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FOURTH AUTUMN COURSE ON MATHEMATICAL ECOLOGY

(24 October - 11 November 1994)

"Discrete-Time Epidemic Models"

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These are preliminary lecture notes, intended only for distribution to participants.

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Discrete-Time Epidemic Models

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Fourth Autumn Course on Mathematical Ecology
International Center for Theoretical Physics
Trieste, Italy
24 October–11 November, 1994

Outline of Talk

Objective: Compare the behavior of some well-known discrete and continuous SI, SIR, and SIS models with and without births and deaths.

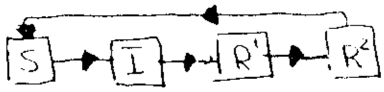
I. Two Applications with Different Discrete-Time SIR Formulations are presented: Discrete I, Discrete II.

II. Discrete and Continuous SI Models with No Births and Deaths Behave Similarly.

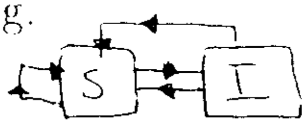
III. Discrete and Continuous SIR Models with No Births and Deaths Behave Similarly.

IV. Discrete Formulation I: SIS Model with No Births and Deaths Exhibits Period Doubling.

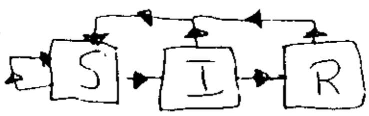
V. Discrete Formulation II: SIR^1R^2S Model with No Births and Deaths Exhibits Quasiperiodic Behavior.



VI. Discrete Formulation I: SI and SIS Models with Births and Deaths Exhibit Period Doubling.



VII. Discrete Formulation I: SIR Model with Births and Deaths Exhibits Period Doubling.



VIII. Final Remarks

Discrete SIR Formulation I

Global Spread of Influenza

- Rvachev and Longini (1985)

- The model was developed to forecast global spread of influenza based on information from the initial city in the transportation network to experience the disease.

- For one city define the following state variables, parameters, and probabilities:

S_n = number of susceptible individuals on day n.

$E_n(\tau)$ = number of latent individuals on day n who were infected on day $n-\tau$.

$I_n(\tau)$ = number of infectious individuals on day n who were infected on day $n-\tau$.

R_n = number of immune individuals on day n .

α = average number of individuals with whom an infectious individual will make sufficient contact (to pass infection) in one day.

τ_1 = maximum length of latent period.

τ_2 = maximum length of latent plus infectious period (infected period).

$\gamma_1(\tau)$ = probability a latent individual becomes infectious on day $\tau+1$, given that the individual was still latent on day τ.

$\gamma_2(\tau)$ = probability an infectious individual recovers on day $\tau+1$ given that the individual was still infectious on day τ.

$g(\tau)$ = fraction of individuals in infectious state at time τ who were all infected at the same time.

The number of new latent individuals due to infectious individuals on day n:

$$E_{n+1}(0) = \frac{\alpha S_n}{N} \sum_{\tau=1}^{\tau_2} E_{n-\tau}(0)g(\tau).$$

Discrete SIR Formulation I

- Note that the number of new infections is

$$E_{n+1}(0) = \frac{\alpha}{N} S_n(\text{Number Infectious}).$$

$$\begin{aligned}
S_{n+1} &= S_n - E_{n+1}(0) \\
E_{n+1}(\tau+1) &= E_n(\tau)[1-\gamma_1(\tau)], \quad \tau = 0, \ldots, \tau_1 - 1 \\
I_{n+1}(\tau+1) &= \begin{cases} \gamma_1(\tau)E_n(\tau) + [1-\gamma_2(\tau)]I_n(\tau), & \tau = 0, \ldots, \tau_1 \\ I_n(\tau)[1-\gamma_2(\tau)], & \tau = \tau_1 + 1, \ldots, \tau_2 - 1. \end{cases}
\end{aligned}$$

$$S_n + \sum_{\tau=0}^{\tau_1} E_n(\tau) + \sum_{\tau=0}^{\tau_2} I_n(\tau) + R_n = N,$$

where N is the total population size.

Initial and boundary conditions before the start of an epidemic:

$$E_{n+1}(0) = \frac{\alpha S_n}{N} \sum_{\tau=1}^{\tau_2} E_{n-\tau}(0) g(\tau), \quad I_n(0) = 0$$

$$S_0 = pN, \quad E_0(\tau) = I_0(\tau) = 0, \quad R_0 = (1-p)N.$$

Discrete SIR Formulation II

Spread of Measles on a University Campus

- Allen, Jones, and Martin (1991)
- The model was developed to predict the spread of measles between dormitory complexes on a university campus.
- The number of new infections has a different formulation in this model.

Define the following parameters:

$\alpha =$ average number of individuals with whom an infectious individual will make sufficient contact to pass infection in one day.

$\alpha SI/N =$ average number of infections per day caused by all infectives.

$\mu = \alpha SI/(SN) = \alpha I/N =$ average number of infections per susceptible individual per day.

The probability of k successful encounters resulting in infection of a susceptible individual by the infective class in one day is assumed to follow a Poisson distribution:

$$p(k) = \frac{\exp(-\mu)\mu^k}{k!}.$$

Thus, the probability that a susceptible individual does not become infective is given by

$$p(0) = \exp(-\mu) = \exp\left(-\frac{\alpha I}{N}\right).$$

The number of new infections is:

$$E_{n+1}(0) = S_n\left[1 - \exp\left(-\frac{\alpha}{N}\sum_{\tau=1}^{T2} I_n(\tau)\right)\right].$$

Discrete SIR Formulation II

The system of equations for S_n, $E_n(\tau)$ and $I_n(\tau)$ is the same as the discrete formulation I:

$$\begin{aligned} S_{n+1} &= S_n - E_{n+1}(0) \\ E_{n+1}(\tau+1) &= E_n(\tau)[1-\gamma_1(\tau)], \quad \tau = 0,\dots,\tau_1 - 1 \\ I_{n+1}(\tau+1) &= \begin{cases} \gamma_1(\tau)E_n(\tau) + [1-\gamma_2(\tau)]I_n(\tau), & \tau = 0,\dots,\tau_1 \\ I_n(\tau)[1-\gamma_2(\tau)], & \tau = \tau_1+1,\dots,\tau_2 - 1. \end{cases} \end{aligned}$$

where

$$E_{n+1}(0) = S_n \left[1 - \exp\left(-\frac{\alpha}{N} \sum_{\tau=1}^{\tau_2} I_n(\tau) \right) \right].$$

Thus,

$$S_{n+1} = S_n \exp\left(-\frac{\alpha}{N} \sum_{\tau=1}^{\tau_2} I_n(\tau) \right).$$

Discrete and Continuous SI Models Behave Similarly

Continuous

$$\dot{S} = -\frac{\alpha}{N} SI$$
$$\dot{I} = \frac{\alpha}{N} SI,$$

where $S(0), I(0) > 0$ and $S(0) + I(0) = N$.

$$S(t) \searrow 0 \quad \text{and} \quad I(t) \nearrow N.$$

Discrete I

$$S_{n+1} = S_n \left(1 - \frac{\alpha \Delta t}{N} I_n\right)$$
$$I_{n+1} = I_n \left(1 + \frac{\alpha \Delta t}{N} S_n\right),$$

where $S_0, I_0 > 0$ and $S_0 + I_0 = N$.

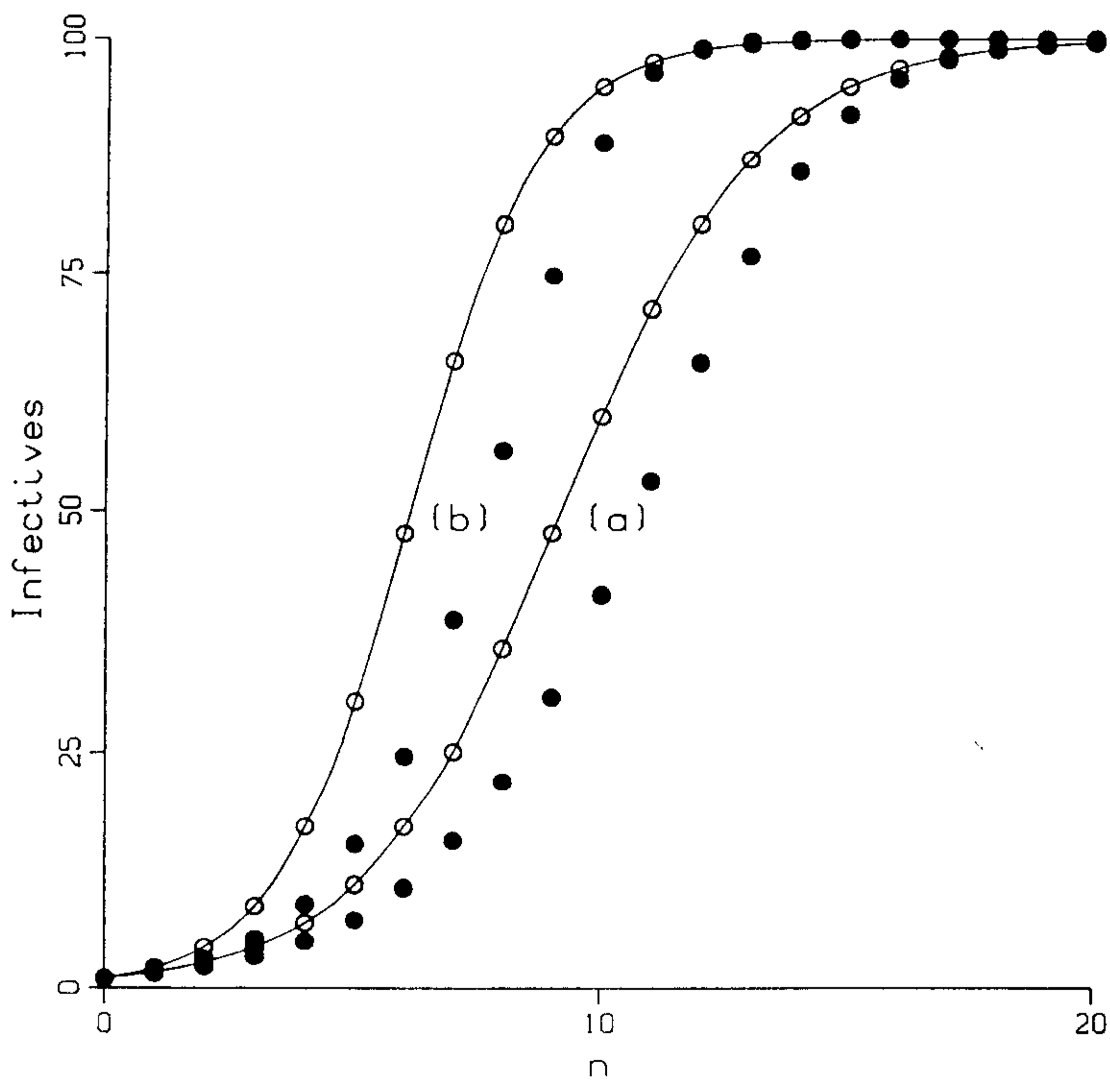
Solutions are positive iff

$$\alpha \Delta t \leq 1.$$

$$S_n \searrow 0 \quad \text{and} \quad I_n \nearrow N.$$

- Parameter α is the average number of individuals with whom an infectious individual will make sufficient contact (to pass infection) per unit time.
- Subscript n represents the time $n\Delta t$.

FIG. 1. Number of infectives in the discrete (• • •) and continuous (o-o-o) SI models, $\Delta t = .25$, $N = 100.$, and I_0=1. **(a)** $\alpha = 2$. **(b)** $\alpha = 3$.



Discrete SI Model

Discrete II

$$
\begin{aligned}
S_{n+1} &= S_n \exp\left(-\frac{\alpha\Delta t}{N} I_n\right) \\
I_{n+1} &= I_n + S_n \left(1 - \exp\left(-\frac{\alpha\Delta t}{N} I_n\right)\right),
\end{aligned}
$$

where $S_0, I_0 > 0$, and $S_n + I_n = N$.

$$
S \searrow 0, \quad I_n \nearrow N
$$

- The expression $\alpha\Delta t SI/N$ is the average number of infections per time Δt caused by the infective class.
- The expression $\mu = \alpha\Delta t SI/SN = \alpha\Delta t I/N$ is the average number of infectious per susceptible individual per time Δt.
- The Poisson distribution gives the probability of k successful encounters: $p(k) = \exp(-\mu)\mu^k/k!$.
- No successful encounters is given by $p(0)$:

$$
p(0) = \exp(-\mu) = \exp\left(-\frac{\alpha\Delta t}{N} I_n\right).
$$

Discrete and Continuous SIR Models Behave Similarly

Continuous

$$\begin{aligned}\dot{S} &= -\frac{\alpha}{N}SI \\ \dot{I} &= \frac{\alpha}{N}SI - \gamma I \\ \dot{R} &= \gamma I,\end{aligned}$$

where $S(0), I(0) > 0$, $R(0) \geq 0$, and $S(0) + I(0) + R(0) = N$.

$$S(t) \searrow S_\infty, \quad I(t) \to 0, \quad R(t) \nearrow R_\infty.$$

$$\mathcal{R}_0 = \frac{\alpha}{\gamma}$$

- $\mathcal{R} = \frac{S_0}{N}\mathcal{R}_0 \leq 1$, $I(t) \searrow 0$. $\mathcal{R} = \frac{S_0}{N}\mathcal{R}_0 > 1$, $\dot{I} > 0$ for $t \epsilon [0, c)$.

Discrete I

$$\begin{aligned}S_{n+1} &= S_n\left(1 - \frac{\alpha\Delta t}{N}I_n\right) \\ I_{n+1} &= I_n\left(1 - \gamma\Delta t + \frac{\alpha\Delta t}{N}S_n\right) \\ R_{n+1} &= R_n + \gamma\Delta t I_n,\end{aligned}$$

where $S_0, I_0 > 0$, $R_0 \geq 0$, and $S_0 + I_0 + R_0 = N$. Solutions are positive iff

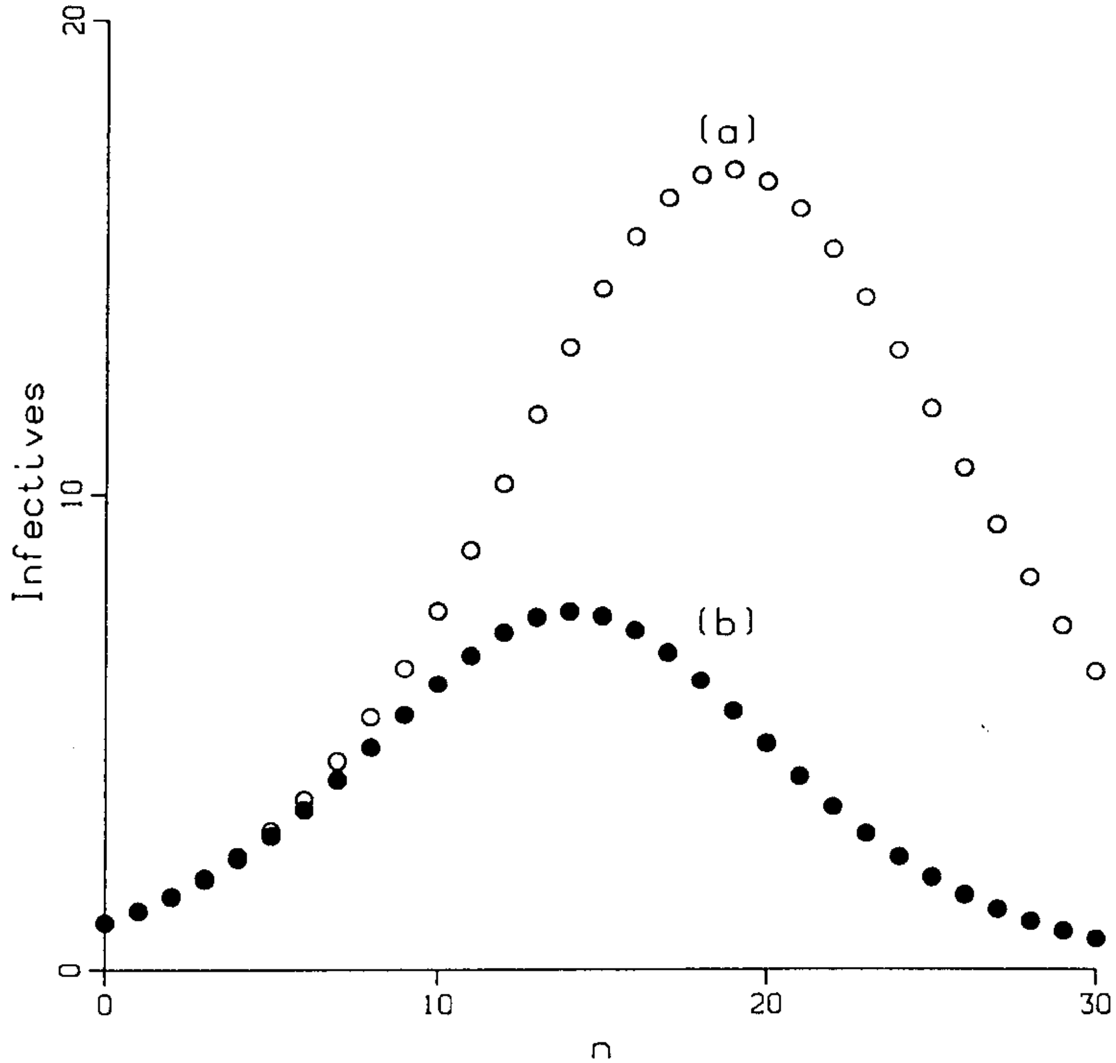
$$\max\{\gamma\Delta t, \alpha\Delta t\} \leq 1.$$

$$S_n \searrow S_\infty, \quad I_n \to 0, \quad R_n \nearrow R_\infty.$$

- $\mathcal{R} \leq 1$, $I_n \searrow 0$. $\mathcal{R} > 1$, $I_{n+1} > I_n$ for $n \epsilon \{0, \ldots, n^*\}$.

Discrete I

FIG. 2. Number of infectives in the discrete SIR model, $\Delta t = .25$, $N = 100.$, $S_0=99.$, and $I_0 = 1$. **(a)** $\alpha=2.$, $\gamma=1.$, and $\mathcal{R} = 1.98$. **(b)** $\alpha = 3.$, $\gamma = 2.$, and $\mathcal{R}=1.485$.



Discrete SIR Model

Discrete II

$$
\begin{aligned}
S_{n+1} &= S_n \exp\left(-\frac{\alpha \Delta t}{N} I_n\right) \\
I_{n+1} &= I_n(1 - \gamma \Delta t) + S_n \left(1 - \exp\left(-\frac{\alpha \Delta t}{N} I_n\right)\right) \\
R_{n+1} &= R_n + \gamma \Delta t I_n
\end{aligned}
$$

where $S_0, I_0 > 0$ and $R_0 \geq 0$ and $S_0 + I_0 + R_0 = N$.

$$S_n \searrow S_\infty \quad I_n \to 0 \quad R_n \nearrow R_\infty$$

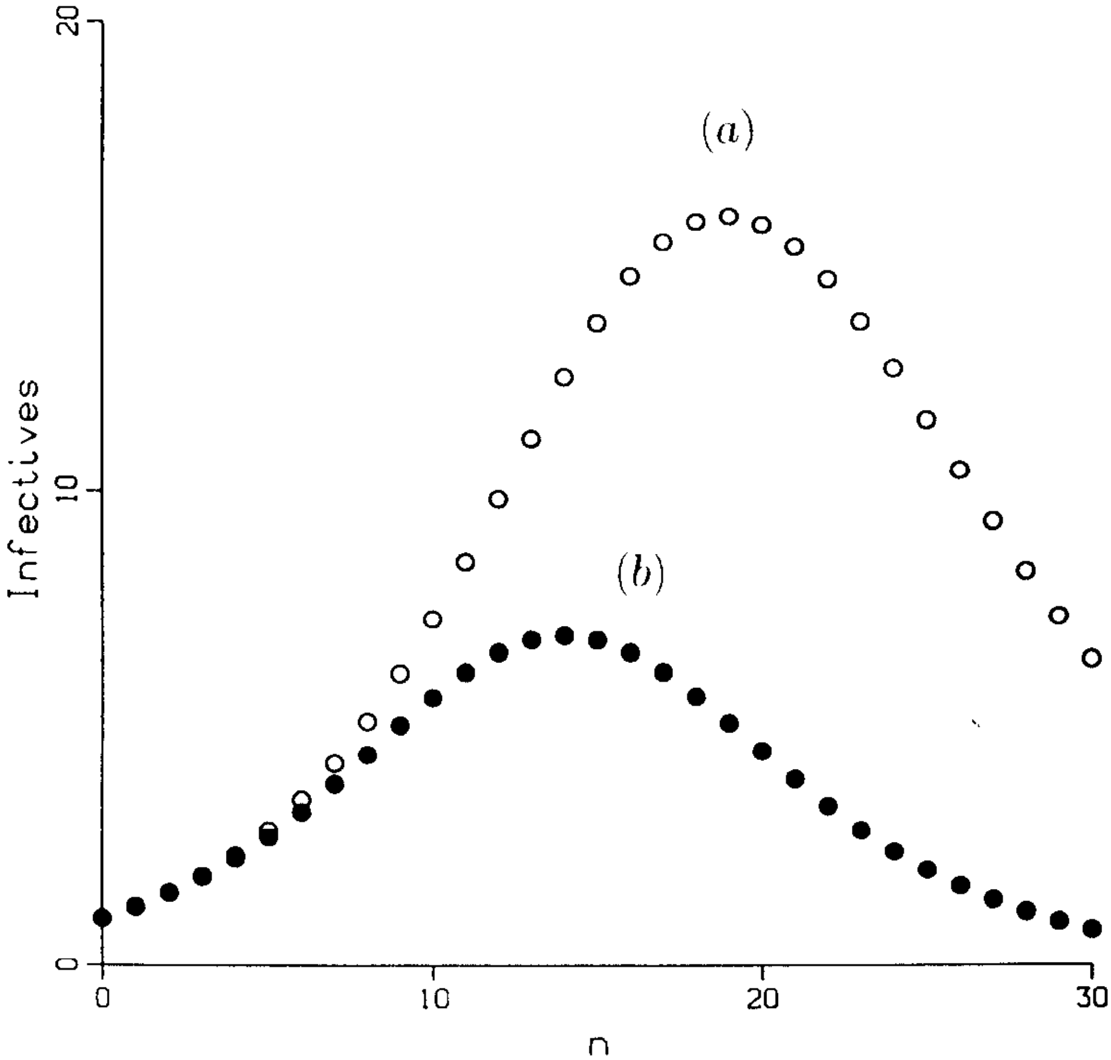
- S_∞ is the unique positive solution $0 < S_\infty < N$ satisfying

$$S_\infty \exp\left(-\frac{\alpha}{N\gamma} S_\infty\right) = S_0 \exp\left(-\frac{\alpha}{\gamma N}(N - R_0)\right)$$

- If $\mathcal{R} = S_0[1 - \exp(-\alpha \Delta t I_0/N)]/(I_0 \gamma \Delta t) > 1$, then $I_1 > I_0$.

Parameters	$\mathcal{R}$	S_∞	Number of Cases after I_0
$\alpha\Delta t = .5$, $\gamma\Delta t = .25$	1.975	19.98	79.02
$\alpha\Delta t = .75$, $\gamma\Delta t = .5$	1.479	40.64	58.36

FIG. 3. Discrete SIR Model with $I_0 = 1$, $S_0 = 99$, $N = 100$ **(a)** $\alpha\Delta t = .5$, $\gamma\Delta t = .25$, **(b)** $\alpha\Delta t = .75$, $\gamma\Delta t = .5$



In the Multipopulation Discrete SIR Model, Epidemic Behavior is Determined by Initial Conditions

Discrete I

$$
\begin{aligned}
S_{n+1}^i &= S_n^i\left(1-\sum_{k=1}^{K}\frac{\alpha_{ik}\Delta t}{N^i}I_n^k\right)\\
I_{n+1}^i &= I_n^i(1-\gamma_i\Delta t)+S_n^i\sum_{k=1}^{K}\frac{\alpha_{ik}\Delta t}{N^i}I_n^k\\
R_{n+1}^i &= R_n^i+\gamma_i\Delta t I_n^i,
\end{aligned}
$$

where $i = 1, ..., K$, $S_0^i > 0$, $I_0^i \geq 0$ ($I_0^k > 0$ for some k), $R_0^i \geq 0$, $S_0^i + I_0^i + R_0^i = N^i$. Solutions are nonnegative iff

$$\max_i\{\sum_{k=1}^{K}\alpha_{ik}\Delta t N^k/N^i, \gamma_i\Delta t\} \leq 1.$$

$$S_n^i \searrow S_\infty^i, \quad I_n^i \to 0, \quad R_n^i \nearrow R_\infty^i.$$

- $\mathcal{R}_i = \frac{S_0^i}{N^i}\frac{\alpha_{ii}}{\gamma_i} > 1, \quad I_{n+1}^i > I_n^i, \quad \text{for} \quad n\epsilon\{0, \ldots, n^*\}.$

Let $I_n = \sum_{i=1}^{K} I_n^i$ be the total number of infectives at time n.

- $\max_k\{\sum_{i=1}^{K} S_0^i\alpha_{ik}/(\gamma_k N^i)\} \leq 1, \quad I_n \searrow 0.$
- $\min_k\{\sum_{i=1}^{K} S_0^i\alpha_{ik}/(\gamma_k N^i)\} > 1, \quad I_{n+1} > I_n \quad \text{for} \quad n\epsilon\{0, \ldots, n^*\}.$

FIG. 4. Number of infectives in a two-population, discrete SIR model (population 1: ●-●-●, population 2: o-o-o), Δt=.25, $\alpha_{11} = 2.$, $\alpha_{12} = .5$, α_{21}=4., $\alpha_{22} = 2.$, $\gamma_1 = 2.$, $\gamma_2 = 1.$, $N^1 = 100$, and $N^2 = 200$. The initial conditions are $I_0^1 = 10.$, $S_0^1 = 90.$, $I_0^2 = 50.$, and $S_0^2 = 150$. Note that $\mathcal{R}_1$=.9, $\mathcal{R}_2$=1.5, and $\min_k\{\Sigma_{i=1}^2 S_0^i\alpha_{ik}/(\gamma_k N^i)\}$=1.95. There is an epidemic in both populations.

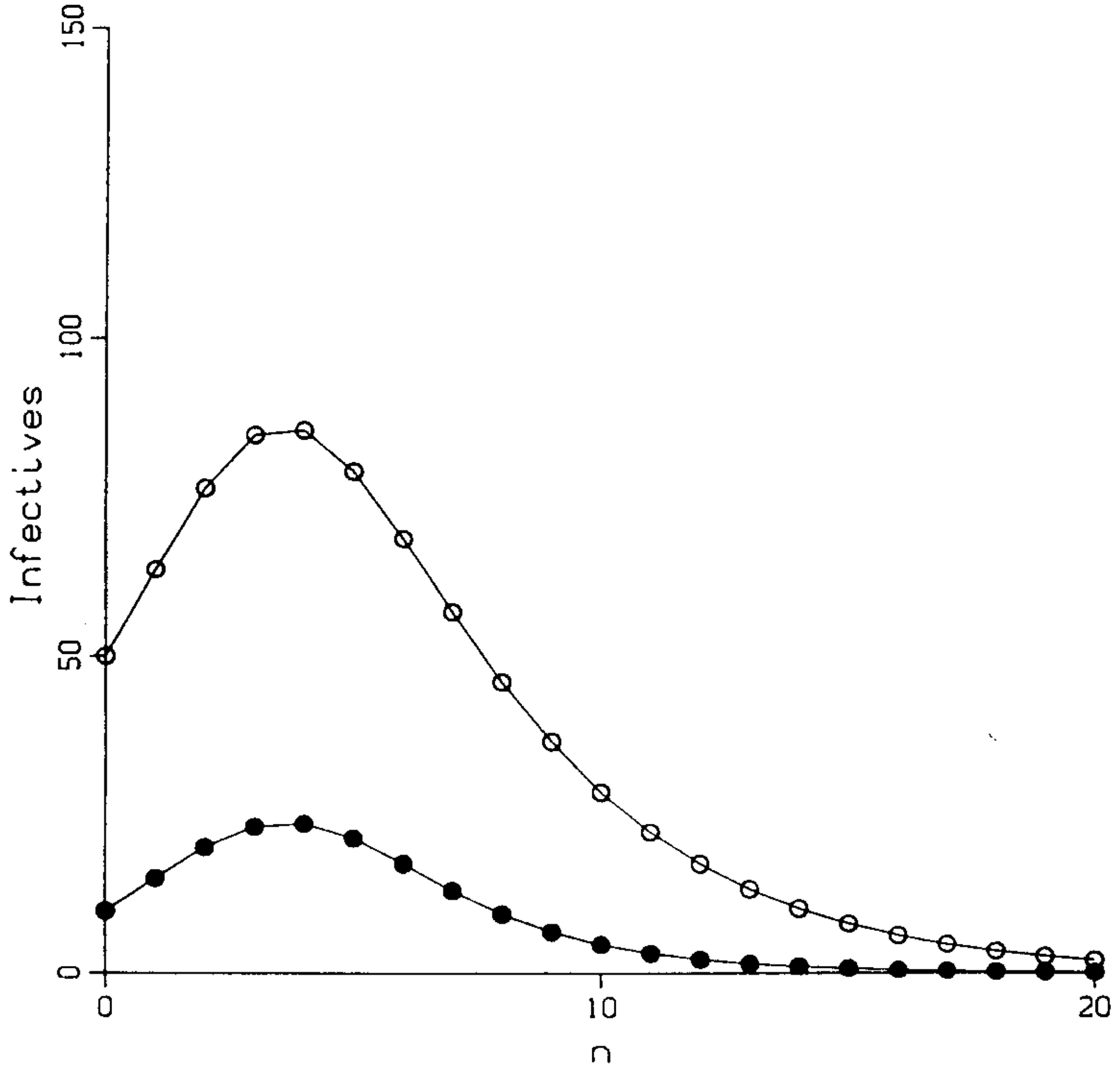


FIG. 5. Number of infectives in a two-population discrete SIR model with the same parameters as in Figure 3. The initial conditions are $I_0^1 = 10.$, $S_0^1 = 90.$, $I_0^2 = 150.$, and $S_0^2 = 50$. Note that $\mathcal{R}_1$=.9 and $\mathcal{R}_2$=.5. There is an epidemic in the first population.

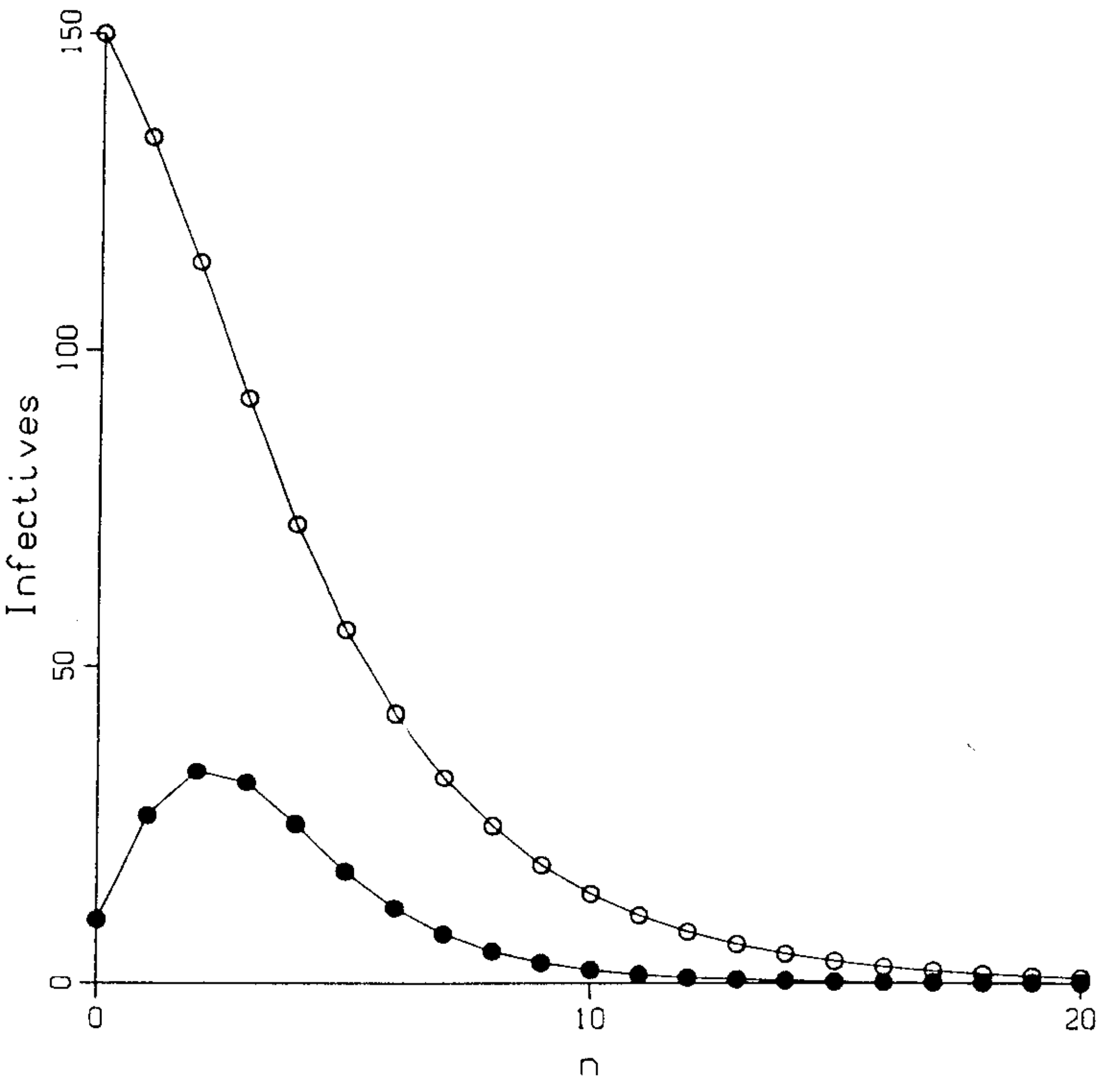
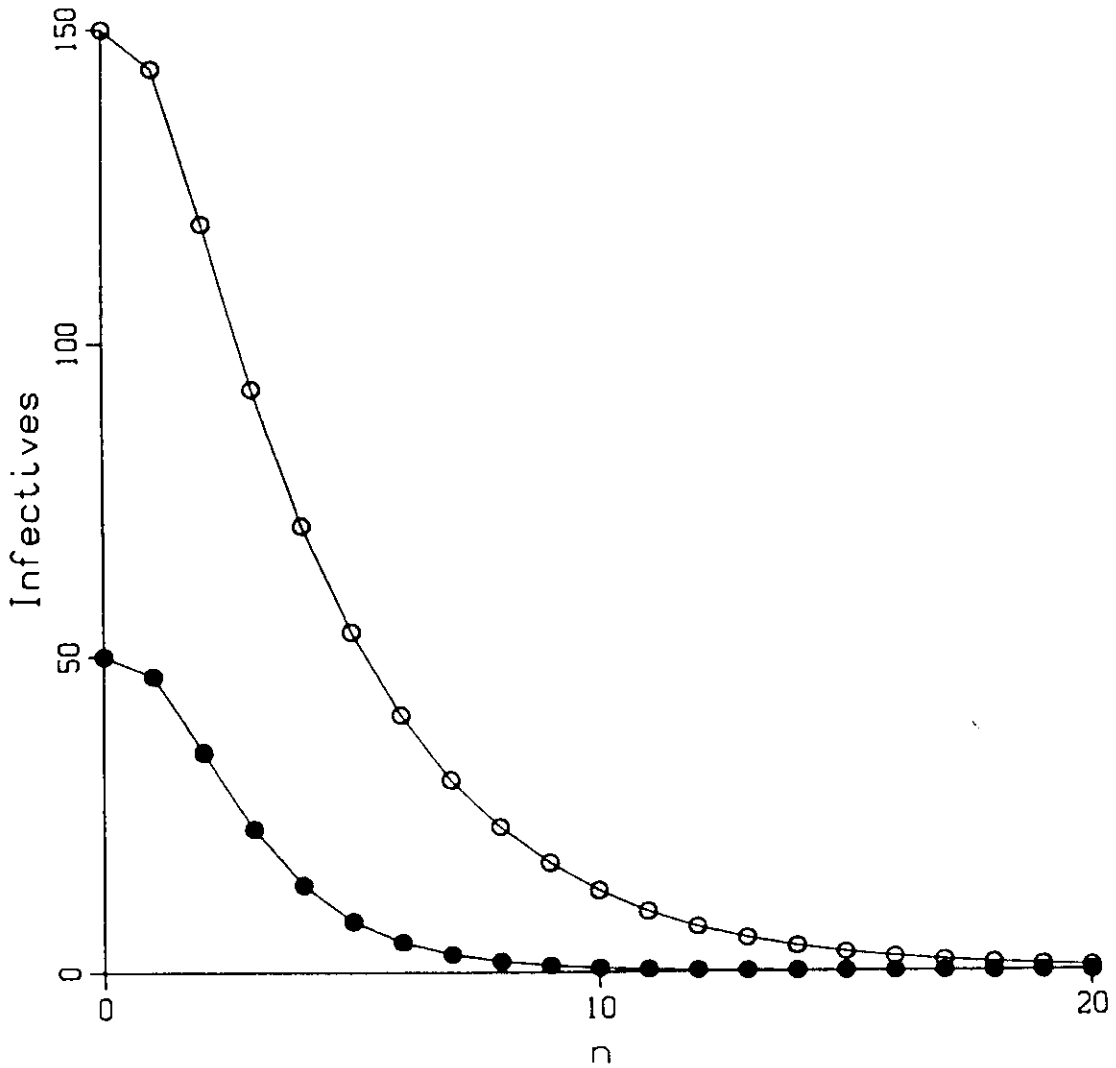


FIG. 6. Number of infectives in a two-population discrete SIR model with the same parameters as in Figure 3. The initial conditions are $I_0^1 = 50.$, $S_0^1 = 50.$, $I_0^2 = 150.$, and $S_0^2 = 50$. Note that $\mathcal{R}_1 = .5$, $\mathcal{R}_2 = .5$, and $\max_k\{\Sigma_{i=1}^2 S_0^i \alpha_{ik}/(\gamma_k N^i)\} = 1$. There is no epidemic in either population.



Discrete SIS Model Exhibits Period Doubling

Continuous

$$\begin{aligned}\dot{S} &= -\frac{\alpha}{N}SI + \gamma I\\ \dot{I} &= \frac{\alpha}{N}SI - \gamma I,\end{aligned}$$

where $S(0), I(0) > 0$ and $S(0) + I(0) = N$.

$$\mathcal{R}_0 = \frac{\alpha}{\gamma}$$

- $\mathcal{R}_0 \leq 1, \quad S \nearrow N, \quad I \searrow 0.$
- $\mathcal{R}_0 > 1, \quad S \to \frac{\gamma N}{\alpha}, \quad I \to N - \frac{\gamma N}{\alpha}.$

Discrete I

$$\begin{aligned}S_{n+1} &= S_n\left(1 - \frac{\alpha\Delta t}{N}I_n\right) + \gamma\Delta t I_n\\ I_{n+1} &= I_n\left(1 - \gamma\Delta t + \frac{\alpha\Delta t}{N}S_n\right),\end{aligned}$$

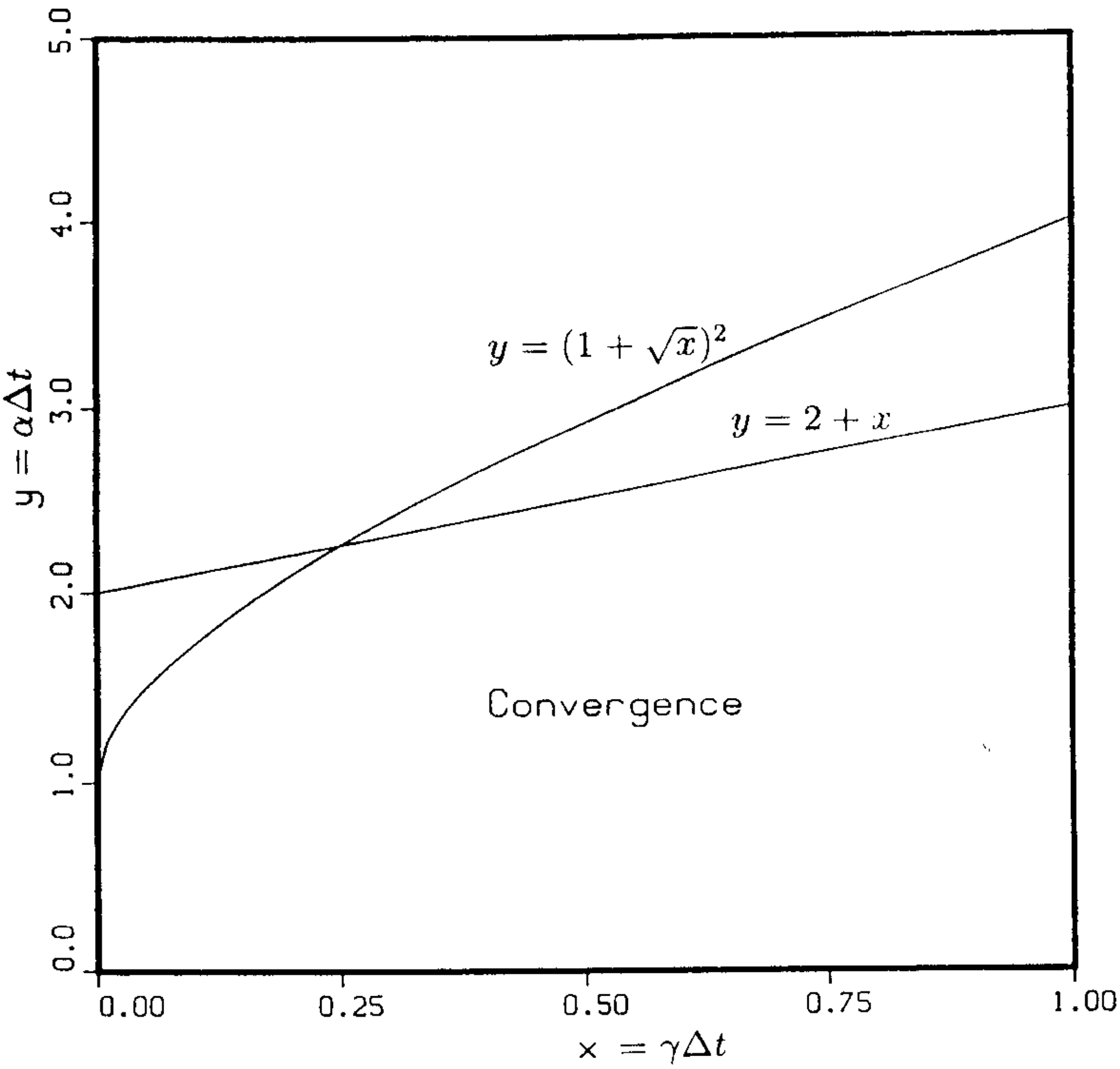
where $S_0, I_0 > 0$ and $S_0 + I_0 = N$. Solutions are positive iff

$$\gamma\Delta t \leq 1 \quad \text{and} \quad \alpha\Delta t < (1 + \sqrt{\gamma\Delta t})^2.$$

- $\mathcal{R}_0 \leq 1, \quad S_n \nearrow N, \quad I_n \searrow 0.$
- $\mathcal{R}_0 > 1$ and $\alpha\Delta t \leq 2 + \gamma\Delta t$, $S_n \to \frac{\gamma N}{\alpha}, \quad I_n \to N - \frac{\gamma N}{\alpha}.$
- $\mathcal{R}_0 > 1$ and $\alpha\Delta t > 2 + \gamma\Delta t$, Period − Doubling.

Discrete I

Fig. 7. Parameter space where solutions to the discrete SIS model are positive: $0 < \gamma\Delta t \leq 1$ and $0 < \alpha\Delta t \leq (1 + \sqrt{\gamma\Delta t})^2$. If, in addition, $\alpha\Delta t \leq 2 + \gamma\Delta t$, solutions converge to an equilibrium value.



Discrete I

FIG. 8. Number of infectives in the discrete (• • •) and continuous (o-o-o) *SIS* models, $\gamma = 2.$, $\Delta t = .5$, $N = 100.$, and I_0=1. **(a)** $\alpha = 7.$, $I^* \approx 71.4$, and $\mathcal{R}_0 = 3.5$, a four-point cycle in the discrete model corresponding to $r = 3.5$ in equation $x_{n+1} = rx_n(1 - x_n)$.

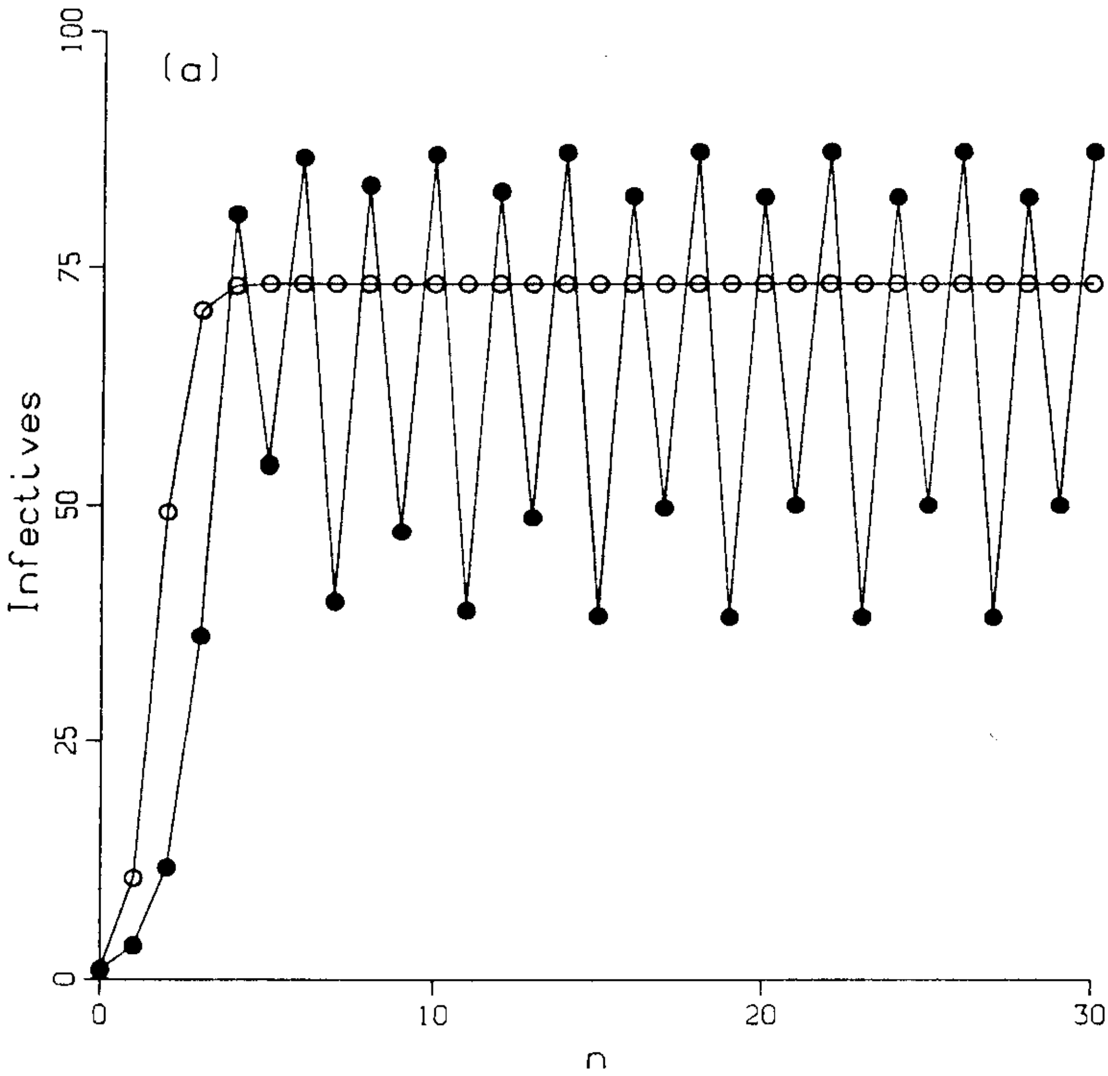
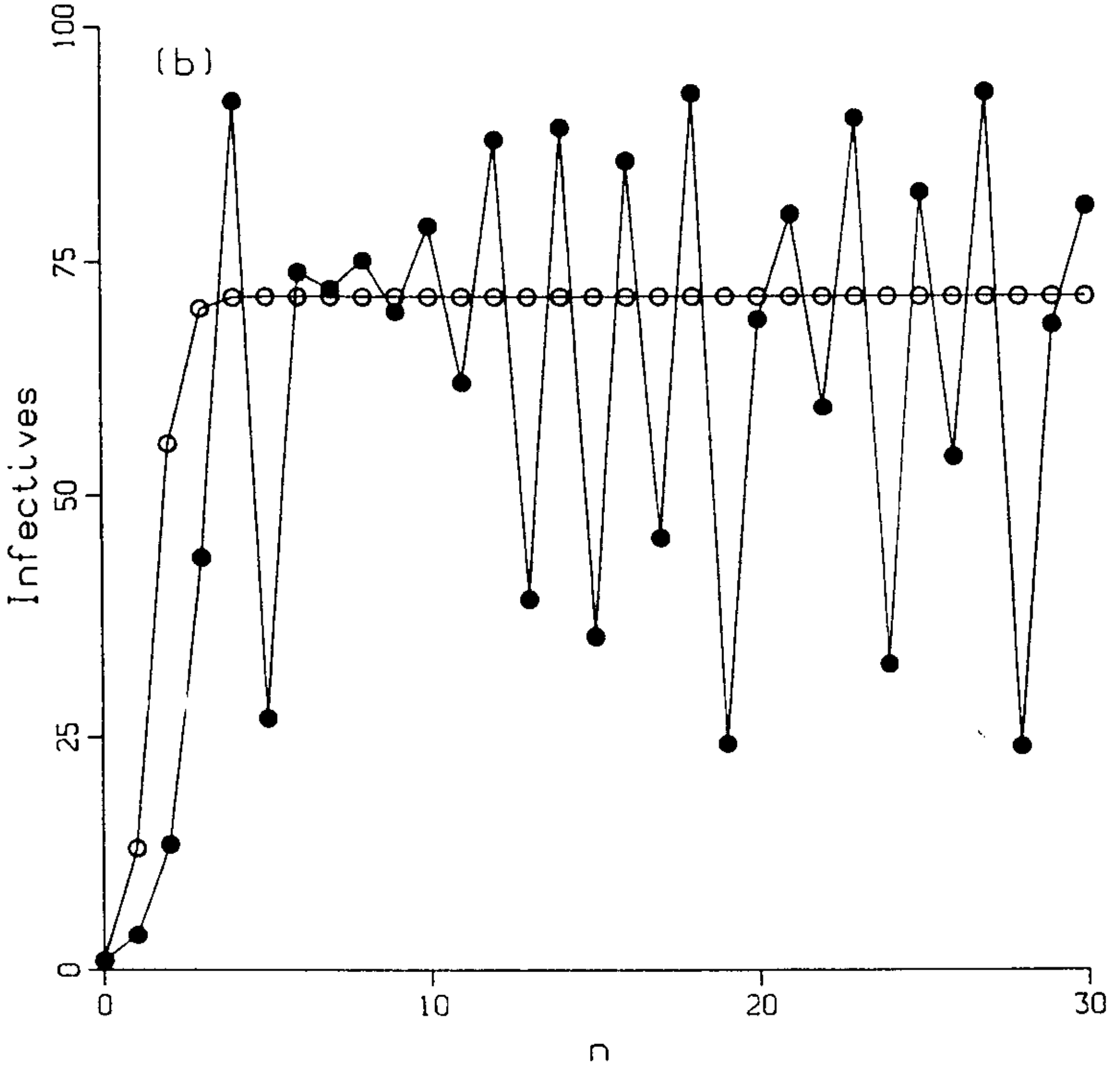


FIG. 8. **(b)** $\alpha = 7.5$, $I^* \approx 73.3$, and $\mathcal{R}_0 = 3.75$, the exact period is difficult to ascertain in the discrete model; it corresponds to $r = 3.75$ in equation $x_{n+1} = rx_n(1 - x_n)$.



Discrete SIS Model

Discrete II

$$
\begin{aligned}
S_{n+1} &= S_n \exp\left(-\frac{\alpha \Delta t}{N} I_n\right) + \gamma \Delta t I_n \\
I_{n+1} &= I_n(1 - \gamma \Delta t) + S_n \left(1 - \exp\left(-\frac{\alpha \Delta t}{N} I_n\right)\right),
\end{aligned}
$$

where $S_0, I_0 > 0$, and $S_n + I_n = N$.

$$
\mathcal{R}_0 = \frac{\alpha}{\gamma}
$$

- $\mathcal{R}_0 \leq 1$, then $S_n \to N$ and $I_n \to 0$.
- $\mathcal{R}_0 > 1$, then $S_n \to S^* > 0$ and $I_n \to I^* > 0$, where I^* satisfies

$$
\exp\left(-\frac{\alpha \Delta t}{N} I^*\right) = \frac{N - (1 + \gamma \Delta t) I^*}{N - I^*}
$$

$$
S^* = N - I^*.
$$

- Cooke, Calef, Level (1977)

Male-Female SIS Model With Only Heterosexual Contact Does Not Exhibit Period-Doubling

Let $I^1 = x$, $I^2 = y$, $N^1 = W = \#$ of Women, and $N^2 = M = \#$ of Men, then

Discrete I

$$\begin{aligned} x_{n+1} &= x_n(1-\gamma_1\Delta t) + \frac{\alpha_{12}\Delta t}{W}(W - x_n)y_n \\ y_{n+1} &= y_n(1-\gamma_2\Delta t) + \frac{\alpha_{21}\Delta t}{M}(M - y_n)x_n. \end{aligned}$$

No homosexual contacts, $\alpha_{ii} = 0$. Solutions are positive iff

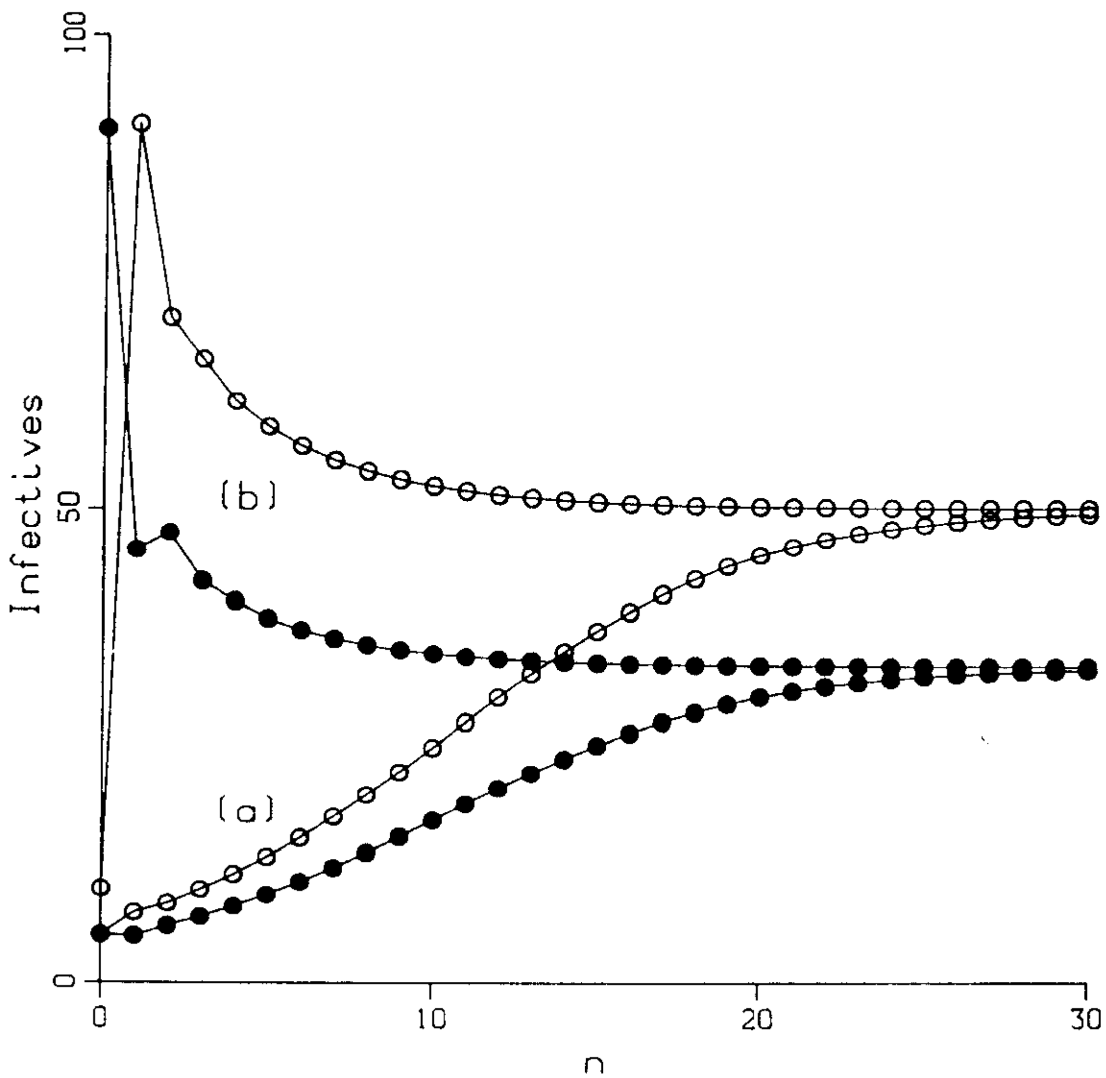
$$\max_{i,i\neq j}\{\alpha_{ij}\Delta t N^j/N^i, \gamma_i\Delta t\} \leq 1.$$

$$\mathcal{R}_0 = \frac{\alpha_{12}\alpha_{21}}{\gamma_1\gamma_2}.$$

- $\mathcal{R}_0 \leq 1$ $x_n \to 0$, $y_n \to 0$.
- $\mathcal{R}_0 > 1$ Numerical Solution $x_n \to x^*$, $y_n \to y^*$.

- Martin, Allen, Stamp (1994)

FIG. 9. Number of infectives in the two-population, discrete SIS model, females (x_n, ●-●-●), males (y_n, ○-○-○), $\Delta t = .25$ $\alpha_{12} = 2.$, $\alpha_{21} = 4.$, $\gamma_1 = 2. = \gamma_2$, $M = 200.$, $W = 100.$, $\mathcal{R}_0 = 2.$, $x^* = 33.3$, and $y^* = 50.$ **(a)** $x_0 = 5.$ and $y_0 = 5.$ **(b)** $x_0 = 90.$ and $y_0 = 10.$ (Note that $\max_{i,i \neq j}\{\alpha_{ij}\Delta t N^j/N^i, \gamma_i \Delta t\} \leq 1.$)



Discrete $SIR^1 \cdots R^m S$ Model

Discrete II

$$
\begin{aligned}
S_{n+1} &= S_n \exp\left(-\frac{\alpha\Delta t}{N} I_n\right) + \gamma_{m+1}\Delta t R_n^m \\
I_{n+1} &= I_n(1-\gamma_1\Delta t) + S_n\left(1-\exp\left(-\frac{\alpha\Delta t}{N} I_n\right)\right) \\
R_{n+1}^1 &= R_n^1(1-\gamma_2\Delta t) + \gamma_1\Delta t I_n \\
&\cdots \\
R_{n+1}^m &= R_n^m(1-\gamma_{m+1}\Delta t) + \gamma_m\Delta t R_n^{m-1},
\end{aligned}
$$

where $S_0, I_0 > 0$, $R_0^i \geq 0$, and $S_n + I_n + \Sigma_i^m R_n^i = N$.

Suppose $\gamma_i\Delta t = 1$ for $i = 1, \ldots, m$, then

$$\mathcal{R}_0 = \alpha\Delta t.$$

- If $\mathcal{R}_0 \leq 1$, then $S_n \to N$ and $I_n \to 0$.

- If $\mathcal{R}_0 > 1$, then the behavior depends on the number of removed states, m.

(i) If $m = 1$, numerical simulations indicate, $S_n \to S^* > 0$ and $I_n \to I^* > 0$.

(ii) If $m \geq 2$, the behavior depends on the magnitude of $\mathcal{R}_0$.

- Cooke, Calef, Level (1977)

Discrete SIR^1R^2S Model

Discrete II

$$
\begin{aligned}
S_{n+1} &= S_n \exp\left(-\frac{\alpha\Delta t}{N} I_n\right) + \gamma_3 \Delta t R_n^2 \\
I_{n+1} &= I_n(1-\gamma_1\Delta t) + S_n\left(1 - \exp\left(-\frac{\alpha\Delta t}{N} I_n\right)\right) \\
R_{n+1}^1 &= R_n^1(1-\gamma_2\Delta t) + \gamma_1 \Delta t I_n \\
R_{n+1}^2 &= R_n^2(1-\gamma_3\Delta t) + \gamma_2 \Delta t R_n^1.
\end{aligned}
$$

Let $x = I/N$ and $\gamma_i \Delta t = 1$, then the system above can be expressed as a third order difference equation:

$$x_{n+1} = (1 - x_n - x_{n-1} - x_{n-2})\left(1 - \exp\left(-\alpha\Delta t x_n\right)\right).$$

$$\mathcal{R}_0 = \alpha\Delta t$$

- Cooke, Calef, Level (1977)

FIG. 10. An SIR^1R^2S model. Initial conditions $x_0 = .1$, $x_1 = .1$, $x_2 = .1$, and $\alpha\Delta t = 4$. An equilibrium is approached $\lim_{n\to\infty} x_n = 0.2101$.

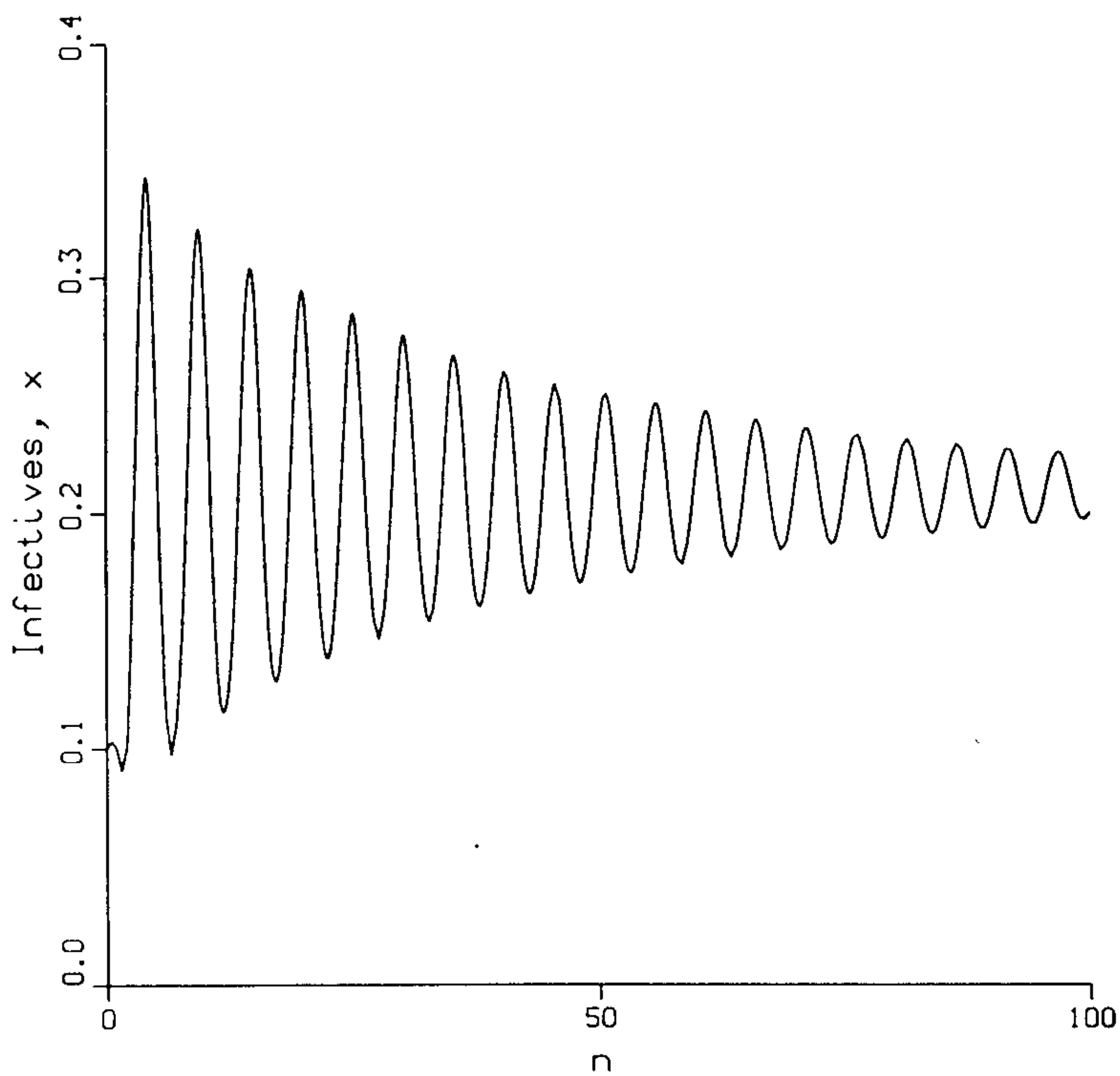


FIG. 11. Quasiperiodic behavior, $\alpha\Delta t = 5$.

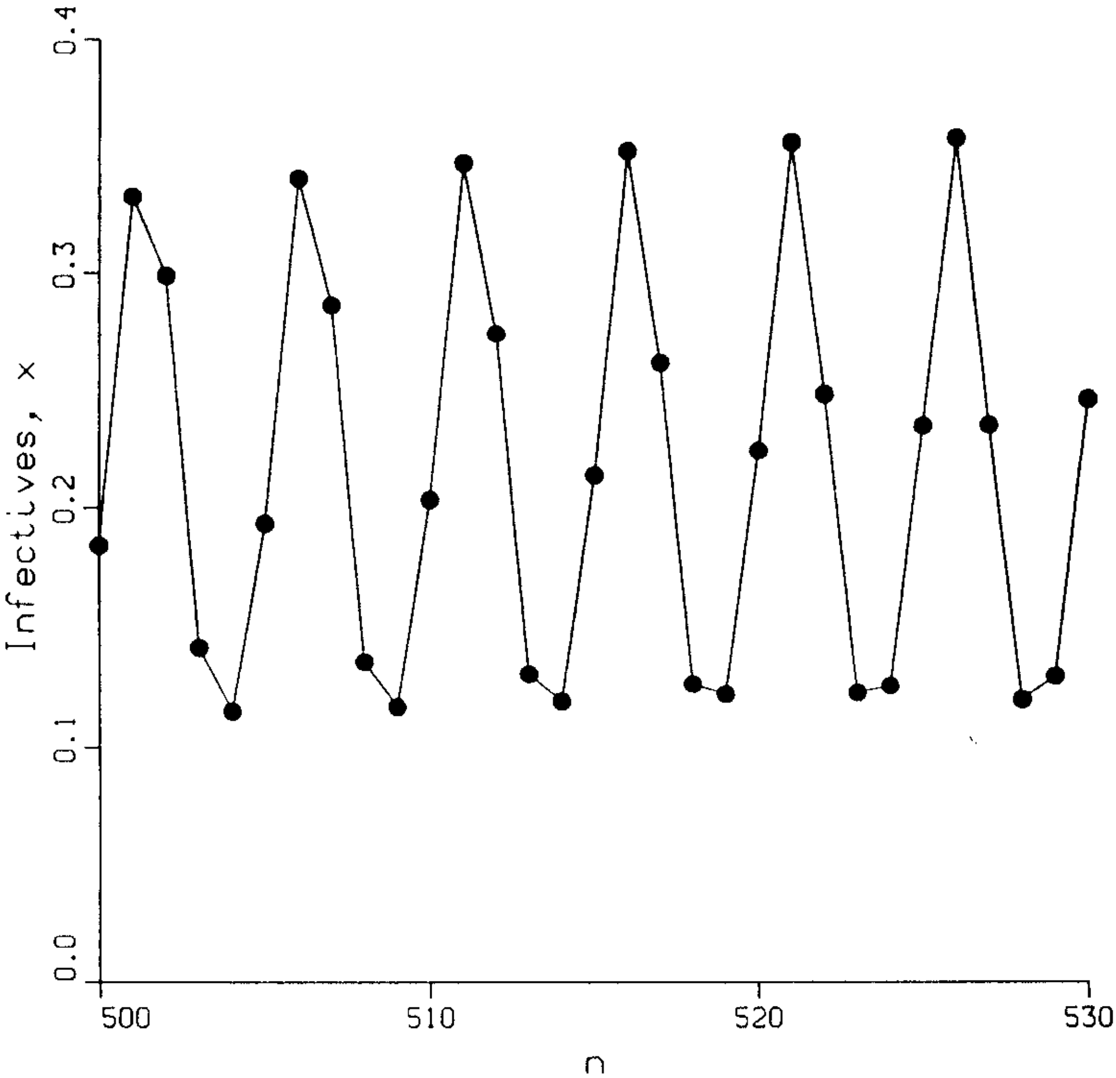
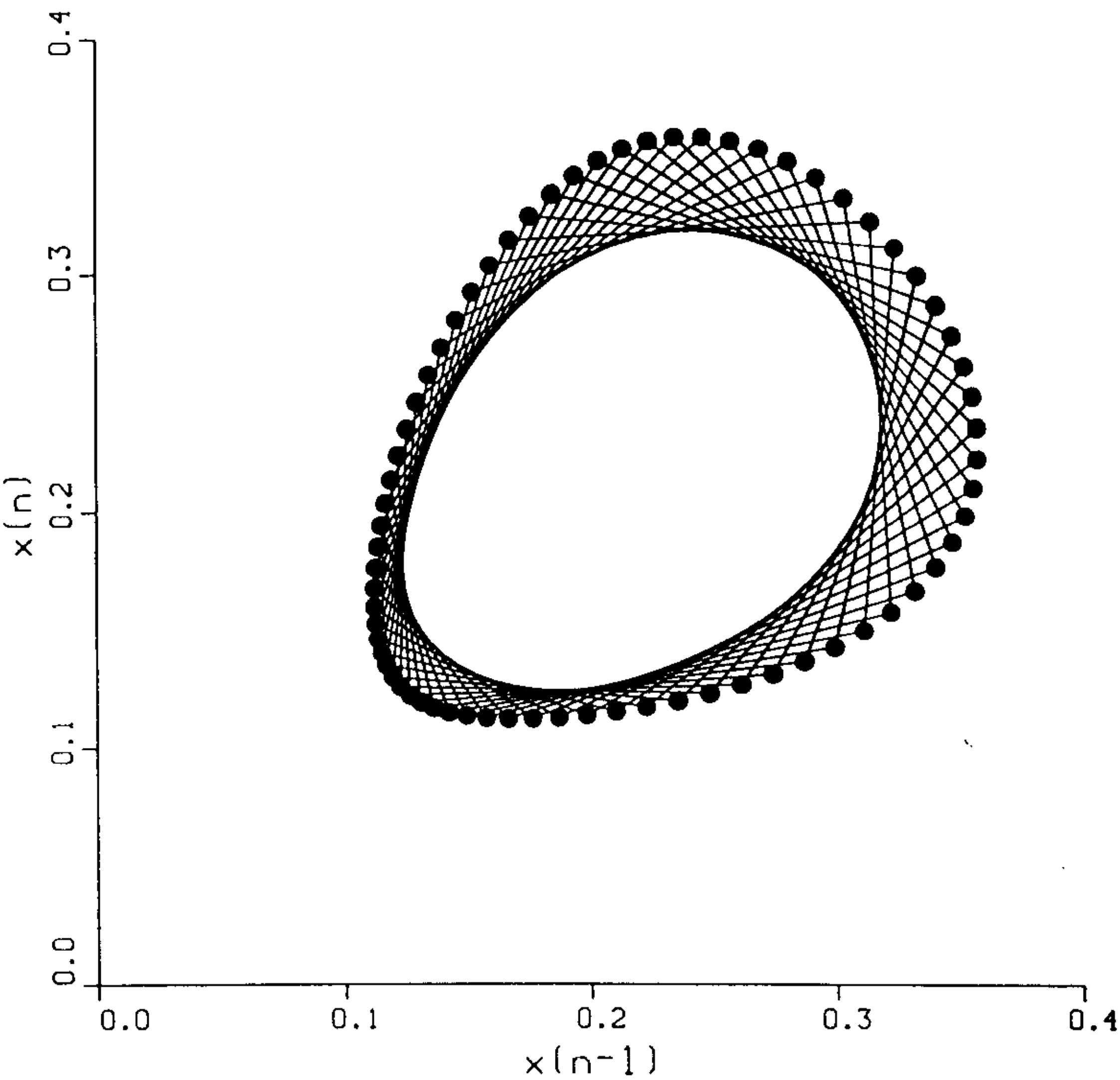


FIG. 12. Quasiperiodic behavior, $\alpha\Delta t = 5$. Solutions (x_{n-1}, x_n) are graphed for 100 points.



VI

Discrete SI and SIS Models With Births and Deaths Exhibit Period-Doubling

Discrete I

$$
\begin{aligned}
S_{n+1} &= S_n\left(1 - \frac{\alpha\Delta t}{N} I_n\right) + \gamma\Delta t I_n + \beta\Delta t(N - S_n) \\
I_{n+1} &= I_n\left(1 - \gamma\Delta t - \beta\Delta t + \frac{\alpha\Delta t}{N} S_n\right),
\end{aligned}
$$

where $S_0 > 0, I_0 > 0$, and $S_0 + I_0 = N$. If $\gamma = 0$, then SI model. Solutions are positive iff

$$(\gamma + \beta)\Delta t \leq 1 \quad \text{and} \quad \alpha\Delta t < (1 + \sqrt{(\beta + \gamma)\Delta t})^2.$$

$$\mathcal{R}_0 = \frac{\alpha}{\gamma + \beta}.$$

- $\mathcal{R}_0 \leq 1, \quad S_n \to N \quad I_n \to 0.$
- $\mathcal{R}_0 > 1$ and $\alpha\Delta t \leq 2 + \gamma\Delta t + \beta\Delta t, \quad S_n \to \frac{(\gamma+\beta)N}{\alpha}, \quad I_n \to N - \frac{(\gamma+\beta)N}{\alpha}.$
- $\mathcal{R}_0 > 1$ and $\alpha\Delta t > 2 + \gamma\Delta t + \beta\Delta t$, Period − Doubling.

Discrete SIS Model with Births and Deaths

Discrete II

$$
\begin{aligned}
S_{n+1} &= S_n \exp\left(-\frac{\alpha \Delta t}{N} I_n\right) + \beta \Delta t (N - S_n) + \gamma \Delta t I_n \\
I_{n+1} &= I_n (1 - \gamma \Delta t - \beta \Delta t) + S_n \left(1 - \exp\left(-\frac{\alpha \Delta t}{N} I_n\right)\right),
\end{aligned}
$$

where $S_0, I_0 > 0$, and $S_n + I_n = N$.

$$
\mathcal{R}_0 = \frac{\alpha}{\gamma + \beta}
$$

- $\mathcal{R}_0 \leq 1$, then $S_n \to N$ and $I_n \to 0$.
- $\mathcal{R}_0 > 1$, then $S_n \to S^* > 0$ and $I_n \to I^* > 0$, where I^* satisfies

$$
\exp\left(-\frac{\alpha \Delta t}{N} I^*\right) = \frac{N - (1 + \gamma \Delta t + \beta \Delta t) I^*}{N - I^*}
$$

$$
S^* = N - I^*.
$$

- Cooke, Calef, Level (1977)

Discrete SIR Model With Births and Deaths Exhibits Periodic Behavior

Discrete I

$$
\begin{aligned}
S_{n+1} &= S_n\left(1-\frac{\alpha\Delta t}{N}I_n\right)+\beta\Delta t(N-S_n)\\
I_{n+1} &= I_n\left(1-\gamma\Delta t-\beta\Delta t+\frac{\alpha\Delta t}{N}S_n\right)\\
R_{n+1} &= R_n(1-\beta\Delta t)+\gamma\Delta t I_n,
\end{aligned}
$$

where $S_0, I_0 > 0$, $R_0 \geq 0$, and $S_0 + I_0 + R_0 = N$. Solutions are nonnegative iff

$$(\gamma+\beta)\Delta t \leq 1 \quad \text{and} \quad \alpha\Delta t \leq (1+\sqrt{\beta\Delta t})^2.$$

$$\mathcal{R}_0 = \frac{\alpha}{\gamma+\beta}.$$

- $\mathcal{R}_0 \leq 1$, $S_n \to S_\infty$, $I_n \to 0$, $R_n \to R_\infty$.
- $\mathcal{R}_0 > 1$ and α, β sufficiently small, $S_n \to \frac{(\gamma+\beta)N}{\alpha}$, $I_n \to \frac{\beta N(\alpha-\gamma-\beta)}{\alpha(\gamma+\beta)}$, $R_n \to \frac{\gamma N(\alpha-\gamma-\beta)}{\alpha(\gamma+\beta)}$.
- $\mathcal{R}_0 > 1$ and α, β sufficiently large, Periodic Behavior.

FIG. 13. Number of infectives in the discrete SIR model with births and deaths, $\gamma = .1$, $\beta = 1.9$, $\Delta t = .5$, $N = 100.$, $S_0 = 99.$, $I_0 = 1.$, and $R_0 = 0$. **(a)** $\alpha = 7.$, $I^* \approx 67.9$, and $\mathcal{R}_0 = 3.5$, a two-point cycle.

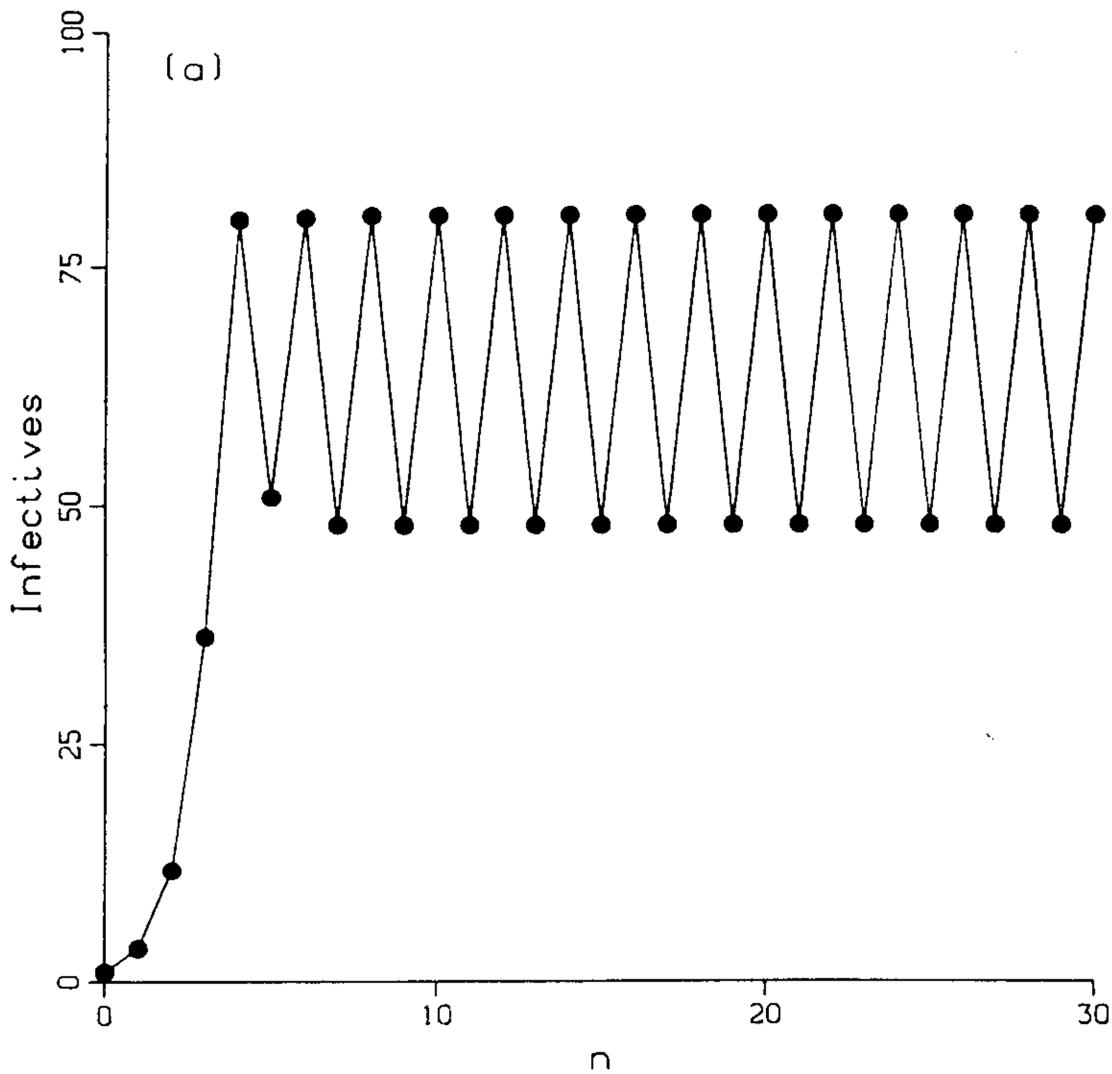


FIG. 13. **(b)** $\alpha = 7.7$, $I^* \approx 70.3$, and $\mathcal{R}_0 = 3.85$, the exact period is difficult to ascertain.

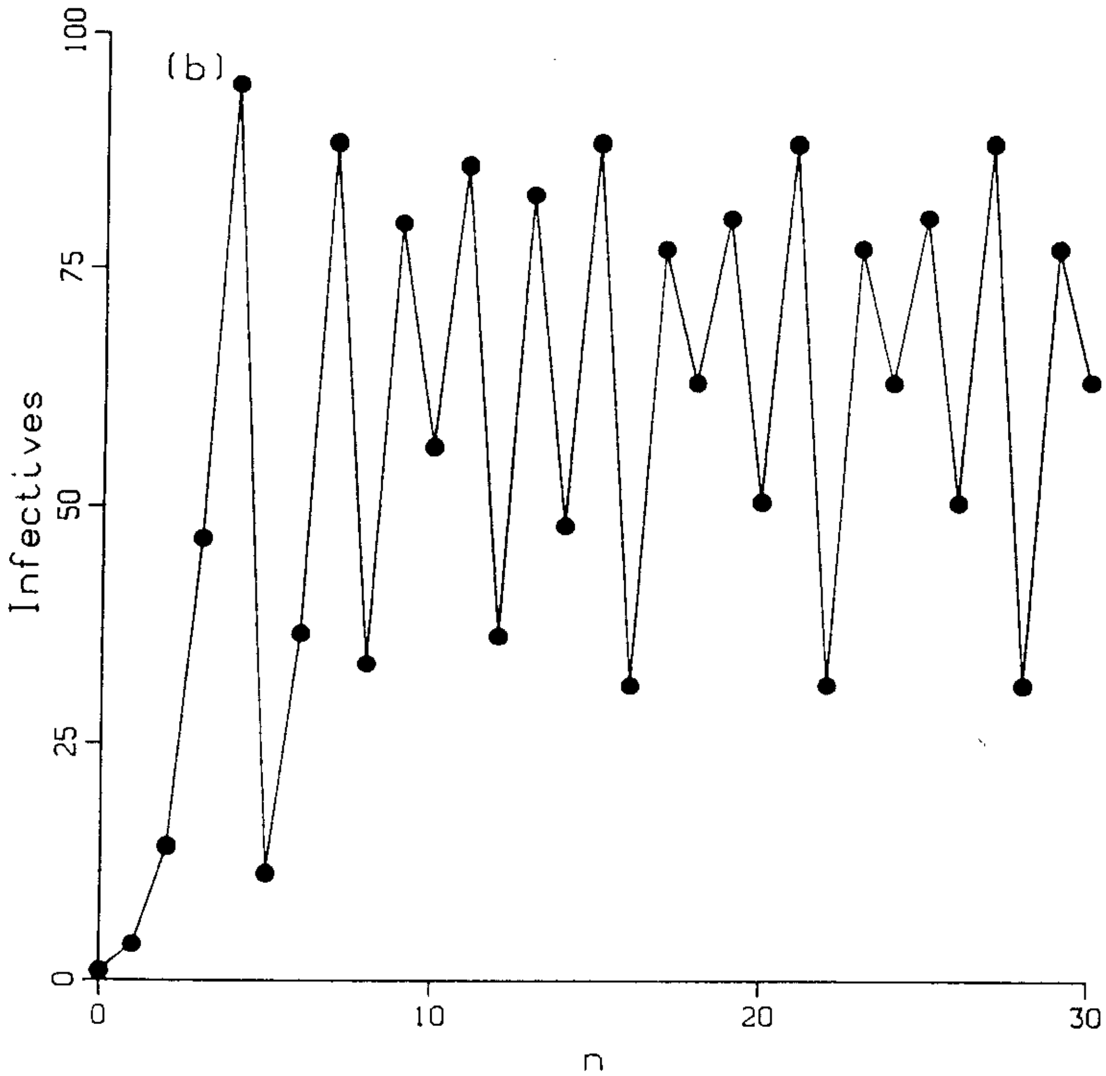
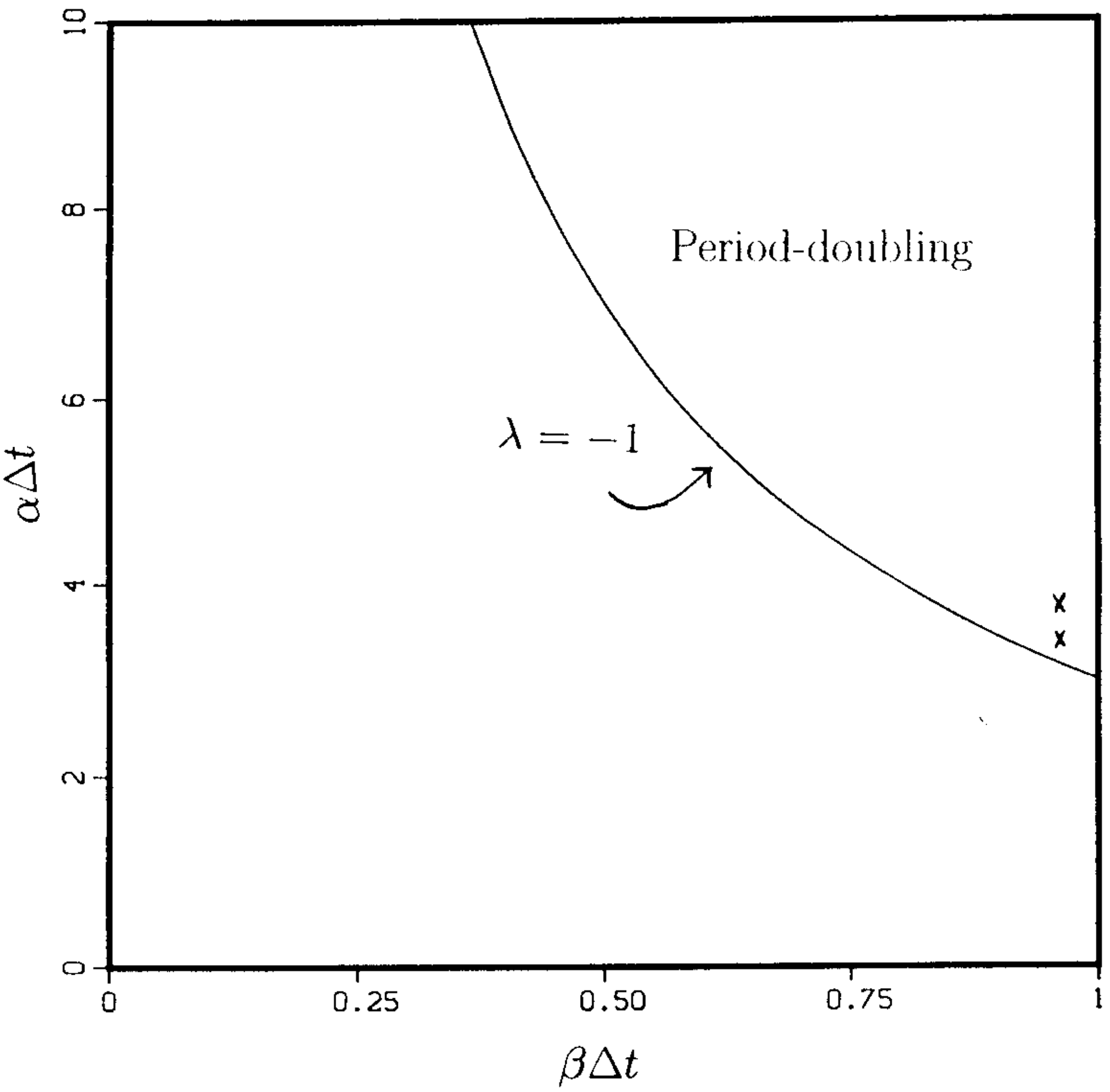


FIG. 14. *SIR* Model with births and deaths exhibits period-doubling behavior if $\alpha\Delta t > [4/\beta\Delta t - (\gamma+\beta)\Delta t][(\gamma+\beta)\Delta t/(2-[\gamma+\beta]\Delta t)]$. In this figure, $(\gamma+\beta)\Delta t = 1$. Thus, period-doubling behavior occurs if

$$\alpha\Delta t > \frac{4}{\beta\Delta t} - 1.$$



Final Remarks

- Discrete SI, SIR Models with no births and deaths, i.e., with no positive feedback, solutions converge to either a disease-free or endemic equilibrium.

- Discrete Formulation I: SIS with or without births and deaths and SI or SIR with births and deaths, i.e., with positive feedback, exhibit periodic behavior.

- Discrete Formulation II: Behaves most similarly to its continuous analogue.

- If $\Delta t \to 0$, then Discrete Formulation I $\to$ Continuous.