact that $Mat_n(k)$ is a finite dimensional algebra on which $GL_n(k)$ acts by automorphisms. Therefore it can be applied in a much more general setting.

3.2. Namely, let A be any finite dimensional algebra, and $G = \mathrm{Aut}\, A$.

Consider the diagonal action of G on $V := A^d = A \oplus \ldots \oplus A$ (d summands). Let $\pi_i : A^d \to A$ be the projection to the i-th summand.

Denote by L_a and R_a resp. the operators of left and right multiplications of A by $a \in A$. They linearly depend on a and

$$T_{g(a)} = g\, T_a\, g^{-1}, \quad g \in G,\ a \in A,\ T \in \{L, R\} \qquad (3.2.1)$$

For any nonassociative polynomial f without constant term in noncommutative variables $t_1, \ldots, t_d$ with the coefficients in k, and any element $v \in V$ the substitution $t_i = \pi_i(v)$ defines an element $f(v) \in A$. One has

$$f(g.v) = g.f(v) \quad , g \in G, \quad v \in V. \qquad (3.2.2)$$

Moreover, let $f, \ldots, h$ be a finite set of such polynomials. Then it follows from (3.2.1) and (3.2.2) that any function

$$V \to k, \ v \mapsto tr\, T^{(f)}_{f(v)} \cdots T^{(h)}_{h(v)}, \text{ where } T^{(f)}, \ldots, T^{(h)} \in \{L, R\} \qquad (3.2.3)$$

is constant on the G-orbits. It is polynomial, i.e. is an element of $k[V]^G$.

Denote by $Tr\, A^d$ the subalgebra with unit in $k[V]^G$ generated by all the functions (3.2.3) (it is enough to take only the functions (3.2.3) such that $f, \ldots, h$ are monomials).

3.3. Example. Let $A = Mat_n(k)$. It follows from the associativity of A that any invariant (3.2.3) has the appearence either

$$v \mapsto tr\, T_{f(v)}, \quad T \in \{L, R\},$$

or

$$v \mapsto tr\, L_{f(v)} R_{h(v)}.$$

It is easy to see that

$$tr\, T_a = n \cdot tr\, a, \quad tr\, L_a R_b = tr\, a \cdot tr\, b, \quad a, b \in A$$

therefore, $Tr\, A^d$ is generated by the invariants (1.4.2), i.e. $Tr\, A^d = k[A^d]^{Aut\, A}$ because of Theorem 1.

3.4. The generality of the construction of n. 3.2, and Example 3.3, lead naturally to the question whether the analogue of M. Artin's conjecture is true for the algebras

A different of $Mat_n(k)$. As the example of algebra with zero multiplication shows, if one expects the positive answer, the certain constraints on A have to be imposed. The algebra $Mat_n(k)$ is simple, and we shall restrict ourself to consideration of simple algebras. The results stated below give an evidence in favour of this restriction.

As a matter of fact, when formulating the analogue of M. Artin's conjecture, one has to distinguish two aspects of it - biregular and birational (cf. n. 1.3).

Let A be a finite dimensional simple algebra, not necessarily associative.

(A) Is it true that $k[A^d]^{Aut\,A} = Tr\,A^d$?

(F) Is it true that $k(A^d)^{Aut\,A} = Frac\ Tr\,A^d$?

As it is pointed out in n. 1.3, if $Aut\,A$ has no nontrivial characters, then the affirmative answer to (A) implies the affirmative answer to (F).

3.5. The affirmative answers to (A) and (F) were obtained for some other types of algebras, except $Mat_n(k)$. For simple Jordan algebra of a nondegenerate symmetric bilinear form the affirmative answers follow from the classical theory of orthogonal invariants. The affirmative answer to [A] was obtained in [S2] for the Cayley-Dickson algebra $A = \mathbb{O}$ (and, since $Aut\,\mathbb{O}$ is the simple connected algebraic group (of the type G_2), to (F) as well). Moreover, it is proved in (PS) that $k(\mathbb{O}^d)^{Aut\,\mathbb{O}}$ is rational if $d \geq 3$. In [I2] the Albert algebra $A = A$ is considered (i.e. 27-dimensional exceptional simple Jordan algebra of hermitian 3×3-matrices over $\mathbb{O}$, in which case Aut A is the connected algebraic group of the type F_4), and (F) is answered in the affirmative. Moreover, it is proved in [I2] that $k[A^d]^{\mathrm{Aut}\,A}$ is integral over $\mathrm{Tr}\,\mathbb{A}^d$ and $\mathrm{Tr}\,\mathbb{A}^d$ separates the closed orbits in $\mathbb{A}^d$. In [Pl] an algebraically independent system of generators of $\mathrm{Tr}\,\mathbb{A}^2$ is found, thus giving the positive answer to (A) for $A = \mathbb{A}$ and $d = 2$. It is shown in [IS] that $k(\mathbb{A}^d)^{\mathrm{Aut}\,\mathbb{A}}$ is rational for any d.

3.6. In the next section I'll show that "in general case" the answer to (A) is <u>negative</u>. It is remarkable that the situation with the question (F) is <u>different</u>. Namely, it was recently proved by Il'tyakov, [I1], the following

<u>Theorem 2</u>. <u>Let</u> A <u>be a finite dimensional simple algebra generated by</u> $\leq d$ <u>elements. Then</u>

$$k(A^d)^{\mathrm{Aut}\,G} = \mathrm{Frac}\;\mathrm{Tr}\,A^d \qquad (3.6.1)$$

In section 6 I'll give the proof of this theorem, simplifying and, I believe, clarifying the original proof of [I1] (but preserving its main idea). For instance, in contrast to [I1], I prove and use several general statements (concerning any subfields of $k(V)$), and avoid using Formanek's central polynomials.

3.7. The condition on d in Theorem 2 is not actually very restrictive in the known examples. For instance, $\mathrm{Mat}_n(k)$ is generated by two, and $\mathbb{O}$ by three elements. Any simple Lie algebra $\mathfrak{g}$ is generated by two elements (if $\mathfrak{g} = \mathfrak{t} \oplus (\bigoplus_{\alpha\in\Delta} \mathfrak{g}_\alpha)$ is the canonical decomposition, take $H \in \mathfrak{t}$ to be an element such that all the values $\alpha(H)$, $\alpha \in \Delta$, are different, and take $X = \sum_{\alpha\in\Delta} e_\alpha$, $e_\alpha \in \mathfrak{g}_\alpha$, $e_\alpha \neq 0$; then H and X are the generators of $\mathfrak{g}$). No nontrivial estimates of the number of generators of a simple algebra are known to me, and it would be interesting to get such estimates. Anyway, the statement

1)

of Theorem 2 is definitely true for any $d \geq \dim A$.

3.8. Although $\operatorname{Tr} A^d$ is not always equal to $k[A^d]^{\operatorname{Aut} A}$, theorem 2 shows that it is "sufficiently close" to it.

<u>Conjecture 1. Let A be a finite dimensional simple algebra. Then</u> $k[A^d]^{\operatorname{Aut} A}$ <u>is integral over</u> $\operatorname{Tr} A^d$.

Using the terminology of [PV], n. 5.1, a point $v \in A^d$ is called <u>nilpotent</u>, if $f(v) = 0$ for any $f \in k[A^d]^{\operatorname{Aut} A}$ such that $f(0) = 0$. If $\operatorname{Aut} A$ is <u>reductive</u> (this is the case if, for instance, $\operatorname{Aut} A$ acts irreducibly on A, cf. nn. 5.5-5.12), then, according to Hilbert's result, [Hl], this conjecture is equivalent to the following one (see the notation in n. 3.2):

<u>Conjecture 1*. A point</u> $v \in A^d$ <u>is nilpotent iff</u> $\operatorname{tr} T^{(d)}_{f(v)} \dots T^{(d)}_{h(v)} = 0$ <u>for all polynomials</u> $f, \dots, h$.

It would be interesting also to know whether $\operatorname{Tr} A^d$ is <u>finitely generated</u> or not, and if it is, to give an upper estimate of the number of its generators, as it is done in Theorem 1 for $A = \operatorname{Mat}_n(k)$.

<u>Conjecture 2.</u> $\operatorname{Tr} A^d$ <u>separates closed orbits of</u> $\operatorname{Aut} A$ <u>in</u> A^d.

3.9. It is worth to mention the analogy between theorem 2 and Conjecture 1, n. 3.8, and Theorem 1 and Conjecture in n. 2 of [Vi], where the invariants of the normalizer of a connected reductive group $G \subset GL_n(k)$, closed in $\operatorname{Mat}_n(k)$, acting diagonally on $G \times \dots \times G$ are considered.

2)

4. Algebras in general position

4.1. Let L be an n-dimensional vector space over k. Fixing a structure of algebra on L is equivalent to fixing a bilinear mapping $L\times L \to L$, which in turn is equivalent to fixing a linear mapping $L\otimes L\to L$. Therefore n^3-dimensional vector space $\mathrm{Hom}(L\otimes L, L)$ has the natural interpretation as the variety of structures of algebras on L. We identify it canonically with $L^*\otimes L^*\otimes L$. An element $\sum f\otimes h\otimes v \in L^*\otimes L^*\otimes L$ corresponds to the structure of algebra on L such that the multiplication is given by the formula

$$ab = \sum f(a)h(b)v \quad , \quad a, b\in L.$$

Denote the algebra defined by ~~the~~ a structure $m\in L^*\otimes L^*\otimes L$ by $\{L, m\}$.

4.2. Using the standard terminology of Invariant theory, cf (PV), we'll say that (n-dimensional) algebra in general position has some property if for any structure m from a certain (depending on the property under consideration) open dense subset of $L^*\otimes L^*\otimes L$ the algebra $\{L, m\}$ has this property.

4.3. Group $G = GL(L)$ acts naturally on $L^*\otimes L^*\otimes L$. It is easy to see that the structures $m_1, m_2 \in L^*\otimes L^*\otimes L$ lie in the same G-orbit iff the algebras $\{L, m_1\}$ and $\{L, m_2\}$ are isomorphic. In particular, stabilizer of a point $m\in L^*\otimes L^*\otimes L$ coinsides with the group of automorphisms of $\{L, \mu\}$.

4.4. <u>Theorem 3</u> <u>Group of automorphisms of n-dimensional algebra in general position is trivial.</u>

<u>Proof</u>. According to n. 4.3, one needs to show that the generic stabilizer (cf. [PV], §7) for the action of G on $L^*\otimes L^*\otimes L$ is trivial.

Subgroup $S = \{\alpha \cdot id_L \mid \alpha \in k^*\}$ of G acts on $L^*\otimes L^*\otimes L$ by scalar multiplication. Therefore if an element $g = st$, where $s \in S$, $t \in SL(L)$, lies in the generic stabilizer of the action of G on $L^*\otimes L^*\otimes L$, then t lies in the generic stabilizer of the action of $SL(L)$ on $P(L^*\otimes L^*\otimes L)$.

The decomposition of $SL(L)$-module $L^*\otimes L^*\otimes L$ into irreducibles has the appearance

$$L^*\otimes L^*\otimes L = R(\varpi_1 + 2\varpi_{n-1}) \oplus R(\varpi_1 + \varpi_{n-2}) \oplus R(\varpi_{n-1}) \oplus R(\varpi_{n-1}). \tag{4.4.1}$$

(one can obtain (4.4.1), say, by means of the tables in [OV.]; if $n=2$ the summand $R(\varpi_1 + \varpi_{n-2})$ is dropped).

Module (4.4.1) does not occur in the tables of $[P_p]$ which give the classification of all connected simple algebraic groups $H \subset GL(U)$ such that the generic stabilizer for the action of H on $P(U)$ does not contain a nonscalar transformation.

Hence, $t \in S$ and therefore $g \in S$ as well. Since S acts on $L^*\otimes L^*\otimes L$ by means of the character χ^{-1}, where $\chi(\alpha \cdot id_L) = \alpha$, and g lies in the stabilizer of some nonzero point of $L^*\otimes L^*\otimes L$, one has $g \in \operatorname{Ker} \chi^{-1} = \{e\}$. ▨

4.5. Let r be an integer, $0 \le r \le n$, and

$$\mathcal{J}_r = \{ m \in L^* \otimes L^* \otimes L \mid \{L, m\} \text{ has an } r\text{-dimensional ideal} \}$$

Theorem 3. $\mathcal{J}_r$ is a closed subset of $L^* \otimes L^* \otimes L$ of dimension $\le n^3 - r(n-r)(2n-r-1)$.

Proof. Let $\{e_i\}$ be a basis of L and $\{e^i\}$ the dual basis of L^*. If $r = 0$ (and n), the assertion is evident, therefore assume that $r \ge 1$, and let L_r be the linear span of the vectors $e_1, \dots, e_r$.

One can easily see that L_r is the ideal of $\{L, m\}$ for $m = \sum c_{ij}^l e^i \otimes e^j \otimes e_l$ iff $c_{ij}^l = 0$ for $l > r$ and either $i \le r$ or $j \le r$. It follows from here that

$$\mathcal{Z}_r = \{ m \in L^* \otimes L^* \otimes L \mid L_r \text{ is the ideal of } \{L, m\} \}$$

is the linear subspace of $L^* \otimes L^* \otimes L$ of dimension $n^3 - r^3 + 3nr^2 - 2n^2 r$.

Further, if I is an ideal of $\{L, m\}$, and $g \in G$, then $g \cdot I$ is an ideal of $\{L, g \cdot m\}$. Since G acts transitively on the set of s-dimensional linear subspaces of L, it follows from here that

$$\mathcal{J}_r = G \cdot \mathcal{Z}_r \tag{4.5.1}$$

Consider now the subgroup $P_r = \{ g \in G \mid g \cdot L_r = L_r \}$ of G. Then

$$P_r \cdot \mathcal{Z}_r = \mathcal{Z}_r \tag{4.5.2}$$

Since P_r is the parabolic subgroup, it follows from ~~…~~ (4.5.1) and (4.5.2) that I_r is closed and $\dim I_r \le \dim G - \dim P_r + \dim \mathcal{Z}_r$. The desired estimate on $\dim I_r$ follows now from the formulas $\dim G = n^2$ and $\dim P_r = r^2 - nr + n^2$. ▨

4.6. It follows from Theorem 3 that n-dimensional algebra in general position is simple. More precisely, one has the following

Theorem 4 The set $\{ m \in L^* \otimes L^* \otimes L \mid \{L, m\} \text{ is simple} \}$ is open and dense.

Proof. The statement follows from Theorem 3 due to the fact that $r(n-r)(2n-r-1) > 0$ for $1 \le r \le n-1$. ▨

4.7. According to Theorems 3 and 4, for each n there exists an n-dimensional algebra A such that

(a) A is simple,
(b) $\mathrm{Aut}\, A = \{e\}$

(and moreover, a "typical" n-dimensional algebra has these properties).

It follows from (b) that for such A one has

$$k[A^d]^{\mathrm{Aut}\, A} = k[A^d] \quad \text{for any } d. \qquad (4.7.1)$$

It follows from the definition of $\mathrm{Tr}\, A^d$ (see n. 3.2) that all homogeneous elements of degree 1 in $\mathrm{Tr}\, A^d \cap \pi_i^*(k[A])$ are the linear combinations of two invariants

$$v \mapsto \mathrm{tr}\, L_{\pi_i(v)} \quad \text{and} \quad v \mapsto \mathrm{tr}\, R_{\pi_i(v)}.$$

Therefore, if $n \geq 3$, then for algebra A in general position one has

$$k[A^d] \neq \mathrm{Tr}\, A^d \qquad (4.7.2)$$

(on the contrary, if $n \leq 2$, then $k[A^d] = \mathrm{Tr}\, A^d$ for algebra A in general position).

Therefore, returning back to n. 3.4, we obtain from (4.7.1) and (4.7.2) that, if $\dim A \geq 3$ and d is arbitrary, then the answer to question (A) is, "as a rule", negative.

5. When is theorem 2 applicable?

5.1. Let L be a finite dimensional vector space and H a subgroup of $GL(L)$. The problem of describing polynomial H-invariants of the system of vectors of L (i.e. H-invariants of the diagonal actions $H : L \oplus \ldots \oplus L$) has classical origin (it is called "the first main theorem of the theory of vector H-invariants" or even "constructing Invariant theory for linear group H", cf. [S2]). As a matter of fact the key case is describing H-invariants of $n = \dim L$ vectors of L, cf. [PV], n. 9.2).

5.2. Classical Invariant theory provides solution to this problem for the classical linear groups, cf., for instance, [PV], § 9. Except this, its complete solution is known only in a few other cases: in [Sim] and [S2] it is obtained for irreducible 7-dimensional representation of G_2; in [S2] for the simplest faithful 8-dimensional irreducible representation of Spin_7; and in [S3] for the representation of SL_2 in the space of cubic binary forms.

5.3. If there exists a structure of algebra $m \in L^* \otimes L^* \otimes L$ on L such that $H \subseteq \mathrm{Aut}\, A$ ~~and~~ for $A = \{L, m\}$, one can construct invariants of the systems of vectors of L by means of the construction of n. 3.2. If, moreover, $H = \mathrm{Aut}\, A$, then theorem 2 gives an essential approximation to the solution of the mentioned problem.

5.4. Thus we arrive in this way to the following question:

(a) When does a nonzero structure of H-invariant algebra A on L exist (i.e. when does an $m \in L^* \otimes L^* \otimes L$ exists such that $H \subset \mathrm{Aut}\, A$ for $A = \{L, m\}$)?

(b) When is A simple?

(c) When is $H = \mathrm{Aut}\, A$?

Such questions were considered before in a series of papers. For instance, in [Di] were investigated by means of transvectants the structures of SL_2-invariant algebras on some SL_2-modules *), and their simplicity was proved (the Cayley-Dickson algebra $\mathbb{O}$ and the exceptional Jordan algebras were constructed on this way). The wide information on the question (a) in general situation was obtained in [E1] (cf. n. 5.10).

5.5. Consider first the question (b).

Theorem 5. Let A be a finitely dimensional algebra

*) If G is a group and M is a G-module, then a structure of algebra on M is called G-invariant if it is invariant with respect to the linear group $\varphi(G)$, where $\varphi: G \to GL(M)$ is the homomorphism defining G-module structure on M.

with a nonzero multiplication. If Aut A contains a connected algebraic subgroup H which acts irreducibly on A, then

(i) A is a simple algebra,

(ii) any algebraic subgroup S such that $H \subseteq S \subseteq \operatorname{Aut} A$, is semisimple and centreless.

Proof (i) Assume that A is not simple. Let I be a minimal proper ideal of A.

Since H acts irreducibly on A, the sum of all ideals $h \cdot I$, where h runs over H, coinsides with A. Since each of the ideals $h \cdot I$ is minimal, and the intersection of any proper ideal with a minimal ideal which is not contained in it, equals 0, it follows from here that there are $e = h_1, \dots, h_\ell \in H$ such that

$$A = h_1 \cdot I \oplus \dots \oplus h_\ell \cdot I. \qquad (5.5.1)$$

Since the product of different minimal ideals is equal to 0, (5.5.1) is the direct sum of algebras.

Let J be another minimal proper ideal of A.

Assume that $J \neq h_i \cdot I$ for each i. Then it follows from (5.5.1) that $ab = ba = 0$ for any $a \in A$, $b \in J$. In particular, J has zero multiplication. In the same way as for I, we can obtain for J the decomposition analogeous to (5.5.1). Since $h \cdot J$ has zero multiplication for any $h \in H$ as well, this shows that A has zero multiplication which is the contradiction.

Therefore, each minimal proper ideal of A coinsides with one of $h_1 I, \dots, h_\ell I$. Hence H acts by permutations of $h_1 I, \dots, h_\ell I$. This action is trivial because H is connected. Hence each $h_i \cdot I$ is an H-invariant subspace in A, which contradicts the assumption that H acts irreducibly on A. ■

(ii). Since $H \leq S$, the action of S on A is irreducible as well. Therefore, S is reductive and its center acts on A by scalar multiplications. Let $s = \alpha \cdot id_A$, $\alpha \in k^*$, be an element of the centre of S. Take $a, b \in A$ such that $ab \neq 0$. Then $s \cdot ab = \alpha ab = (s \cdot a)(s \cdot b) = (\alpha a)(\alpha b) = \alpha^2 ab$. Hence $\alpha = 1$ and $s = id_A$. ■

5.6. As it will be clear from the further discussion of the questions (a) and (c), ~~...~~ Theorem 5 shows that one can use the technique of traces and Theorem 2 for describing H-invariants of the systems of vectors of L rather frequently, including the cases of some representations $H : L$ of nonclassical algebraic groups H (for instance, exceptional simple) in which almost nothing was known on invariants as yet.

Having in mind theorem 5, we assume further that H is reductive (but will not assume that H acts irreducibly on L; there are some constructions of simple H-invariant algebras generalizing constructions of (Di) in which H acts on L reducibly, cf. [E1], [Hr]).

5.7. Discuss now question (a) from n. 5.4.

Existence on L of a nonzero structure of H-invariant algebra is equivalent to existence of a nonzero H-equivariant bilinear mapping $L \times L \to L$ or, equivalently, of nonzero morphism of H-modules $L \otimes L \to L$. Since H is reductive, this latter condition is equivalent to the property that $L \otimes L$ and L have a mutual nonzero simple submodule. In particular, if H acts irreducibly on L, then a nonzero structure of the invariant algebra on L exists iff L occurs in the decomposition of $L \otimes L$ into irreducibles.

5.8. Assume now that H is connected and semi-simple. Let Λ be the semigroup of the dominant weights of H written additively. According to nn. 5.7 and 5.5,

$$\Lambda_{alg} := \{ \lambda \in \Lambda \mid R(\lambda) \text{ has a nonzero structure of } H\text{-invariant algebra} \} =$$
$$= \{ \lambda \in \Lambda \mid R(\lambda) \text{ is a submodule of } R(\lambda) \otimes R(\lambda) \}$$

It follows from Theorem 5, (ii), that Λ_{alg} is contained in the semigroup Λ_{rad} of radical weights of Λ.

Theorem 6. Λ_{alg} is the finitely generated subsemigroup of Λ.

The closedness of Λ_{alg} with respect to addition was proved in [Kr], and, independently, in [E2]. In [E2] it was conjectured that Λ_{alg} is finitely generated. This conjecture was proved later on by M. Brion and F. Knop by the method which simultaneously gives the new proof of the closedness of Λ_{alg} with respect to addition (see their proof in [E1]).

5.9. By virtue of Theorem 6, the set of irreducible

H-modules admitting a nonzero structure of a simple H-invariant algebra is described in principle by the finite data (namely, by a system of generators of Λ_{alg}).

One can show, [E2], that finding of such a system is reduced to the case of simple group H.

The problem of explicit describing of Λ_{alg} and its generators for simple groups H was considered in [E1], [E2], where one can find a lot of concrete information (many of the results in [E1] and [E2] are just announced, apparently because the proofs are based on the explicit calculations). I'll give the formulation of some of these results as an illustration of the ~~c~~ character of available information.

5.10. The complete description of Λ_{alg} is known at this writing for the groups of types B_r, E_8 and E_7:

<u>If</u> $H = B_r$, <u>then the system of generators of</u> Λ_{alg} <u>is</u>

$$\begin{cases} \varpi_i \text{ for } 1 \le i \le r-1, \text{ and } i \text{ is even if } i \le \frac{2r+1}{3}; \\ \varpi_i + \varpi_{i+2s}, \text{ if } i \text{ is odd and } i+2s < \frac{2r+1}{3}; \\ \varpi_i + \varpi_j, \text{ if } i < \frac{2r+1}{3} \le j \le r-1; \\ \varpi_i + 2\varpi_j, \text{ if } 1 \le i \le r-1, \\ 2\varpi_r \end{cases}$$

<u>If</u> $H = E_7$, <u>then</u> $\Lambda_{alg} = \Lambda_{2ad}$,

<u>If</u> $H = E_8$, then $\Lambda_{alg} = \Lambda$

2).

5.11. For the other groups there are only partial results. Namely:

<u>If</u> $H = C_r, D_{2l}, G_2$ <u>or</u> F_4, <u>then</u> $2\Lambda \subset \Lambda_{alg}$,

<u>If</u> $H = D_{2l+1}$ <u>or</u> E_6, <u>then selfcontragredient weights from</u> Λ_{rad} <u>lie in</u> Λ_{alg}.

5.12. The case $H = A_r$ appears to be especially complicated, cf. the details in [E1], [E2], where, for instance the systems of generators of Λ_{alg} are found for small r.

5.13. Finally, I'll make several remarks concerning question (c) of n. 5.4, assuming that $H \subset GL(L)$ is a <u>connected semisimple</u> group acting <u>irreducibly</u> on L.

Generally speaking, it is <u>not true</u> that $H = \mathrm{Aut}\, A$.

<u>Example</u>. Let L be the irreducible module of the group D_4 with the highest weight $2\omega_3$. There is a structure of D_4-invariant algebra on L. It is known that the subgroup B_3 in D_4 acts irreducibly on L (cf. the tables in [MP]). Therefore, if one takes H to be this subgroup, then $H = B_3 \subsetneq D_4 \subseteq \mathrm{Aut}\, L$.

5.14. If, using the notation of the proof of Theorem 3, the structure of algebra $A = \{L, m\}$ has the appearance

$$m = \sum c_{ij}^{l} e^i \otimes e^j \otimes e_l ,$$

then $\mathrm{Lie}\,(\mathrm{Aut}\, H)$ consists of the linear operators X on L

such that

$$\sum c_{ij}^{l}\,(X^{*}e^{i}\otimes e^{j}\otimes e_{l}+e^{i}\otimes X^{*}e^{j}\otimes e_{l}+e^{i}\otimes e^{j}\otimes Xe_{l})=0, \qquad (5.14.1)$$

where X^{*} is the conjugate to X operator on L^{*}.

The system of linear equations (5.14.1) on X makes it possible in principle to find $\mathrm{Lie}(\mathrm{Aut}\,A)$ and hence decide whether H coincides with $(\mathrm{Aut}\,A)^{0}$ or not.

5.15. As for the group $\mathrm{Aut}\,A$ itself, it appears to be connected under some restrictions:

Theorem 7. Let A be a finite dimensional algebra with a nonzero multiplication. If

(i) $\mathrm{Lie}(\mathrm{Aut}\,A)$ acts irreducibly on A,

(ii) $\mathrm{Lie}(\mathrm{Aut}\,A)$ has no outer automorphisms,

then $\mathrm{Aut}\,A$ is a connected semisimple group.

Proof. Semisimplicity of $\mathrm{Aut}\,A$ follows from (i) and Theorem 6.

Let $g\in\mathrm{Aut}\,A$. It follows from (ii) that the conjugation by g is an inner automorphism of $(\mathrm{Aut}\,A)^{0}$. Hence there exists $g_0\in(\mathrm{Aut}\,A)^{0}$ such that gg_0 lies in the centre of $(\mathrm{Aut}\,A)^{0}$. Since this centre is trivial by Theorem 6, one has $g\in(\mathrm{Aut}\,A)^{0}$. ∎

5.16. Remark. The same arguments show that without the assumption (ii) the natural homomorphism $G/G^{0}\to\mathrm{Aut}\,G^{0}/\mathrm{Int}\,G^{0}$, where $G=\mathrm{Aut}\,A$, is injective.

6. Proof of Theorem 2

6.1. We keep the notation of n. 3.2.
Until specially specified, we will not assume that A is simple.

6.2. Let

$$n = \dim A$$

and $e_1, \ldots, e_n$ be a basis of A and $x_i^{(s)}$ the i-th coordinate function on the s-th summand of $V := A^{\oplus d}$ with respect to this basis.
We identify $x_i^{(s)}$ with $\pi_s^*(x_i^{(s)}) \in k[V]$. Then

$$k[V] = k[\ldots, x_i^{(s)}, \ldots], \quad k(V) = k(\ldots, x_i^{(s)}, \ldots)$$

6.3. Consider $k(V)$-algebra

$$A_{k(V)} = k(V) \otimes_k A$$

and identify A with $1 \otimes A$. We identify $A_{k(V)}$ with the $k(V)$-algebra of all rational mappings $V \to A$ considering an element $\sum f_j e_j \in A_{k(V)}$ as the mapping $v \mapsto \sum f_j(v) e_j$.
Then the element

$$y_s = \sum_{j=1}^{n} x_j^{(s)} e_j \qquad (6.3.1)$$

is the projection $\pi_s : V \to A$.

6.4. Denote by B the k-subalgebra of $A_{k(V)}$

generated over k by the elements (6.3.1). One can obtain the following interpretation of $\operatorname{Tr} A^d$ by means of B.

Denote by $M(A_{k(V)})$ the k-algebra generated over k by the operators T_p, $T \in \{L, R\}$, of left and right multiplications of $A_{k(V)}$ by the elements $p \in A_{k(V)}$. We identify them with their matrices in the basis $e_1, \dots, e_n$ of $A_{k(V)}$ over $k(V)$, thus assuming that

$$M(A_{k(V)}) \subset \operatorname{Mat}_n(k(V))$$

As a matter of fact $M(A_{k(V)})$ is the $k(V)$-subalgebra of $k(V)$-algebra $\operatorname{Mat}_n(k(V))$ since the dependence of T_p on p is $k(V)$-linear.

Denote by $M(B)$ the k-subalgebra of $M(A_{k(V)})$ generated over k by the operators T_p, $p \in B$. It follows from the definition of B that

$$M(B) \subset \operatorname{Mat}_n(k[V]). \qquad (6.4.1)$$

Any element of B is obtained from an appropriate nonassociative polynomial $f = f(t_1, \dots, t_d)$ in noncommutative variables $t_1, \dots, t_d$ with the coefficients in k by means of the substitution $t_i = y_i$. Considered as a mapping $V \to A$, this element $f(y_1, \dots, y_d)$ has the appearance $v \mapsto f(v)$ (cf. nn 3.2 and 6.3). Therefore the polynomial (3.2.3) is the trace of the polynomial matrix

$$T^{(f)}_{f(y_1,\dots,y_d)} \cdots T^{(h)}_{h(y_1,\dots,y_d)}$$

Therefore $\operatorname{Tr} A^d$ is the subalgebra with unit in

$k[V]^G$ generated by the traces of all the matrices from $M(B)$.

6.5. We explain now in nn 6.5–6.7 plays an important role in separation of orbits by means of invariants.

<u>Theorem 8</u>. <u>Assume</u> that algebra A <u>is generated</u> by $\le d$ <u>elements</u>. Then

$$k(V)\cdot B = A_{k(V)} \tag{6.5.1}$$

<u>Proof</u>. Let $a_1, \dots, a_d$ be a system of generators of algebra A. Then

$$e_j = h_j(a_1, \dots, a_d) \tag{6.5.2}$$

for some nonassociative polynomials $h_j(t_1, \dots, t_d)$ in noncommutative variables $t_1, \dots, t_d$ with the coefficients in k.

Show that n elements $h_i(y_1, \dots, y_d) \in B$ are linearly independent over $k(V)$. Since $k(V)$-algebra $A_{k(V)}$ is n-dimensional, this will complete the proof.

Since $\{e_j\}$ is a basis,

$$a_s = \sum_j \alpha_j^{(s)} e_j, \quad \alpha_j^{(s)} \in k. \tag{6.5.3}$$

It follows from (6.3.1) that

$$h_i(y_1, \dots, y_d) = \sum_j f_{ij} e_j, \quad f_{ij} \in k[V].$$

The formulas (6.5.3) and (6.5.2) show that substituting $\alpha_j^{(s)}$ instead on $x_j^{(s)}$ in the right hand side of (6.5.4) is equivalent to substituting a_s instead of y_s in the left hand side, which gives e_i by virtue of (6.5.2). Hence the polynomial matrix (t_{ij}) is specializing into E_n under this specialization of the variables $x^{(s)}$. Therefore (t_{ij}) is nondegenerate. This completes the proof by virtue of (6.5.4). ▨

Corollary 1. Assume that algebra A is generated by $\leq d$ elements. Then

$$\dim_K K \cdot B \geq n$$

for any subfield K of $k(V)$.

Corollary 2. Assume that algebra A is generated by $\leq d$ elements and K is a subfield of $k(V)$ such that

$$\dim_K K \cdot B = n$$

Let $b_1, \dots, b_n$ be a basis of $K \cdot B$ over K. Then, if $b_i = \sum_j t_{ij} e_j$, $t_{ij} \in k(V)$, one has $t := \det(t_{ij}) \neq 0$

Proof. It follows from the condition, theorem 8 and Corollary 1 that $b_1, \dots, b_n$ is a basis of $A_{k(V)}$ over $k(V)$. The claim now follows, since $e_1, \dots, e_n$ is such a basis as well. ▨

Corollary 3. Assume that algebra A is generated by $\leq d$ elements. Then

$$k(V) \cdot M(B) = M(A_{k(V)}).$$

6.6. Let H be an algebraic group acting unipotically on an algebraic variety X.

We say that a subfield K of $k(X)$ separates the H-orbits in general position in X, if for the points in general position in X each level variety of K is contained in an H-orbit. In another words, there exists an open dense subset of X such that any two points of it which are not separated by K (i.e. such that the values of each function from K at these points are either both defined and equal or both are not defined) lie in the same H-orbit.

We do not assume in this definition that $K \subset k(X)^H$.

The following statement is the general result of the algebraic transformation group theory (cf., for instance, [PV], n. 2.3):

Theorem 9. Let $k \subset K \subset k(X)^H$. Then K separates G-orbits in general position in X iff $K = k(X)^H$.

6.7. In our case there is the following number criterion:

Theorem 10. Assume that algebra A is generated by $\leq d$ elements and that K is a subfield of $k(V)$ such that

$$\dim_K K \cdot B = n$$

then K separates the G-orbits in general position in X.

Proof. Let $b_1, \ldots, b_n \in B$ be a basis of $K \cdot B$ over K.

Since B is a k-algebra, $K.B$ is a K-algebra. Therefore

$$b_i b_j = \sum_s f_{ij}^{(s)} b_s \quad , \quad f_{ij}^{(s)} \in K \qquad (6.7.1)$$

Since $y_j \in B$, one has

$$y_j = \sum_s h_j^{(s)} b_s \quad , \quad h_j^{(s)} \in K \qquad (6.7.2)$$

According to Corollary 2 of Theorem 8 (the notation of which we shall use), there exists an open dense subset $\mathcal{U}$ of V such that for any ~~function~~ point $v \in \mathcal{U}$ all the functions $f_{ij}^{(s)}$ and $h_j^{(s)}$ are regular at v, and $t(v) \neq 0$.

If $v \in \mathcal{U}$, it follows from $b_i(v) = \sum_j t_{ij}(v) e_j$ and $t(v) = \det(t_{ij}(v)) \neq 0$ that $b_1(v), \dots, b_n(v)$ is a basis of A over k. It follows from (6.7.1) that $b_i(v)\, b_j(v) = \sum_s f_{ij}^{(s)}(v)\, b_s(v)$, therefore the elements $f_{ij}^{(s)}(v) \in k$ are the structural constants of A in this basis.

Assume now that the points v and $u \in \mathcal{U}$ are not separated by K. Then $f_{ij}^{(s)}(v) = f_{ij}^{(s)}(u)$, i.e. the structural constants of A in the basises $\{b_i(v)\}$ and $\{b_i(u)\}$ are equal. Therefore the k-linear mapping

$$g : A \to A \quad , \quad g \cdot b_i(v) = b_i(u) \qquad (6.7.3)$$

is an automorphism of algebra A, i.e. $g \in G$.

Show now that $g.v = u$. We have $h_j^{(s)}(v) = h_j^{s}(u)$ because of the condition on u and v. Therefore, taking into account the interpretation of π_j as y_j (cf. n. 6.3) and formulas (6.7.2), (6.7.3), we get

$$g \cdot \pi_j(v) = g \cdot y_j(v) = g \cdot \sum_s h_j^{(s)}(v)\, b_s(v) = \sum_s h_j^{(s)}(v)\, g \cdot b_s(v) =$$

$$= \sum_s h_j^{(s)}(u)\, b_s(u) = y_j(u) = \pi_j(u),$$

and we are done ■

6.8. Assume that we can find in $k(V)$ a subfield K such that

(a) $K \subset \operatorname{Frac} \operatorname{Tr} A^d$,

(b) $\dim_K K \cdot B = n$

By virtue of (b) and Theorem 10, K separates the orbits in general position in V. Hence the same property has the field $\widetilde{K}$ generated by k and K. Since $k \subset \widetilde{K} \subset \operatorname{Frac} \operatorname{Tr} A^d \subset k(V)^G$, it follows from Theorem 9 that $\widetilde{K} = \operatorname{Frac} \operatorname{Tr} A^d = k(V)^G$.

Therefore, to prove Theorem 2 it suffices to find a field K which has properties (a), (b). This is done in nn. 6.9–6.11. The methods which are used for that purpose belong entirely to noncommutative ring theory which indicates, from my point of view, that the role of this theory for Invariant theory is not understood in a proper way as yet.

6.9. Using the notation of n. 6.4, consider in $M(A_{k(V)})$ the k-subalgebra $M(A)$ generated over k by the operators T_p, $T \in \{l, r\}$, $p \in A$. Since $k(V) \cdot A = A_{k(V)}$, one has $k(V) \cdot M(A) = M(A_{k(V)})$.

Theorem 11. (a) Assume that algebra A is simple. Then

$$M(A) = \mathrm{Mat}_n(k), \tag{6.9.1}$$

$$M(A_{k(V)}) = \mathrm{Mat}_n(k(V)). \tag{6.9.2}$$

(b) Assume moreover that A is generated by $\le d$ elements. Then:

1)

(i) $k(V)\cdot M(B) = Mat_n(k(V))$ (6.9.3)

(ii) $K\cdot B$ and $K\cdot M(B)$ are prime K-algebras for any subfield K of $k(V)$.

Proof. (1) The subspace A of $A_{k(V)}$ is $M(A)$-invariant and is a faithful $M(A)$-module (because A is simple). Therefore (6.9.1) follows from the density theorem and algebraic closedness of k, [J1], [He]. The equality (6.9.2) follows from (6.9.1).

(2)(i). (6.9.3) follows from (6.9.2) and Corollary 3 of Theorem 8.

(ii) Let I_1, I_2 be nonzero ideals in $K\cdot B$. It follows from Theorem 8 that $\tilde{I}_j = k(V)\cdot I_j$ are the nonzero ideals of $k(V)$-algebra $A_{k(V)}$. Since A is simple, this algebra is simple as well, [J2]. Hence $\tilde{I}_1 = \tilde{I}_2 = A_{k(V)}$. Therefore it will follow from $I_1 I_2 = 0$ that A has zero multiplication. This contradiction shows that $K\cdot B$ is prime.

Using (i) and simplicity of $k(V)$-algebra $Mat_n(k(V))$, one can prove that $K\cdot M(B)$ is prime in the same way. ▨

6.10. From now on we'll assume that A is simple and generated by $\leq d$ elements.

It follows from (6.4.1) that $M(B)$ is a PI-algebra. It is prime according to Theorem 11. It follows from these properties that $M(B)$ has a nonzero centre Z, cf. [J3] (in our case this fact also immediately follows from n. (2)(i) of Theorem 11 and existence of polylinear central polynomials for matrix rings, cf. [Ro] and n. 6.13).

We assume that $k(V) \subset Mat_n(k(V))$ identifying $f \in k(V)$ with fE_n. Then $k(V)$ is the centre of $Mat_n(k(V))$

It follows from n.(i) of Theorem 11 and the description of $Tr\, A^d$ given in n. 6.4 that

$$Z \subset Tr\, A^d \qquad (6.10.1)$$

6.11. We'll show that $K := Frac\, Z \subset k(V)$ has the properties (a) and (b), n. 6.8.

The property (a) follows from (6.10.1), therefore one needs only to prove property (b).

Theorem 12.

(i) $K \cdot M(B)$ is a finitely dimensional central simple K-algebra,

(ii) $\dim_K K \cdot M(B) = n^2$,

(iii) $K \cdot B$ is a simple K-algebra,

(iv) $\dim_K K \cdot B = n$.

Proof. Since $M(B)$ is a prime PI-algebra, (i) follows from the Posner-Rowen theorem, [J3].

By virtue of (i), there exists a finite extension of fields K'/K such that for a certain s one has an isomorphism of K'-algebras (cf. [J1], [He]):

$$K \cdot M(B) \otimes_K K' \cong Mat_s(K') \qquad (6.11.1)$$

Hence,

$$\dim_K K \cdot M(B) = s^2.$$

It is well known, [J1] [He], that F being a field, the standard polynomial $[x_1, \dots, x_{2m}]$ is the polylinear identity of degree $2m$ of F-algebra $Mat_m(F)$ (Amitsur-Levitski theorem), and $Mat_m(F)$ does not satisfy an identity of a smaller degree

Now (ii) follows from here and (6.11.1), (6.9.3) and (6.11.2).

As a vector space over K, $A_{k(V)}$ is a $K\cdot M(B)$-module and $K\cdot B$ is its submodule. The submodules of $K\cdot M(B)$-module $K\cdot B$ are precisely the ideals of K-algebra $K\cdot B$. Assume that this algebra is not simple, and let I be its proper ideal. It follows from (i) that any $K\cdot M(B)$-module is completely reducible, [He], therefore $K\cdot B = I \oplus J$ for a certain ideal $J \neq 0$. Therefore $IJ \subset I \cap J = 0$; this contradicts the property that $K\cdot B$ is prime (see theorem 11, n. (ii)), whence (iii).

Let $M(K\cdot B)$ and C be resp. multiplication algebra and centroid of K-algebra $K\cdot B$. It follows from (iii), cf. [J1], [He], that C is a field (containing K) and $M(K\cdot B)$ is a dense subring of C-algebra $\mathrm{End}_C K\cdot B$ of all linear transformations of $K\cdot B$ over C. It follows from (6.5.1) that the restriction of operators on $K\cdot B$ is an isomorphism of K-algebras

$$K\cdot M(B) \xrightarrow{\sim} M(K\cdot B) \qquad (6.11.3)$$

It follows from (6.11.3) and (ii) that

$$\dim_K M(K\cdot B) = n^2 \qquad (6.11.4)$$

In particular, $M(K\cdot B)$ is finitely dimensional over K. Since $K \subset C$, $C\cdot M(K\cdot B)$ is finite dimensional over C. It follows from here and from the density property that

$$\dim_C K\cdot B = s < \infty \qquad (6.11.5)$$

and $M(K\cdot B) = C\cdot M(K\cdot B) = \mathrm{End}_C K\cdot B$. Therefore, $M(K\cdot B)$ is isomorphic to $\mathrm{Mat}_s(C)$. Using the same arguments as in the proof of (ii) one derives from here and

(6.11.3), (6.9.3) that

$$s = n \qquad (6.11.6)$$

and hence

$$\dim_C M(K.B) = n^2 \qquad (6.11.7)$$

It follows from (6.11.4) and (6.11.7) that

$$C = K$$

Now (iv) follows from (6.11.5), (6.11.6) and (6.11.8). ∎

6.12. Remark. Since $k.Z = Z$, one has $k \subset K$. Therefore it is proved (cf. n. 6.8) that $K = k(V)^G$.

6.13. Remark. There is a construction giving "explicit" expressions of the traces of elements of $M(B)$ as the fractions of the traces of elements of Z (this construction was used in the proof of Theorem 2 given in [I 1]). Namely, let $f = f(x_1, \dots, x_s)$, $s \geq n^2$, be a central polynomial (the existence of which was proved by Formanek, cf. [Ro]), i.e. ~~such~~ a polylinear skewsymmetric polynomial with integer coefficients in the noncommutative variables x_i such that for any commutative ring R one has the properties:

(a) $f(Q_1, \dots, Q_s)$ is a scalar matrix for any $Q_i \in Mat_n(R)$;

(b) $f(P_1, \dots, P_s) \neq 0$ for certain $P_i \in Mat_n(R)$;

35)

(c) $n \cdot \operatorname{tr} Q \cdot f(Q_1, \ldots, Q_s) = \sum_{i=1}^{n^2} f(Q_1, \ldots, Q_{i-1}, Q_i Q, Q_{i+1}, \ldots, Q_s)$, for any $Q_i, Q \in Mat_n(R)$.

Let $R = k(V)$. It follows from (a) that $f(Q_1, \ldots, Q_s) \in \in Z$ for any $Q_i \in M(B)$, and from (6.9.3) that one can take $P_i \in M(B)$ in (b). Now it follows from (c) that for any $Q \in M(B)$ one has $\operatorname{tr} Q = = z'/nz$, where z' is the value of the right hand side of (c) for $Q_i = P_i$.

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