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**On the path model of representations**

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# On the path model of representations

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The aim of this note is to present an elementary construction that can be viewed as a generalization of the Young tableaux into the setting of symmetrizable Kac-Moody algebras. Details can be found in [7,8], for the relation to the standard monomial theory see [5,6], and for the relation to quantum groups see [1,3,6].

Denote by $X$ the weight lattice of a symmetrizable Kac-Moody algebra $\mathfrak{g}$ and set $X_\mathbb{Q} := X \otimes \mathbb{Z}$. As a replacement for the Young tableaux we consider the set $\Pi$ of all piecewise linear paths $\pi : [0,1] \to X_\mathbb{Q}$ such that $\pi(0) = 0$ and $\pi(1) \in X$. For each simple root $\alpha$ we introduce operators $e_\alpha$ and $f_\alpha$ on $\Pi \cup \{0\}$, which roughly speaking just replace some parts of a path by its image with respect to a reflection at some affine hyperplane. It turns out that the set of paths $B_\pi$ obtained by applying the operators $e_\alpha$ and $f_\alpha$ to a path $\pi$ has some remarkable properties that are closely related to the representation theory of the Kac-Moody algebra $\mathfrak{g}$.

### The root operators

We write $[0,1]$ for the set $\{t \in \mathbb{Q} \mid 0 \le t \le 1\}$. Denote by $\Pi$ the set of all piecewise linear paths $\pi : [0,1] \to X_\mathbb{Q}$ such that $\pi(0) = 0$ and $\pi(1) \in X$. We consider two paths $\pi_1, \pi_2$ as identical if there exists a piecewise linear, nondecreasing, surjective, continuous map $\phi : [0,1] \to [0,1]$ such that $\pi_1 = \pi_2 \circ \phi$. Let $\mathbb{Z}\Pi$ be the free $\mathbb{Z}$-module with basis $\Pi$. For each simple root $\alpha$ we define linear operators $e_\alpha$ and $f_\alpha$ (the root operators) on $\mathbb{Z}\Pi$.

Let $\pi, \pi_1, \pi_2 \in \Pi$ be paths. For a simple root $\alpha$ let $s_\alpha(\pi)$ be the path given by $s_\alpha(\pi)(t) := s_\alpha(\pi(t))$. By $\pi := \pi_1 * \pi_2$ we mean the concatenation of the paths, i.e. $\pi$ is the path defined by

$$\pi(t) := \begin{cases} \pi_1(2t), & \text{if } 0 \le t \le 1/2; \\ \pi_1(1) + \pi_2(2t-1), & \text{if } 1/2 \le t \le 1. \end{cases}$$

Fix a simple root $\alpha$ and let $\alpha^\vee$ be its coroot. To define the operator $e_\alpha$ we cut a path $\pi \in \Pi$ into several parts according to the behavior of the function

$$h_\alpha : [0,1] \to \mathbb{Q}, \quad t \mapsto \langle \pi(t), \alpha^\vee \rangle.$$

Let $m_\alpha := \min\{h_\alpha(t) \mid t \in [0,1]\}$ be the minimal value attained by $h_\alpha$.

If $m_\alpha \le -1$, then fix $t_1 \in [0,1]$ minimal such that $h_\alpha(t_1) = m_\alpha$ and let $t_0 \in [0,t_1]$ be maximal such that $h_\alpha(t) \ge m_\alpha + 1$ for $t \in [0, t_0]$.

Choose $t_0 = s_0 < s_1 < \ldots < s_r = t_1$ such that either

(1) $h_\alpha(s_{i-1}) = h_\alpha(s_i)$ and $h_\alpha(t) \ge h_\alpha(s_{i-1})$ for $t \in [s_{i-1}, s_i]$;

(2) or $h_\alpha$ is strictly decreasing on $[s_{i-1}, s_i]$ and $h_\alpha(t) \ge h_\alpha(s_{i-1})$ for $t \le s_{i-1}$.

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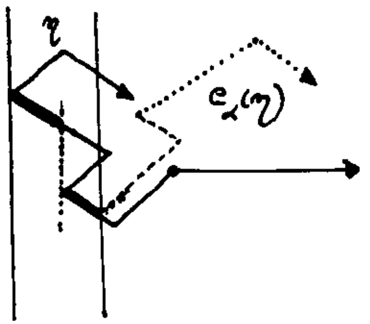



Set $s_{-1} := 0$ and $s_{r+1} := 1$, and denote by $\pi_i$ the path defined by

$$\pi_i(t) := \pi((s_{i-1} + t(s_i - s_{i-1})) - \pi(s_{i-1}), \quad i = 0, \dots, r+1.$$

It is clear that $\pi = \pi_0 * \pi_1 * \dots * \pi_{r+1}$.

**Definition** [8] If $m_\alpha \leq -1$, then set $e_\alpha \pi := \pi_0 * \eta_1 * \eta_2 * \dots * \eta_r * \pi_{r+1}$, where $\eta_i = \pi_i$ if the function $h_\alpha$ behaves on $[s_{i-1}, s_i]$ as in (1), and $\eta_i = s_\alpha(\pi_i)$ if the function $h_\alpha$ behaves on $[s_{i-1}, s_i]$ as in (2). If $m_\alpha > -1$, then we set $e_\alpha \pi := 0$.

The definition of the operator $f_\alpha$ is similar. Let $t_0 \in [0,1]$ be maximal such that $h_\alpha(t_0) = m_\alpha$. If $h_\alpha(1) - m_\alpha \geq 1$, then fix $t_1 \in [t_0, 1]$ minimal such that $h_\alpha(t) \geq m_\alpha + 1$ for $t \in [t_1, 1]$. Choose $t_0 = s_0 < s_1 < \dots < s_r = t_1$ such that either

(1) $h_\alpha(s_i) = h_\alpha(s_{i-1})$ and $h_\alpha(t) \geq h_\alpha(s_{i-1})$ for $t \in [s_{i-1}, s_i]$;

(2) or $h_\alpha$ is strictly increasing on $[s_{i-1}, s_i]$ and $h_\alpha(t) \geq h_\alpha(s_i)$ for $t \geq s_i$.

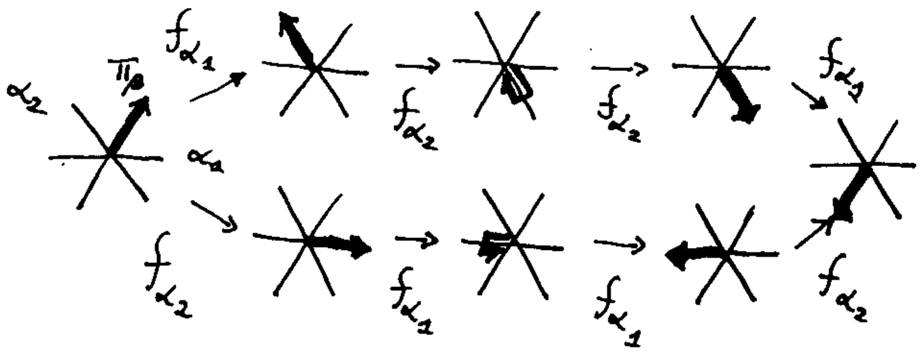
Set $s_{-1} := 0$ and $s_{r+1} := 1$, and denote by $\pi_i$ the path defined by

$$\pi_i(t) := \pi((s_{i-1} + t(s_i - s_{i-1})) - \pi(s_{i-1}), \quad i = 0, \dots, r+1.$$

It is clear that $\pi = \pi_0 * \pi_1 * \dots * \pi_{r+1}$.

**Definition** [8] If $h_\alpha(1) - 1 \geq 1$, the set $f_\alpha \pi := \pi_0 * \eta_1 * \eta_2 * \dots * \eta_r * \pi_{r+1}$, where $\eta_i = \pi_i$ if the function $h_\alpha$ behaves on $[s_{i-1}, s_i]$ as in (1), and $\eta_i = s_\alpha(\pi_i)$ if the function $h_\alpha$ behaves on $[s_{i-1}, s_i]$ as in (2). If $h_\alpha(1) - m_\alpha < 1$, then we set $f_\alpha \pi := 0$.

**Example** Suppose $\mathfrak{g} = \mathfrak{sl}_3$ and $\beta$ is the highest root. The eight paths obtained from $\pi_\beta : t \mapsto t\beta$ by applying the operators $f_\alpha, e_\alpha$ are the paths $\pi_\gamma(t) := t\gamma$, where $\gamma$ is an arbitrary root, and for $\alpha$ simple one gets two paths in addition:





## Some simple properties

Denote by $\mathcal{A}$ the subalgebra of $\mathrm{End}_{\mathbb{Z}}\mathbb{Z}\Pi$ generated by the root operators and let $m_\alpha := \min\{h_\alpha(t) \mid t \in [0,1]\}$ be the minimal value attained by the function $h_\alpha$ for $\pi \in \Pi$ and a fixed simple root $\alpha$.

**Lemma** [8] *a) If $e_\alpha \pi \neq 0$, then $e_\alpha \pi(1) = \pi(1) + \alpha$ and $f_\alpha e_\alpha \pi = \pi$, and if $f_\alpha \pi \neq 0$, then $f_\alpha \pi(1) = \pi(1) - \alpha$ and $e_\alpha f_\alpha \pi = \pi$.*

*b) $e_\alpha^n \pi = 0$ if and only if $n > |m_\alpha|$, and $f_\alpha^n \pi = 0$ if and only if $n > \langle \pi(1), \alpha^\vee \rangle - m_\alpha$.*

*c) For $\pi \in \Pi$ let $n_1, n_2$ be maximal such that $e_\alpha^{n_1} \pi \neq 0$ and $f_\alpha^{n_2} \pi \neq 0$. Then $\langle \pi(1), \alpha^\vee \rangle = n_2 - n_1$.*

*d) The $\mathcal{A}$-module $\mathcal{A}\pi \subset \mathbb{Z}\Pi$ generated by $\pi$ has as basis the set of all paths $\eta \in \Pi$ contained in $\mathcal{A}\pi$.*

These results show a certain resemblance with standard results in the representation theory of the Lie algebra $\mathfrak{sl}_2$. Since the root operators are locally nilpotent, the operators

$$x_\alpha := \sum_{i \geq 1} e_\alpha^i f_\alpha^{i-1}, \quad y_\alpha := \sum_{i \geq 1} f_\alpha^i e_\alpha^{i-1}, \quad h_\alpha := \sum_{i \geq 1} (e_\alpha^i f_\alpha^i - f_\alpha^i e_\alpha^i)$$

make sense. The following proposition follows easily:

**Proposition** [8] *If $\pi$ is an element of $\Pi$, then $h_\alpha \pi = \langle \pi(1), \alpha^\vee \rangle \pi$. Further,*

$$[x_\alpha, y_\alpha] = h_\alpha, \quad [h_\alpha, x_\alpha] = 2x_\alpha, \quad [h_\alpha, y_\alpha] = -2y_\alpha,$$

*so the elements $x_\alpha, y_\alpha$ and $h_\alpha$ span a Lie subalgebra of $\mathrm{End}_{\mathbb{Z}}\mathbb{Z}\Pi$ isomorphic to $\mathfrak{sl}_2(\mathbb{Z})$.*

The $x_\alpha$ respectively $y_\alpha$ do not satisfy the Serre relations, but the $h_\alpha$ commute. Let $\mathfrak{h}$ be the subalgebra of $\mathrm{End}_{\mathbb{Z}}\mathbb{Z}\Pi$ spanned by the $h_\alpha$. So if we define for $\pi \in \Pi$ the "character" of the the $\mathcal{A}$-module $M_\pi := \mathcal{A}\pi$ as

$$\mathrm{Char}\, M_\pi := \sum_{\eta \in M_\pi} e^{\eta(1)}$$

the sum over the endpoints of all paths in $M_\pi$, then this can also be viewed as the (usual) character of $M_\pi$ as an $\mathfrak{h}$-module.

## The main results

Let $\mathcal{A}$ be the algebra in $\mathrm{End}_{\mathbb{Z}}\mathbb{Z}\Pi$ generated by the root operators $e_\alpha$ and $f_\alpha$ and denote by $\Pi^+$ the set of paths $\pi$ such that the image is contained in the dominant Weyl chamber. For $\pi \in \Pi^+$ let $M_\pi$ be the $\mathcal{A}$-module $\mathcal{A}\pi$. Clearly the set $B_\pi$ of paths contained in $M_\pi$ is a basis for $M_\pi$. We show that the $\mathcal{A}$-module structure of $M_\pi$ is invariant under those deformations of $\pi$ which stay inside the dominant Weyl chamber and fix the starting point and the endpoint of the path:

**Isomorphism Theorem** [8] *For $\pi, \pi' \in \Pi^+$ the $\mathcal{A}$-modules $M_\pi$ and $M_{\pi'}$ are isomorphic if and only if $\pi(1) = \pi'(1)$.*

In particular, the isomorphism theorem shows that we get always the same "character" for $M_\pi$. The character can be calculated using Weyl's character formula. Let $\rho \in X$ be such that $\langle \rho, \alpha^\vee \rangle = 1$ for all simple roots.

**Character formula** [8] *For $\pi \in \Pi^+$ let $\mathrm{Char}\, M_\pi$ be the character $\sum_{\eta \in B_\pi} e^{\eta(1)}$ of the $\mathcal{A}$-module $M_\pi$. Then:*

$$\sum_{\sigma \in W} \mathrm{sgn}(\sigma) e^{\sigma(\rho)}\, \mathrm{Char}\, M_\pi = \sum_{\sigma \in W} \mathrm{sgn}(\sigma) e^{\sigma(\rho+\lambda)}.$$

*In particular, $\mathrm{Char}\, M_\pi$ is equal to the character of the irreducible, integrable $\mathfrak{g}$-module $V_\lambda$ of highest weight $\lambda := \pi(1)$.*

To define an analogue of a tensor product for $\mathcal{A}$-modules, let for $\pi, \eta \in \Pi^+$ the concatenation $M_\pi * M_\eta$ of two modules be just the span of all concatenations of path $\pi' * \eta'$, where $\pi' \in B_\pi$ and $\eta' \in B_\eta$.

**Tensor product rule** [8] *For $\pi_1, \pi_2 \in \Pi^+$ one has*

$$M_{\pi_1} * M_{\pi_2} = \bigoplus_\pi M_\pi,$$

*where $\pi$ runs over all paths in $\Pi^+$ of the form $\pi = \pi_1 * \eta$ for some $\eta \in B_{\pi_2}$.*

By the character formula we get immediately the following Littlewood-Richardson type decomposition rule:

**Decomposition formula** [8] *If $\pi_1, \pi_2 \in \Pi^+$ are such that $\lambda = \pi_1(1)$ and $\mu = \pi_2(1)$, then the tensor product $V_\lambda \otimes V_\mu$ of irreducible $\mathfrak{g}$-modules decomposes into the direct sum*

$$V_\lambda \otimes V_\mu \simeq \bigoplus_\pi V_{\pi(1)},$$

*where $\pi$ runs over all paths in $\Pi^+$ of the form $\pi = \pi_1 * \eta$ for some $\eta \in B_{\pi_2}$.*

For an appropriate choice of $\pi_2$ this rule is for $\mathfrak{g} = \mathfrak{gl}_n$ the Littlewood-Richardson rule. It should be interesting to find a direct correspondence to Lusztig's decomposition formula [9].

For a Levi subalgebra $\mathfrak{l}$ of $\mathfrak{g}$ let $\mathcal{A}_\mathfrak{l}$ be the subalgebra generated by those $e_\alpha, f_\alpha$ such that $\alpha$ is a simple root of $\mathfrak{l}$. Denote by $\Pi_\mathfrak{l}^+$ the set of paths contained in the dominant Weyl chamber of the root system of $\mathfrak{l}$, and for $\eta \in \Pi_\mathfrak{l}^+$ denote by $N_\eta$ the $\mathcal{A}_\mathfrak{l}$-module generated by $\eta$.

**Restriction rule** [8] *The $\mathcal{A}$-module $M_\pi$, $\pi \in \Pi^+$, decomposes as $\mathcal{A}_\mathfrak{l}$-module into the direct sum $M_\pi = \bigoplus_\eta N_\eta$, where $\eta$ runs over all paths in $B_\pi$ contained in $\Pi_\mathfrak{l}^+$.*

By the character formula we get for $\lambda = \pi(1)$: $V_\lambda$ decomposes as $\mathfrak{l}$-module into the direct sum $\bigoplus_\eta U_{\eta(1)}$ of simple $\mathfrak{l}$-modules, where $\eta$ runs over all paths in $B_\pi$ contained in $\Pi_\mathfrak{l}^+$.



Another connection between the $\mathcal{A}$-modules $M_\pi$ and the $\mathfrak{g}$-module $V_{\pi(1)}$ is given as follows: Let $\mathcal{G}(\pi)$ be the oriented, colored graph having as points the elements of the basis $B_\pi$, and we put an arrow $\pi_1 \xrightarrow{\alpha} \pi_2$ with color $\alpha$ if and only if $f_\alpha(\pi_1) = \pi_2$. Joseph [1], Lakshmibai [6] and Kashiwara [3] have proved (independently):

**The Crystal Graph** *For $\pi = \pi_\lambda$ the graph $\mathcal{G}(\pi_\lambda)$ is isomorphic to the crystal graph of the representation $V_\lambda$ of the q-analogue $U_q(\mathfrak{g})$ of the enveloping algebra of $\mathfrak{g}$.*

For a simple root $\alpha$ set:

$$\tilde{s}_\alpha(\pi) := \begin{cases} f_\alpha^n \pi; & \text{if } n := \langle \pi(1), \alpha^\vee \rangle \geq 0, \\ e_\alpha^{-n} \pi; & \text{if } n := \langle \pi(1), \alpha^\vee \rangle < 0, \end{cases}$$

Note that $\tilde{s}_\alpha^2 = 1$ and $\tilde{s}_\alpha(\pi)(1) = s_\alpha(\pi(1))$ for the ususal simple reflection $s_\alpha$ in the Weyl group. In fact:

**Weyl group action** [8] *The map $s_\alpha \mapsto \tilde{s}_\alpha$ on the simple reflections extends to a representation map $W \to \mathrm{End}_\mathbb{Z}\mathbb{Z}\Pi$ of the Weyl group of $\mathfrak{g}$ such that $w(\pi)(1) = w(\pi(1))$ for $\pi \in \Pi$ and $w \in W$.*

## The Lakshmibai-Seshadri paths

We give now a description of the basis of the $\mathcal{A}$-module $\mathcal{A}\pi_\lambda$, where $\lambda$ is a dominant weight and $\pi_\lambda$ is the path $t \mapsto t\lambda$ that connects the origin with $\lambda$ by a straight line. In the following let $V_\lambda$ be the irreducible highest weight module of $\mathfrak{g}$ of highest weight $\lambda$.

In $X_\mathbb{Q}$ let $\mathcal{C}(\lambda)$ be the convex hull of the orbit $W \cdot \lambda$. We consider pairs of sequences representing a path in $X_\mathbb{Q}$:

Let $W_\lambda$ be the stabilizer of $\lambda$, and let "$\leq$" be the Bruhat order on $W/W_\lambda$. Suppose

- $\underline{\tau} : \tau_1 > \tau_2 > \ldots > \tau_r$ is a sequence of linearly ordered cosets in $W/W_\lambda$ and
- $\underline{a} : a_0 := 0 < a_1 < \ldots < a_r := 1$ is a sequence of rational numbers.

We call the pair $\pi = (\underline{\tau}, \underline{a})$ a *rational W-path of shape* $\lambda$. We identify $\pi$ with the path $\pi : [0,1] \to X_\mathbb{Q}$ given by

$$\pi(t) := \sum_{i=1}^{j-1} (a_i - a_{i-1}) \tau_i(\lambda) + (t - a_{j-1}) \tau_j(\lambda) \quad \text{for} \quad a_{j-1} \leq t \leq a_j.$$

Recall that a weight $\mu$ in $X$ is a weight of $V_\lambda$ if and only if $\mu \in \mathcal{C}(\lambda)$ and $\lambda - \mu$ is a sum of positive roots. Since the $\tau_i$ are linearly ordered, the differences $\tau_{i+1}(\lambda) - \tau_i(\lambda)$ are sums of positive roots. Note that

$$\lambda - \pi(1) = \lambda - \sum_{i=1}^{r} (a_i - a_{i-1}) \tau_i(\lambda) = (\lambda - \tau_r(\lambda)) + \sum_{i=1}^{r-1} a_i (\tau_{i+1}(\lambda) - \tau_i(\lambda)),$$

so if the $a_i$ are chosen such that the $a_i(\tau_{i+1}(\lambda) - \tau_i(\lambda))$ are still in the root lattice, then $\pi(1)$ is a weight of $V_\lambda$. To ensure that $\pi(1)$ is a weight of $V_\lambda$, we introduce now the notion of an $a$-chain. Note that the condition below is stronger than just demanding that the $a_i(\tau_{i+1}(\lambda) - \tau_i(\lambda))$ are in the root lattice. We use the usual notation $l(\cdot)$ for the length function on $W/W_\lambda$ and $\beta^\vee$ for the coroot of a positive real root $\beta$:

Let $\tau > \sigma$ be two elements of $W/W_\lambda$ and let $0 < a < 1$ be a rational number. By an *a-chain* for the pair $(\tau, \sigma)$ we mean a sequence of cosets in $W/W_\lambda$: (see [5])

$$\kappa_0 := \tau > \kappa_1 := s_{\beta_1}\tau > \kappa_2 := s_{\beta_2}s_{\beta_1}\tau > \ldots > \kappa_s := s_{\beta_s}\cdot\ldots\cdot s_{\beta_1}\tau = \sigma,$$

where $\beta_1, \ldots, \beta_s$ are positive real roots such that for all $i = 1, \ldots, s$:

$$l(\kappa_i) = l(\kappa_{i-1}) - 1 \quad \text{and} \quad a\langle\kappa_i(\lambda), \beta_i^\vee\rangle \in \mathbb{Z}.$$

The last condition can be expressed as follows: Each summand in

$$a\big(\tau(\lambda) - \sigma(\lambda)\big) = \sum_{i=0}^{s-1} a\big(\kappa_i(\lambda) - \kappa_{i-1}(\lambda)\big) = \sum_{i=1}^{s} a\langle\kappa_i(\lambda), \beta_i^\vee\rangle\beta_i$$

is an element in the root lattice. This is obviously stronger than just to demand that $a(\tau(\lambda) - \sigma(\lambda))$ is an element of the root lattice. The following definition is a reformulation of the definition in [5] into the language of paths.

**Definition** A rational $W$-path $\pi$ of shape $\lambda$ is called a *Lakshmibai-Seshadri* path, if for all $i = 1, \ldots, r-1$ there exists an $a_i$-chain for the pair $(\tau_i, \tau_{i+1})$.

**Remark** If $\pi = (\underline{\tau}, \underline{a})$ is a rational $W$-path of shape $\lambda$, then there exists an $n \geq 1$ such that $\pi$ is a Lakshmibai-Seshadri path of shape $n\lambda$.

**Theorem** [7] *The $\mathcal{A}$-module $\mathcal{A}\pi_\lambda$ has as basis the set of all Lakshmibai-Seshadri paths $\pi = (\underline{\tau}, \underline{a})$ of shape $\lambda$.*

Let $\rho \in X$ be such that $\langle\rho, \alpha^\vee\rangle = 1$ for all simple roots $\alpha$. In the following we denote by $\Lambda_\alpha$ the Demazure operator on $\mathbb{Z}[X]$:

$$\Lambda_\alpha(e^\mu) := \frac{e^{\mu+\rho} - e^{s_\alpha(\mu+\rho)}}{1 - e^{-\alpha}} e^{-\rho}$$

The following character formula had been conjectured by Lakshmibai and Seshadri [5]:

**Theorem** [7] *Let $\tau = s_{\alpha_1}\ldots s_{\alpha_r}$ be a reduced decomposition of $\tau \in W/W_\lambda$, and denote by $B_\lambda(\tau)$ the set of Lakshmibai Seshadri paths $\pi = (\underline{\tau}, \underline{a})$ of shape $\lambda$ such that the initial term satisfies $\tau_1 \leq \tau$. Then*

$$\Lambda_{\alpha_1}\ldots\Lambda_{\alpha_r}(e^\lambda) = \sum_{\pi \in B_\lambda(\tau)} e^{\pi(1)}.$$

## The P–R–V conjecture

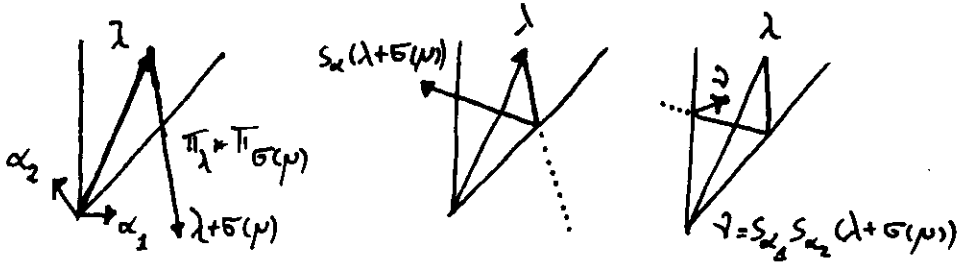
Consider the tensor product $V_\lambda \otimes V_\mu$ of two $\mathfrak{g}$-modules of highest weight $\lambda$ and $\mu$. The Parthasaraty Ranga-Rao Varadarajan conjecture states that



**P–R–V conjecture** *If $\sigma, \tau \in W$ are such that $\nu := \tau(\lambda) + \sigma(\mu)$ is a dominant weight, then the module $V_\nu$ occurs in $V_\lambda \otimes V_\mu$.*

Proofs of the conjecture have been given independently in [4] and [10]. To show how the paths can be used to prove existence results, we sketch how to give a new proof of this conjecture. Starting with the paths $\pi_\lambda$ and $\pi_\mu$, it is easy to see that the path $\pi_{\sigma(\mu)}$ connecting 0 with the weight $\sigma(\mu)$ by a straight line, is in fact in $\mathcal{A}\pi_\mu$.

The figure below shows how to construct successively a path $\pi'$ such that (see [7]) $\pi_\lambda * \pi' \in \Pi^+$, $\pi' \in \mathcal{A}\pi_\mu$ and the endpoint is equal to $\nu := \lambda + \pi'(1) = w(\lambda + \sigma(\mu))$.



It follows by the decomposition formula that $V_\nu$ occurs in the tensor product for any $w, \sigma \in W$ such that $w(\lambda + \sigma(\mu))$ is a dominant weight, but this is just a reformulation of the P–R–V conjecture.

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