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Second Workshop on
Three-Dimensional Modelling of Seismic Waves
Generation, Propagation and their Inversion

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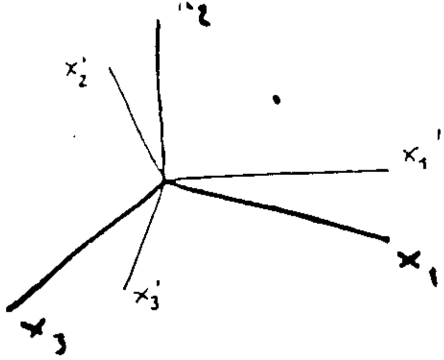
Fundamentals of Dynamic Elasticity

F.J. Sánchez-Sesma

Universidad Nacional Autonoma de Mexico
Instituto de Ingenieria
Coyoacan, Mexico

MAIN BUILDING STRADA COSTIERA, 11 TEL. 22401 TELEFAX 224163 TELEX 460392 ADRIATICO GUEST HOUSE VIA GRIGNANO, 9 TEL. 224241 TELEFAX 224531 TELEX 460449
MICROPROCESSOR LAB. VIA BEIRUT, 31 TEL. 224471 TELEFAX 224600 TELEX 460392 GALILEO GUEST HOUSE VIA BEIRUT, 7 TEL. 22401 TELEFAX 2240310 TELEX 460392



$$x_i' = \beta_{ij}\, x_j$$

$$\beta_{ij} = \cos(x_i', x_j)$$

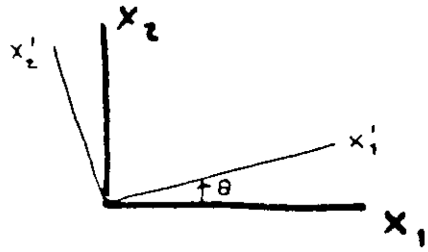
$$x_i = \beta_{ji}\, x_j'$$

$$u_i' = \beta_{ij}\, u_j$$

$$\sigma_{ij}' = \beta_{ik}\beta_{jl}\sigma_{kl}$$

$$a'_{ijk} = \beta_{il}\beta_{jm}\beta_{kn}a_{lmn}$$

$$\vdots$$



$$\beta_{11} = \cos\theta$$
$$\beta_{12} = \sin\theta$$
$$\beta_{21} = -\sin\theta$$
$$\beta_{22} = \cos\theta$$

Delta de Kronecker

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

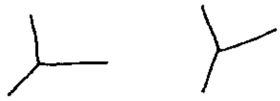
$$e_{ijk} = \begin{cases} 1 & \begin{matrix} i \\ j \quad k \end{matrix} \qquad \begin{matrix} 1 \\ 2 \quad 3 \end{matrix} \\ 0 & i = j,\ 0 \quad i = k,\ 0 \quad j = k \\ & 0 \quad i = j = k \\ -1 & \begin{matrix} i \\ j \quad k \end{matrix} \qquad \begin{matrix} 1 \\ 3 \quad 2 \end{matrix} \end{cases}$$

$$\delta_{ij} = \beta_{li}\beta_{lj} = \beta_{ik}\beta_{jk}$$

$$u_i = \delta_{ij} u_j = \delta_{i1} u_1 + \delta_{i2} u_2 + \delta_{i3} u_3$$

$$(\bar{u} \times \bar{v})_i = w_i = e_{ijk}\, u_j v_k$$

isotropie $\quad a_{ij} \to a'_{kl} \qquad a'_{kl} = a_{kl}$



ordre 2

$$a_{ij} = c\,\delta_{ij}$$

$$a'_{kl} = \beta_{ki}\beta_{lj}\,a_{ij} = c\,\beta_{ki}\beta_{lj}\,\delta_{ij} = c\,\beta_{ki}\beta_{li} = c\,\delta_{kl} = a_{kl}$$

ordre 3

$$b_{ijk} = c\,e_{ijk}$$

ordre 4

$$c_{ijkl} = \lambda\,\delta_{ij}\,\delta_{kl} + \mu\left(\delta_{ik}\,\delta_{jl} + \delta_{il}\,\delta_{jk}\right)$$

$$\operatorname{grad} f = \nabla f = \frac{\partial f}{\partial x_1}\hat{e}_1 + \frac{\partial f}{\partial x_2}\hat{e}_2 + \frac{\partial f}{\partial x_3}\hat{e}_3 \qquad \nabla = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right)$$

$$(\operatorname{grad} f)_i = \frac{\partial f}{\partial x_i} = f_{,i}$$

$$\frac{\partial u_i}{\partial x_j} = u_{i,j} \quad \longleftarrow \quad \operatorname{grad} \bar{u} = \nabla \bar{u}$$

$$\frac{\partial u'_i}{\partial x'_j} = \frac{\partial(\beta_{il} u_l)}{\partial x'_j} = \beta_{il}\frac{\partial u_l}{\partial x'_j} = \beta_{il}\frac{\partial u_l}{\partial x_k}\frac{\partial x_k}{\partial x'_j} = \beta_{il}\beta_{jk}\frac{\partial u_l}{\partial x_k}$$

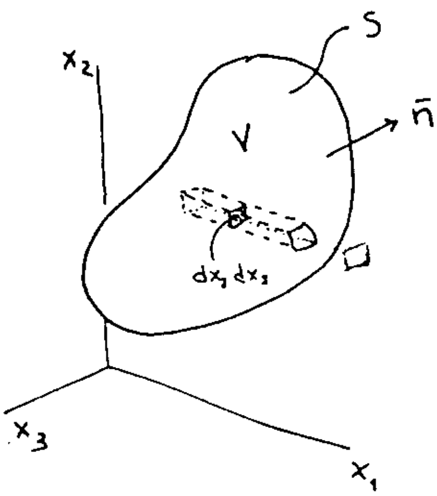
$$\frac{\partial u_i}{\partial x_i} = u_{i,i} \quad \longleftrightarrow \quad \operatorname{div} \bar{u} = \nabla \cdot \bar{u}$$

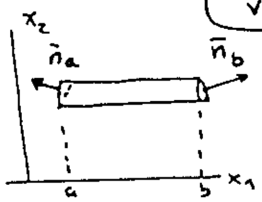
$$e_{ijk} u_{k,j} \quad \longleftrightarrow \quad \operatorname{rot} \bar{u} = \nabla \times \bar{u}$$

$$f_{,ii} \quad \longleftrightarrow \quad \operatorname{div} \operatorname{grad} f = \nabla \cdot \nabla f = \nabla^2 f$$

$$u_{i,jj} \quad \longleftrightarrow \quad \nabla^2 \bar{u}$$

Théorème de Gauss $\left\{ \begin{array}{l} \text{de Green ou} \\ \text{de la divergence} \end{array} \right.$ $\boxed{\int_V \frac{\partial f}{\partial x_i}\, dV = \int_S n_i f \, dS}$





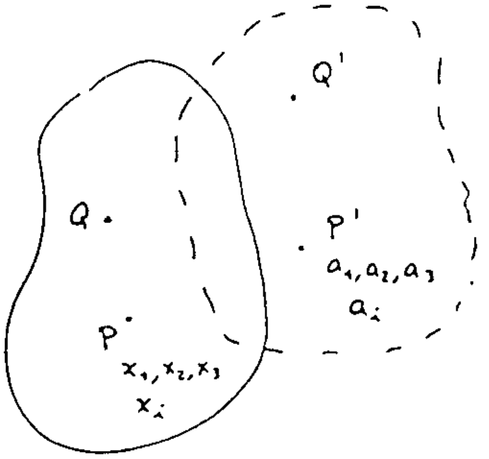
$$\int_a^b dx_2 dx_3 \frac{\partial f}{\partial x_1} dx_1 =$$

$$= dx_2 dx_3 \left[f(b) - f(a) \right] =$$

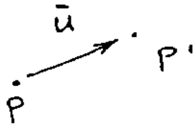
$$\boxed{\begin{aligned} dx_2 dx_3 &= dS(b)\, n_1(b) \\ &= -dS(a)\, n_1(a) \end{aligned}}$$

$$= (n_1 f \, dS)_b + (n_1 f \, dS)_a$$

$$\Rightarrow \int_V \frac{\partial f}{\partial x_1}\, dV = \int_S n_1 f \, dS$$



$$\underbrace{P(x_1, x_2, x_3)}_{x_i} \qquad \underbrace{P'(a_1, a_2, a_3)}_{a_i}$$

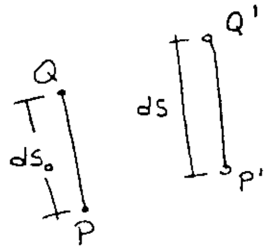


$$\boxed{u_i = a_i - x_i}$$

$$a_i = a_i(x_1, x_2, x_3)$$

$Q \leftrightarrow x_i + dx_i$

$Q' \leftrightarrow a_i + da_i$



$$(ds_0)^2 = dx_i\, dx_i$$

$$(ds)^2 = da_i\, da_i$$

$$(ds)^2 - (ds_0)^2 = da_i\, da_i - dx_i\, dx_i$$

$$a_i = x_i + u_i$$

$$da_i = \frac{\partial a_i}{\partial x_j} dx_j =$$

$$= \left(\delta_{ij} + \frac{\partial u_i}{\partial x_j}\right) dx_j$$

$$(ds)^2 - (ds_0)^2 = \underbrace{\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_j}\right)}_{2E_{ij}} dx_i\, dx_j$$

$$E_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_j}\right)$$

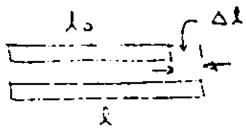
Tensor de deformación de Green

$$\boxed{e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)}$$

Tensor de deformación infinitesimal de Cauchy

$u_1 = u_1(x_1) \quad u_2 = u_3 = 0$

$$E_{11} = \frac{\partial u_1}{\partial x_1} + \frac{1}{2}\left(\frac{\partial u_1}{\partial x_1}\right)^2$$



$\Delta l = l - l_0$

$$l^2 - l_0^2 = (l - l_0)(l + l_0) =$$

$$= \frac{\Delta l}{l_0}\, (2 l_0 + \Delta l)\, l_0$$

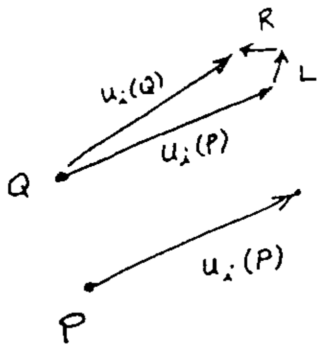
$$l^2 - l_0^2 = 2\underbrace{\left[\frac{\Delta l}{l_0} + \frac{1}{2}\left(\frac{\Delta l}{l_0}\right)^2\right]}_{E_{11}} l_0^2$$

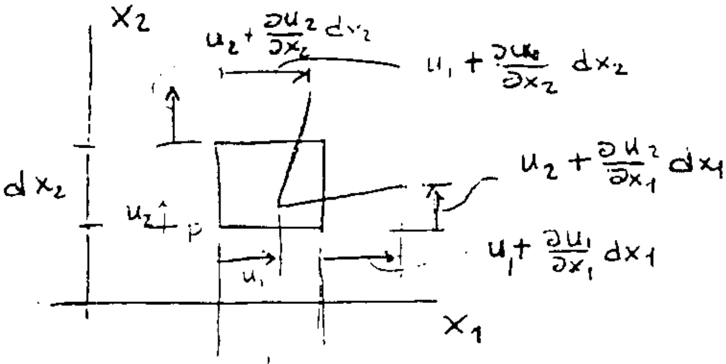
$$u_i(Q) = u_i(x_1 + dx_1,\ x_2 + dx_2,\ x_3 + dx_3)$$

$$u_i(P) = u_i(x_1,\ x_2,\ x_3)$$

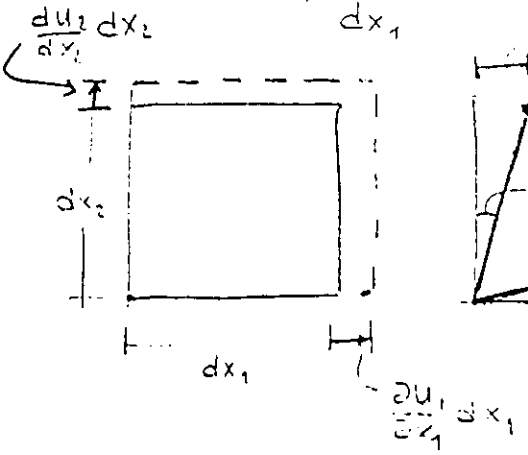
$$u_i(Q) = u_i(P) + \left.\frac{\partial u_i}{\partial x_j}\right|_P dx_j + \dots$$

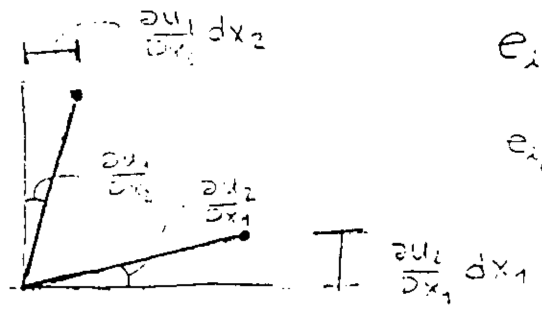
$$u_i(Q) = u_i(P) + \underbrace{\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)_P}_{e_{ij}} dx_j + \underbrace{\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)_P}_{\omega_{ij}} dx_j$$





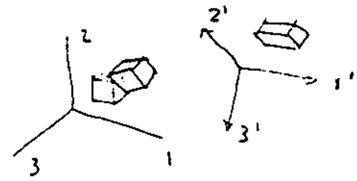
$$e_{ij}=\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)$$





$$e_{ij}\,n_j=\varepsilon_i$$

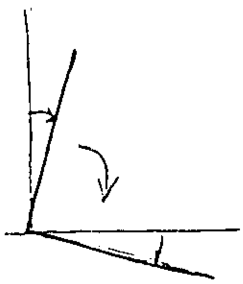
$$e_{ij}\,n_j=\varepsilon\,n_i$$



$$e_{11}=\frac{\partial u_1}{\partial x_1}\qquad\qquad e_{12}=\frac{1}{2}\left(\frac{\partial u_1}{\partial x_2}+\frac{\partial u_2}{\partial x_1}\right)$$

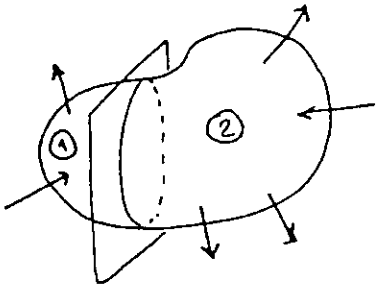
. Soit par example

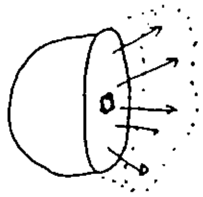
$$\frac{\partial u_2}{\partial x_1}=-\frac{\partial u_1}{\partial x_2}\qquad \frac{\partial u_1}{\partial x_2}>0\qquad e_{12}=0$$

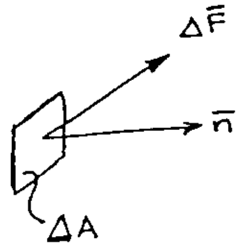


$$\omega_{12}=\frac{1}{2}\left(\frac{\partial u_1}{\partial x_2}-\frac{\partial u_2}{\partial x_1}\right)$$
$$=\frac{\partial u_1}{\partial x_2}$$

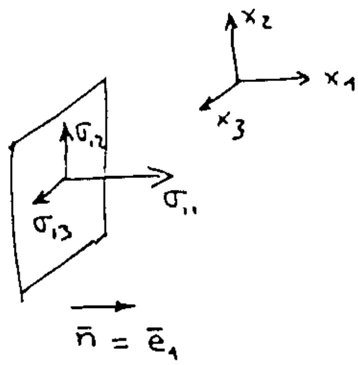
$$\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}-\frac{\partial u_j}{\partial x_i}\right)dx_j=\omega_{ij}\,dx_j=-\frac{1}{2}(\mathrm{rot}\,\bar{u})\times d\bar{x}$$



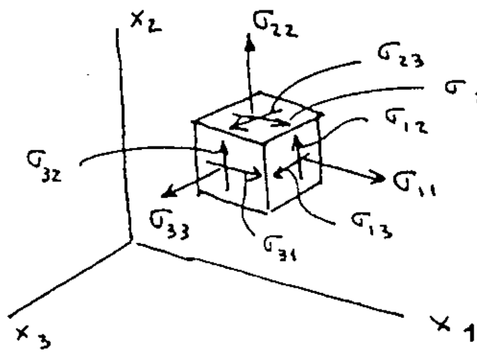


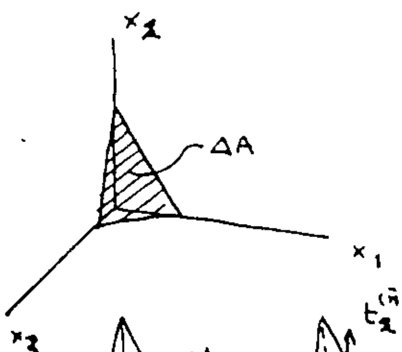


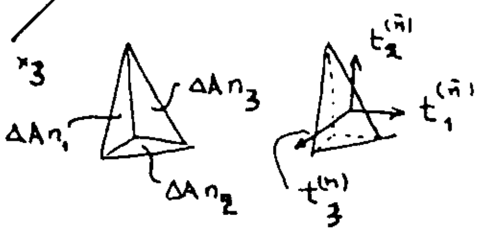
$$t_i^{(\bar n)} = \lim_{\Delta A \to 0} \frac{\Delta F_i}{\Delta A}$$



$$t_1^{(\bar e_1)} = \sigma_{11}$$
$$t_2^{(\bar e_1)} = \sigma_{12}$$
$$t_3^{(\bar e_1)} = \sigma_{13}$$



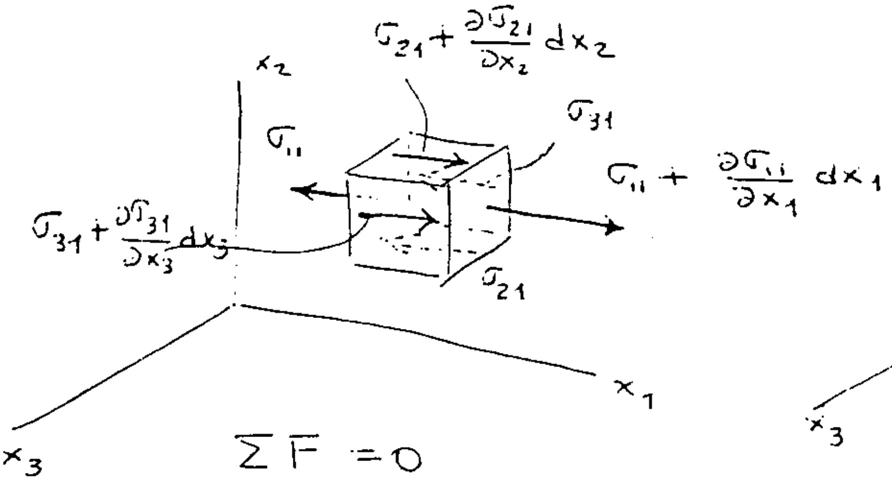




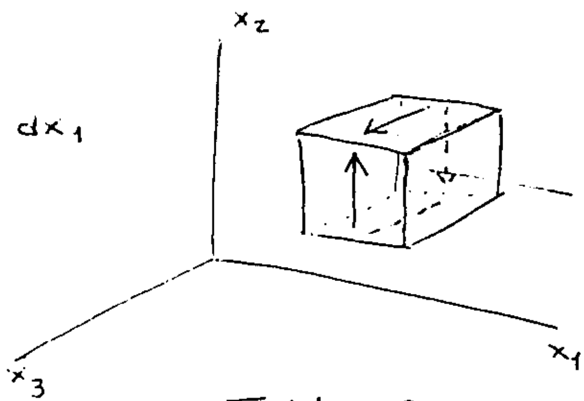
$$t_1^{(\bar n)} \Delta A = \sigma_{11} \Delta A n_1 + \sigma_{21} \Delta A n_2 + \sigma_{31} \Delta A n_3$$

$$\Rightarrow \quad t_1^{(\bar n)} = \sigma_{k1} n_k$$

$$\Rightarrow \quad \boxed{t_i^{(\bar n)} = \sigma_{ki} n_k}$$ Cauchy



$\Sigma F = 0$



$\Sigma M = 0$

$$\left(\sigma_{11} + \frac{\partial \sigma_{11}}{\partial x_1} dx_1\right) dx_2 dx_3 - \sigma_{11}\, dx_2 dx_3$$

$$+ \left(\sigma_{21} + \frac{\partial \sigma_{21}}{\partial x_2} dx_2\right) dx_1 dx_3 - \sigma_{21}\, dx_1 dx_3$$

$$+ \left(\sigma_{31} + \frac{\partial \sigma_{31}}{\partial x_3} dx_3\right) dx_1 dx_2 - \sigma_{31}\, dx_1 dx_2$$

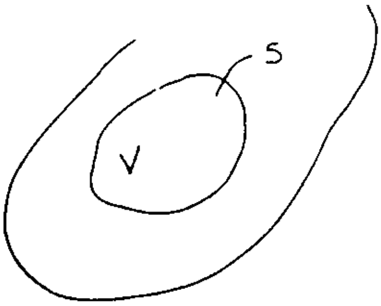
$$+ f_1\, dx_1 dx_2 dx_3 = \rho\, dx_1 dx_2 dx_3 \frac{\partial^2 u_1}{\partial t^2}$$

$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{21}}{\partial x_2} + \frac{\partial \sigma_{31}}{\partial x_3} + f_1 = \rho \frac{\partial^2 u_1}{\partial t^2}$$

$i = 1$

$$\frac{\partial \sigma_{1i}}{\partial x_1} + \frac{\partial \sigma_{2i}}{\partial x_2} + \frac{\partial \sigma_{3i}}{\partial x_3} + f_i = \rho \frac{\partial^2 u_i}{\partial t^2} \qquad i = 1, 2, 3$$

$$\boxed{\frac{\partial \sigma_{ji}}{\partial x_j} + f_i = \rho \frac{\partial^2 u_i}{\partial t^2}} \qquad \boxed{\sigma_{ij} = \sigma_{ji}}$$



$$\int_S \bar{t}\, dS + \int_V \rho \bar{f}\, dV = \int_V \rho \ddot{\bar{u}}\, dV$$

$$\int_S t_i^{(\bar{n})}\, dS + \int_V \rho f_i\, dV = \int_V \rho \ddot{u}_i\, dV$$

$$t_i^{(\bar{n})} = \sigma_{ki} n_k$$

$$\int_S \sigma_{ki} n_k\, dS = \int_V \sigma_{ki,k}\, dV$$

$$\Rightarrow \quad \int_V \left(\sigma_{ki,k} + \rho f_i - \rho \ddot{u}_i\right) dV = 0$$

$$\Rightarrow \quad \boxed{\frac{\partial \sigma_{ki}}{\partial x_k} + \rho f_i = \rho \frac{\partial^2 u_i}{\partial t^2}}$$

$$\int_S (\bar{x} \times \bar{t})\, dS + \int_V (\bar{x} \times \bar{f}) \rho\, dV = \int_V \rho \frac{\partial}{\partial t} (\bar{x} \times \dot{\bar{u}})\, dV$$

$$\underbrace{\int_S e_{ijk} x_j t_k^{(n)}\, dS}_{} + \int_V e_{ijk} x_j f_k \rho\, dV = \int_V \rho\, e_{ijk} x_j \ddot{u}_k\, dV$$

$$\hookrightarrow \int_S e_{ijk} x_j \sigma_{lk} n_l\, dS = \int_V e_{ijk} \left(x_j \sigma_{lk,l} + \delta_{jl} \sigma_{lk}\right) dV$$

$$\Rightarrow \int_V e_{ijk} \left(x_j \sigma_{lk,l} + \sigma_{jk} + x_j f_k \rho - \rho x_j \ddot{u}_k \right) dV = 0$$

$$\int_V e_{ijk} \sigma_{jk}\, dV = 0$$

$$e_{ijk} \sigma_{jk} = 0$$

$$\boxed{\sigma_{jk} = \sigma_{kj}}$$

Loi de Hooke

$$\sigma_{ij} = C_{ijkl}\, e_{kl}$$

$$\sigma'_{mn} = C_{mnpq}\, e'_{pq}$$

$$\Rightarrow \quad C_{mnpq} = \beta_{mi}\,\beta_{nj}\,\beta_{pk}\,\beta_{ql}\, C_{ijkl}$$

$$C_{ijkl} = \lambda\,\delta_{ij}\,\delta_{kl} + \mu\,(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$$

$$\boxed{\sigma_{ij} = \lambda\, e_{kk}\,\delta_{ij} + 2\mu\, e_{ij}}$$

$$e_{ij} = -\frac{\lambda}{2\mu(3\lambda+2\mu)}\,\sigma_{kk}\,\delta_{ij} + \frac{1}{2\mu}\,\sigma_{ij}$$

$$\boxed{e_{ij} = -\frac{\nu}{E}\,\sigma_{kk}\,\delta_{ij} + \frac{1+\nu}{E}\,\sigma_{ij}}$$

$$E = \frac{\mu(3\lambda+2\mu)}{\lambda+\mu} \qquad \lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}$$

$$\nu = \frac{\lambda}{2(\lambda+\mu)} \qquad \mu = \frac{E}{2(1+\nu)} = G$$

$$\frac{\partial \sigma_{ki}}{\partial x_k} + f_i = \rho \frac{\partial^2 u_i}{\partial t^2}$$

$$\sigma_{ij} = \lambda e_{kk} \delta_{ij} + 2\mu e_{ij}$$

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

$$\sigma_{ij} = \lambda \frac{\partial u_k}{\partial x_k} \delta_{ij} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

$$\sigma_{ki} = \lambda \frac{\partial u_j}{\partial x_j} \delta_{ki} + \mu\left(\frac{\partial u_k}{\partial x_i} + \frac{\partial u_i}{\partial x_k}\right)$$

$$\lambda \frac{\partial^2 u_j}{\partial x_j \partial x_k} \delta_{ki} + \mu\left(\frac{\partial^2 u_k}{\partial x_i \partial x_k} + \frac{\partial^2 u_i}{\partial x_k \partial x_k}\right) + f_i = \rho \frac{\partial^2 u_i}{\partial t^2}$$

$$\lambda \frac{\partial^2 u_j}{\partial x_j \partial x_i} + \mu \frac{\partial^2 u_k}{\partial x_i \partial x_k} + \mu \frac{\partial^2 u_i}{\partial x_k \partial x_k} + f_i = \rho \frac{\partial^2 u_i}{\partial t^2}$$

$$\mu \frac{\partial^2 u_i}{\partial x_k \partial x_k} + (\lambda+\mu) \frac{\partial^2 u_k}{\partial x_i \partial x_k} + f_i = \rho \frac{\partial^2 u_i}{\partial t^2}$$

$$\boxed{\mu \nabla^2 \bar{u} + (\lambda+\mu) \nabla(\nabla \cdot \bar{u}) + \bar{f} = \rho \frac{\partial^2 \bar{u}}{\partial t^2}}$$

ou

$$\boxed{(\lambda+2\mu) \nabla(\nabla \cdot \bar{u}) - \mu \nabla \times (\nabla \times \bar{u}) + \bar{f} = \rho \frac{\partial^2 \bar{u}}{\partial t^2}}$$

$$(\lambda+2\mu)\nabla(\nabla\cdot\bar{u}) - \mu\nabla\times(\nabla\times\bar{u}) + \bar{f} = \rho\frac{\partial^2\bar{u}}{\partial t^2}$$ Navier

Soit $\bar{u}(\bar{x},t)$ solution.

Si

$$\bar{f} = \nabla\bar{\Phi} + \nabla\times\bar{\Psi}$$ Helmholtz

$$\dot{\bar{u}}(\bar{x},0) = \nabla A + \nabla\times\bar{B}$$

$$\bar{u}(x,0) = \nabla C + \nabla\times\bar{D}$$

avec

$$\nabla\cdot\bar{\Psi} = \nabla\cdot\bar{B} = \nabla\cdot\bar{D} = 0$$

$\Rightarrow$ Existent ϕ y $\bar{\psi}$ tels que

1) $\bar{u} = \nabla\phi + \nabla\times\bar{\psi}$

2) $\nabla\cdot\bar{\psi} = 0$

3) $\alpha^2\nabla^2\phi + \dfrac{\bar{\Phi}}{\rho} = \dfrac{\partial^2\phi}{\partial t^2}$ $\left(\alpha = \sqrt{\dfrac{\lambda+2\mu}{\rho}}\right)$

4) $\beta^2\nabla^2\bar{\psi} + \dfrac{\bar{\Psi}}{\rho} = \dfrac{\partial^2\bar{\psi}}{\partial t^2}$ $\left(\beta = \sqrt{\dfrac{\mu}{\rho}}\right)$

Théorème de Lamé

Si $\bar{f} = 0$

$$\bar{u} = \nabla\phi + \nabla\times\bar{\psi}, \quad \nabla\cdot\bar{\psi} = 0$$

$$\nabla^2\phi = \frac{1}{\alpha^2}\frac{\partial^2\phi}{\partial t^2}$$

$$\nabla^2\bar{\psi} = \frac{1}{\beta^2}\frac{\partial^2\bar{\psi}}{\partial t^2}$$

$$\boxed{\mu \nabla^2 \bar{u} + (\lambda+\mu)\nabla(\nabla\cdot\bar{u}) = \rho \frac{\partial^2 \bar{u}}{\partial t^2}}$$

$$\nabla^2 \bar{u} = \nabla(\nabla\cdot\bar{u}) - \nabla\times(\nabla\times\bar{u})$$

$$\nabla\cdot\bar{u} = 0 \quad \rightarrow \quad \nabla^2\bar{u} = \frac{1}{\beta^2}\frac{\partial^2\bar{u}}{\partial t^2} \qquad \beta = \sqrt{\frac{\mu}{\rho}}$$

$$\nabla\times\bar{u} = 0 \quad \rightarrow \quad \nabla^2\bar{u} = \frac{1}{\alpha^2}\frac{\partial^2\bar{u}}{\partial t^2} \qquad \alpha = \sqrt{\frac{\lambda+2\mu}{\rho}}$$

$$u_1 = u \quad u_2 = v \quad u_3 = w \qquad x_1 = x \quad x_2 = y \quad x_3 = z$$

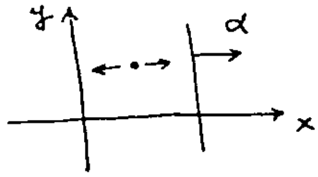
$$\boxed{\begin{aligned}
&\mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) + (\lambda+\mu)\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = \rho\frac{\partial^2 u}{\partial t^2}\\
&\mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) + (\lambda+\mu)\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = \rho\frac{\partial^2 v}{\partial t^2}\\
&\mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right) + (\lambda+\mu)\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = \rho\frac{\partial^2 w}{\partial t^2}
\end{aligned}}$$

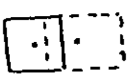
$$u \neq 0 \quad v = w = 0 \quad \rightarrow \quad \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) + (\lambda+\mu)\frac{\partial^2 u}{\partial x^2} = \rho\frac{\partial^2 u}{\partial t^2}$$

$$\frac{\partial^2 u}{\partial x \partial y} = 0 \quad ; \quad \frac{\partial^2 u}{\partial z \partial x} = 0$$

$$u = F(x,t) + G(y,t) + H(z,t)$$

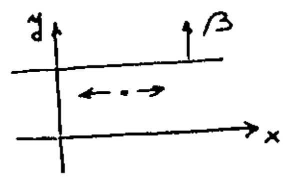
1) $u = u(x,t) \rightarrow (\lambda+2\mu)\dfrac{\partial^2 u}{\partial x^2} = \rho\dfrac{\partial^2 u}{\partial t^2}$

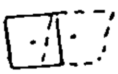




$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{\alpha^2}\frac{\partial^2 u}{\partial t^2}$$

2) $u = u(y,t) \rightarrow \mu\dfrac{\partial^2 u}{\partial y^2} = \rho\dfrac{\partial^2 u}{\partial t^2}$

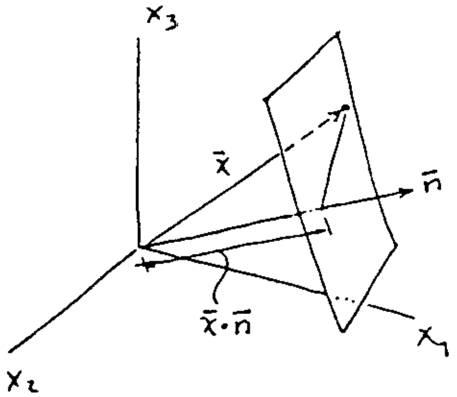




$$\frac{\partial^2 u}{\partial y^2} = \frac{1}{\beta^2}\frac{\partial^2 u}{\partial t}$$

$$\beta = \sqrt{\frac{\mu}{\rho}} < \sqrt{\frac{\lambda+2\mu}{\rho}} = \alpha$$

$$q = x_j n_j - ct$$



1) t, q constantes

$$q + ct = x_j n_j$$

2) q, $t+\Delta t$ constantes

$$q + ct + c\Delta t = x_j n_j$$

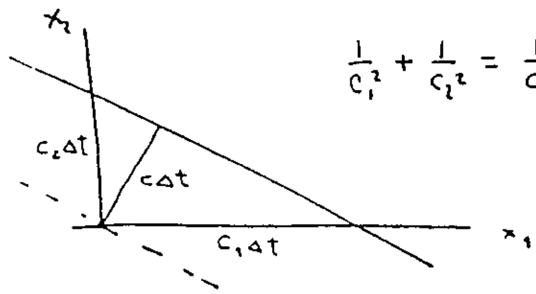
$$\phi = f(x_j n_j - ct)$$

$$\nabla^2 \phi = \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2}$$

$$\nabla^2\phi = \underbrace{n_k n_k}_{1} f''(\ ) \quad ; \quad \frac{\partial^2\phi}{\partial t^2} = c^2 f''(\ )$$

$$g(t - s_i x_i)$$

$$s_i = \frac{1}{c} n_i$$



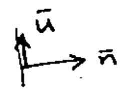
$$\frac{1}{c_1^2} + \frac{1}{c_2^2} = \frac{1}{c^2}$$

$u_i = U_i(x_j n_j - ct) \to$ Ec Navier

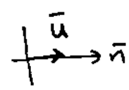
$$\left[\rho c^2 \delta_{jk} - \mu n_i n_i \delta_{jk} - (\lambda+\mu) n_j n_k\right] U_j'' = 0$$

$$\Rightarrow \left[\rho c^2 - (\lambda + 2\mu)\right] \bar{u}''\cdot\bar{n} = 0 \quad ; \left[\rho c^2 - \mu\right] \bar{u}''\times\bar{n} = 0$$

$$\bar{u}\cdot\bar{n} = 0 \to c = \beta = \sqrt{\mu/\rho}$$



$$\bar{u}\times\bar{n} = 0 \to c = \alpha = \sqrt{\frac{\lambda+2\mu}{\rho}}$$



$$\nabla^2 \phi = \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2}$$

$$\phi = X(x)\, Y(y)\, Z(z)\, T(t)$$

$$c^2 X'' Y Z T + c^2 X Y'' Z T + c^2 X Y Z'' T - X Y Z T'' = 0$$

$$\underbrace{c^2 \frac{X''}{X}}_{-k_x^2} + \underbrace{c^2 \frac{Y''}{Y}}_{-k_y^2} + \underbrace{c^2 \frac{Z''}{Z}}_{-k_z^2} - \underbrace{\frac{T''}{T}}_{-\omega^2} = 0$$

$$-k_x^2 - k_y^2 - k_z^2 + \omega^2 = 0$$

$$X = e^{\pm i k_x x/c} \; ; \; Y = e^{\pm i k_y y/c} \; ; \; Z = e^{\pm i k_z z/c} \; ; T = e^{\pm i \omega t}$$

$$\phi = \iiint_{-\infty}^{\infty} A(k_x, k_y, \omega) \exp\left[i\omega \left(t - \frac{k_x}{\omega c} x - \frac{k_y}{\omega c} y - \frac{k_z}{\omega c} z \right)\right] dk_x\, dk_y\, d\omega$$

$$k_z = \sqrt{\omega^2 - k_x^2 - k_y^2}$$

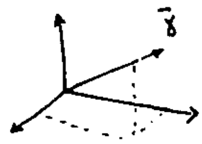
$$t - \frac{k_x}{\omega c} x - \frac{k_y}{\omega c} y - \frac{k_z}{\omega c} z \longleftrightarrow t - \frac{1}{c} n_i x_i$$

$$n_i n_i = 1 \qquad \left(\frac{k_x}{\omega}\right)^2 + \left(\frac{k_y}{\omega}\right)^2 + \left(\frac{k_z}{\omega}\right)^2 = 1$$

si $(k_x/\omega)^2 + (k_y/\omega)^2 \leqslant 1$; k_z/ω real $\longrightarrow$ onde plan homogène

si $(k_x/\omega)^2 + (k_y/\omega)^2 > 1$; k_z/ω imaginaire $\rightarrow$ onde plan inhomogène

Ondes sphériques



Soit $\quad u_i = u(r,t)\,\gamma_i \quad , \quad \gamma_i = \frac{x_i}{r}$

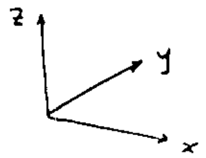
$$\Rightarrow \quad \frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\left(\frac{\partial u}{\partial r} - \frac{u}{r}\right) = \frac{1}{\alpha^2}\frac{\partial^2 u}{\partial t^2} \qquad \alpha^2 = \frac{\lambda + 2\mu}{\rho}$$

avec $\quad u = \frac{\partial \phi}{\partial r} \quad \longrightarrow \quad \frac{\partial^2 (r\phi)}{\partial r^2} = \frac{1}{\alpha^2}\frac{\partial^2 (r\phi)}{\partial t^2}$

$$\phi = \underbrace{\frac{1}{r} f(t - r/\alpha)} + \frac{1}{r} g(t + r/\alpha)$$

$$u(r,t) = \frac{1}{r}\frac{\partial f(t - r/\alpha)}{\partial r} - \frac{1}{r^2} f(t - r/\alpha)$$

Ondes cylindriques



$u = w = 0$

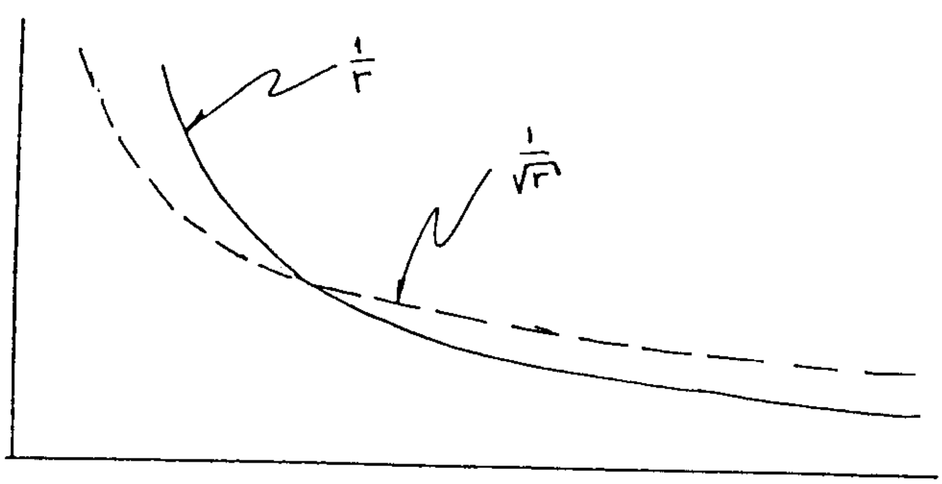
$v = v(r,t)$

$r = \sqrt{x^2 + z^2}$

$$\Rightarrow \quad \nabla^2 v = \frac{1}{\beta^2}\frac{\partial^2 v}{\partial t^2} \quad , \quad \beta^2 = \frac{\mu}{\rho}$$

$$\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} = \frac{1}{\beta^2}\frac{\partial^2 v}{\partial t^2}$$

$$v \simeq \frac{1}{\sqrt{r}} f(t - r/\beta) \quad , \quad \left|\frac{\beta^2 f}{4 r^2 f''}\right| \ll 1$$



$$\mu \frac{\partial^2 u_i}{\partial x_k \partial x_k} + (\lambda+\mu) \frac{\partial^2 u_k}{\partial x_i \partial x_k} = \rho \frac{\partial^2 u_i}{\partial t^2} \qquad \text{Navier}$$

Soit $\quad u_i = u(r)\, \frac{x_i}{r} = u\, \gamma_i$

$$\frac{\partial u_i}{\partial x_j} = u(r)\left[\frac{x_i}{-r^2}\frac{x_j}{r} + \frac{\delta_{ij}}{r}\right] + \frac{\partial u}{\partial r}\frac{x_j}{r}\frac{x_i}{r}$$

$$= \frac{u}{r}\left(\delta_{ij} - \gamma_i \gamma_j\right) + \frac{\partial u}{\partial r}\gamma_i \gamma_j$$

$$\frac{\partial \gamma_i}{\partial x_\ell} = \frac{\partial}{\partial x_\ell}\left(\frac{x_i}{r}\right) = \frac{1}{r}\left(\delta_{i\ell} - \gamma_i \gamma_\ell\right)$$

$$\frac{\partial^2 u_i}{\partial x_j \partial x_\ell} = \frac{u}{r}\left(-\gamma_i \frac{\partial \gamma_j}{\partial x_\ell} - \gamma_j \frac{\partial \gamma_i}{\partial x_\ell}\right) + \left(\frac{-u}{r^2}\gamma_\ell + \frac{1}{r}\frac{\partial u}{\partial r}\gamma_\ell\right)\left(\delta_{ij} - \gamma_i\gamma_j\right)$$

$$+ \frac{\partial u}{\partial r}\left(\gamma_i \frac{\partial \gamma_j}{\partial x_\ell} - \gamma_j \frac{\partial \gamma_i}{\partial x_\ell}\right) + \frac{\partial^2 u}{\partial r^2}\gamma_\ell \gamma_i \gamma_j$$

$$\frac{\partial^2 u_i}{\partial x_k \partial x_k} = \frac{u}{r}\left(-\gamma_i \frac{1}{r}(3-1) - \frac{1}{r}\cancel{\gamma_k\left(\delta_{ik} - \gamma_i\gamma_k\right)}\right)$$

$$+ \left(\frac{-u}{r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right)\cancel{\left(\gamma_i - \gamma_i\right)}$$

$$+ \frac{\partial u}{\partial r}\left(\frac{2\gamma_i}{r}\right) + \frac{\partial^2 u}{\partial r^2}\gamma_i = \left[\frac{2}{r}\left(\frac{\partial u}{\partial r} - \frac{u}{r}\right) + \frac{\partial^2 u}{\partial r^2}\right]\gamma_i$$

$$\frac{\partial^2 u_k}{\partial x_i \partial x_k} = \frac{\partial}{\partial x_i}\left(\frac{2u}{r} + \frac{\partial u}{\partial r}\right) = 2u\left(-\frac{1}{r^2}\gamma_i\right) + \frac{2}{r}\frac{\partial u}{\partial r}\gamma_i + \frac{\partial^2 u}{\partial r^2}\gamma_i = \left[\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\left(\frac{\partial u}{\partial r} - \frac{u}{r}\right)\right]\gamma_i$$

$$(\lambda+2\mu)\left[\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\left(\frac{\partial u}{\partial r} - \frac{u}{r}\right)\right] = \rho \frac{\partial^2 u}{\partial t^2}$$

$$u = \frac{\partial \phi}{\partial r} \quad \rightarrow (\lambda+2\mu)\frac{\partial^2}{\partial r^2}(r\phi) = \rho \frac{\partial^2 (r\phi)}{\partial t^2} \qquad \phi = \frac{1}{r} f\left(t \pm \frac{r}{\alpha}\right)$$

avec $\alpha^2 = \frac{\lambda+2\mu}{\rho}$

$$u_r = \frac{1}{r}\frac{\partial f}{\partial r} - \frac{f}{r^2}$$

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = \frac{1}{\beta^2}\frac{\partial^2 u}{\partial t^2}$$

$$u = R(r)\,\Theta(\theta)\,T(t)$$

$$\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{\Theta''}{\Theta} = \frac{1}{\beta^2}\underbrace{\frac{T''}{T}}_{-\omega^2} \quad \Rightarrow \quad T = e^{\pm i\omega t}$$

$$\Downarrow$$

$$\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{\Theta''}{\Theta} + k^2 = 0 \qquad k = \omega/\beta$$

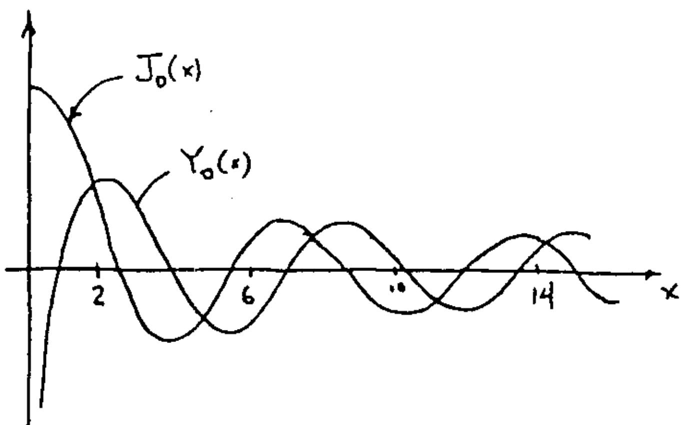
$$r^2\frac{R''}{R} + r\frac{R'}{R} + r^2k^2 + \underbrace{\frac{\Theta''}{\Theta}}_{-\nu^2} = 0 \quad \Rightarrow \quad \Theta = e^{\pm i\nu\theta}$$

$$\Downarrow$$

$$r^2R'' + rR' + (r^2k^2 - \nu^2)R = 0 \qquad \text{Ec de Bessel}$$

$$\hookrightarrow \quad R = A\,J_\nu(kr) + B\,Y_\nu(kr)$$

$$u = A\,J_\nu(kr)\,e^{\pm i\nu\theta}\,e^{\pm i\omega t} + B\,Y_\nu(kr)\,e^{\pm i\nu\theta}\,e^{\pm i\omega t}$$



$$J_\nu(x) \sim \sqrt{\frac{2}{\pi x}}\cos\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right)$$

$$Y_\nu(x) \sim \sqrt{\frac{2}{\pi x}}\sin\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right)$$

$$\boxed{x \gg \nu}$$

$$\left[J_\nu(kr) \pm i\,Y_\nu(kr)\right] e^{\pm i\omega t} e^{\pm i\nu\theta} \rightarrow \sqrt{\frac{2}{\pi kr}}\exp\left[\pm i\left(kr - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) \pm i\omega t \pm i\nu\theta\right]$$

$$\Downarrow$$

$$\sqrt{\frac{2}{\pi kr}}\exp(\pm i\nu\theta)\,\exp\left[\pm i\omega(t \mp r/\beta) \mp i\left(\frac{\nu}{2} + \frac{1}{4}\right)\pi\right]$$

$$\exp[\pm i\omega\,(t - r/\beta)]$$

$$\exp[\pm i\omega\,(t + r/\beta)]$$

$$e^{+i\omega t}$$

$$H_\nu^{(2)}(kr)\; e^{i\omega t}\; e^{\pm i\nu\theta}$$

$$H_\nu^{(1)}(kr)\; e^{i\omega t}\; e^{\pm i\nu\theta}$$

$$H_\nu^{(1,2)}(kr) = J_\nu(kr) \pm i\, Y_\nu(kr)$$

Hankel

si $\quad u(r,\theta,t) = u(r,\theta+2\pi,t) \;\Rightarrow\; \nu = 0, 1, 2, \dots, n, \dots$

$$J_n(kr)\;\cos n\theta$$

$$J_n(kr)\;\sin n\theta$$

$$Y_n(kr)\;\cos n\theta$$

$$Y_n(kr)\;\sin n\theta$$

$\boxed{\nabla^2 u + k^2 u = 0}$ ec de Helmholtz

$$\boxed{z^2 \frac{d^2 w}{dz^2} + z \frac{dw}{dz} + (z^2 - \nu^2) w = 0}$$

Equation de Bessel

$$J_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{k=0}^{\infty} \frac{(-z^2/4)^k}{k!\,\Gamma(\nu+k+1)}$$

fonction de Bessel de 1ère épèce et ordre ν

$$Y_\nu(z) = \left\{ J_\nu(z) \cos \nu\pi - J_{-\nu}(z) \right\} / \sin \nu\pi$$

2^{me} épèce (ou de Neumann)

si $\nu \to n$ lim

$$J_n(z) = \left(\frac{z}{2}\right)^n \sum_{k=0}^{\infty} \frac{(-z^2/4)^k}{k!\,(n+k)!}$$

$$Y_n(z) = -\frac{1}{\pi}\left(\frac{z}{2}\right)^{-n} \sum_{k=0}^{n-1} \frac{(n-k-1)!}{k!}\left(\frac{z}{2}\right)^{2k} + \frac{2}{\pi} \ln\left(\frac{z}{2}\right) J_n(z)$$

$$- \frac{1}{\pi}\left(\frac{z}{2}\right)^n \sum_{k=0}^{\infty} \left\{ \psi(k+1) + \psi(n+k+1) \right\} \frac{(-z^2/4)^k}{k!\,(n+k)!}$$

$\psi(k) = -0.5772157\ldots + 1 + 1/2 + 1/3 + \ldots + 1/k$

fonctions de Hankel

$$H_\nu^{(1)}(z) = J_\nu(z) + i\, Y_\nu(z)$$

$$H_\nu^{(2)}(z) = J_\nu(z) - i\, Y_\nu(z)$$

$$\boxed{z \to 0}$$

$$J_\nu(z) \sim \frac{(z/2)^\nu}{\Gamma(\nu+1)} \qquad Y_0(z) \sim \frac{2}{\pi}\ln z$$

$(\nu \neq -1, -2, \dots)$

$$Y_\nu(z) \sim \frac{-1}{\pi}\Gamma(\nu)\left(\frac{z}{2}\right)^{-\nu}$$

$(\mathrm{Re}(\nu) > 0)$

$$\boxed{|z| \to \infty}$$

$$J_\nu(z) \sim \sqrt{\frac{2}{\pi z}}\cos\left(z - \frac{(2\nu+1)\pi}{4}\right)$$

$$H_\nu^{(1,2)}(z) \sim \sqrt{\frac{2}{\pi z}}\, e^{\pm i\left(z - \frac{(2\nu+1)\pi}{4}\right)} \qquad Y_\nu(z) \sim \sqrt{\frac{2}{\pi z}}\sin\left(z - \frac{(2\nu+1)\pi}{4}\right)$$

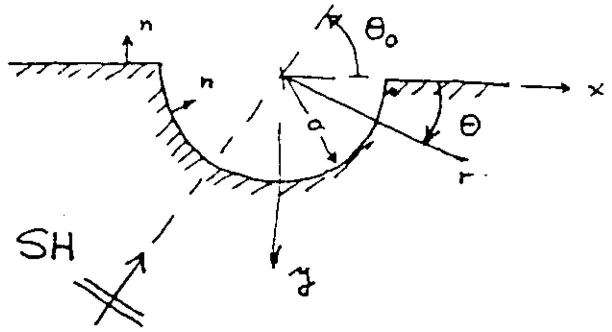
$$\boxed{e^{\frac{z}{2}(t-1/t)} = \sum_{k=-\infty}^{\infty} t^k J_k(z)}$$

$$1 = J_0(z) + 2J_2(z) + 2J_4(z) + \dots$$

$$\cos z = J_0(z) - 2J_2(z) + 2J_4(z) - + \dots$$

$$\sin z = 2J_1(z) - 2J_3(z) + 2J_5(z) - + \dots$$

$$e^{iz\cos\theta} = J_0(z) + 2iJ_1(z)\cos\theta + 2i^2J_2(z)\cos 2\theta + \dots$$
$$= \sum_{m=0}^{\infty} \varepsilon_m (i)^m J_m(z)\cos m\theta$$

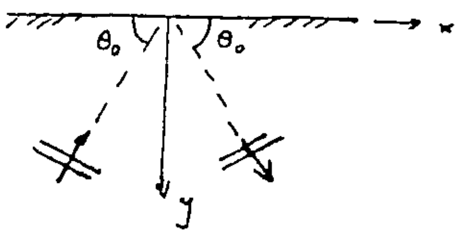


$$w = w(r,\theta,\omega)\, e^{i\omega t}$$

$$\nabla^2 w + k^2 w = 0$$

$$k = \frac{\omega}{\beta} \qquad \beta = \sqrt{\frac{\mu}{\rho}}$$

$$w = w^{(0)} + w^{(d)}$$



$$w^{(0)} = w_0 \exp\left[i\omega\left(t - \frac{x\cos\theta_0 - y\sin\theta_0}{\beta}\right)\right] + w_0 \exp\left[i\omega\left(t - \frac{x\cos\theta_0 + y\sin\theta_0}{\beta}\right)\right]$$

$$= w_0 \exp\left[-ikr\cos(\theta_0-\theta)\right] e^{i\omega t} + w_0 \exp\left[-ikr\cos(\theta_0+\theta)\right] e^{i\omega t}$$

$$w^{(0)} = w_0 \sum_{m=0}^{\infty} A_m^{(0)} J_m(kr)\cos m\theta \; e^{i\omega t}$$

$$A_m^{(0)} = 2(-i)^m (\cos m\theta_0)\,\varepsilon_m$$

$$w^{(s)} = w_0 \sum_{m=0}^{\infty} A_m H_m^{(2)}(kr)\cos m\theta \; e^{i\omega t}$$

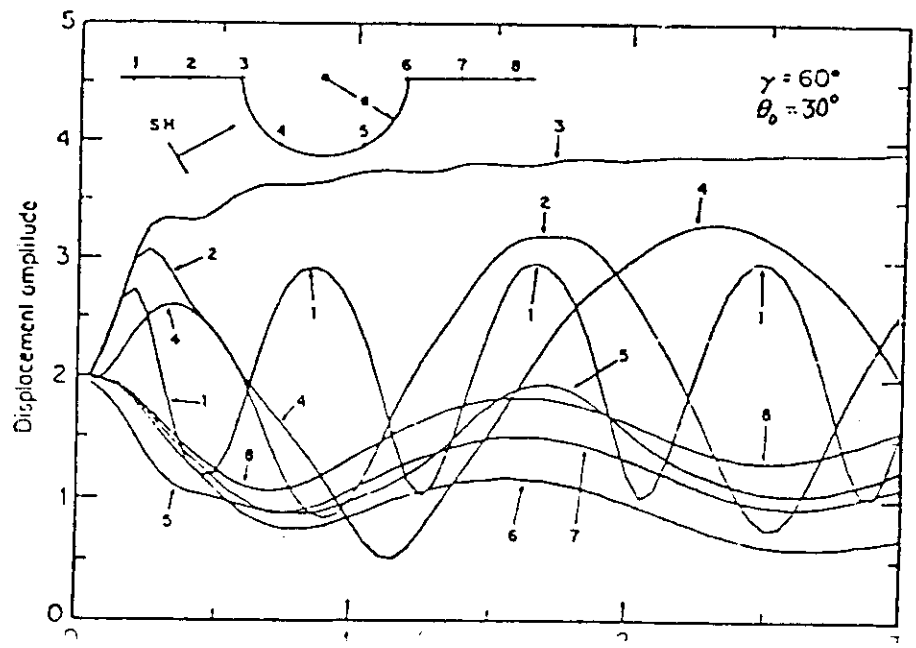
$$\frac{\partial w}{\partial \theta} = 0 \quad r \geq a \quad \theta = 0, \pi$$

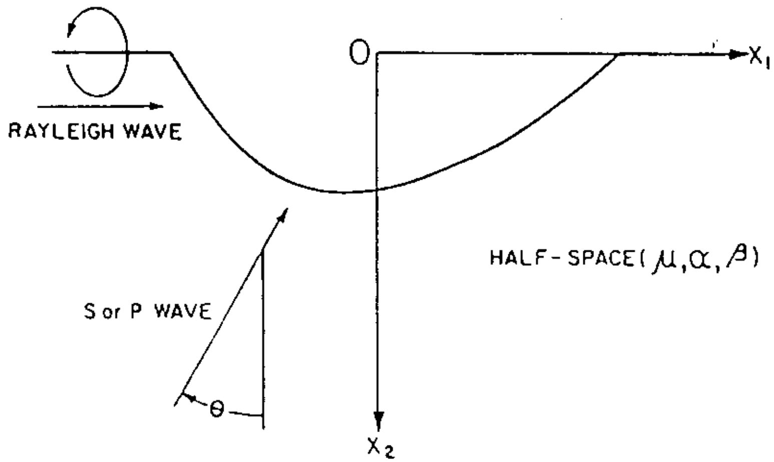
$$A_m = -A_m^{(0)} \frac{J_m'(ka)}{H_m^{(1)'}(ka)}$$

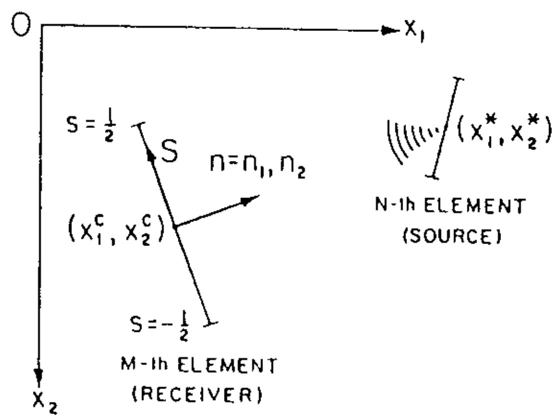
$$\left.\frac{\partial w}{\partial r}\right|_{r=a} = 0 \quad 0 \leq \theta \leq \pi$$

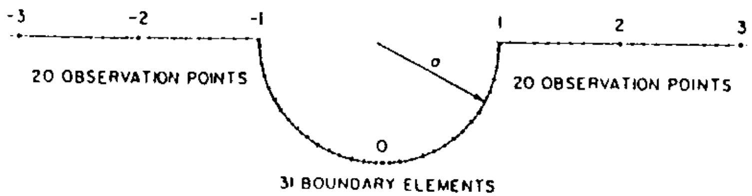
$$w = w_0 \sum_{m=0}^{\infty} A_m^{(0)} \left[J_m(kr) - \frac{J_m'(ka)}{H_m^{(2)'}(ka)} H_m(kr) \right] \cos m\theta \; e^{i\omega t}$$

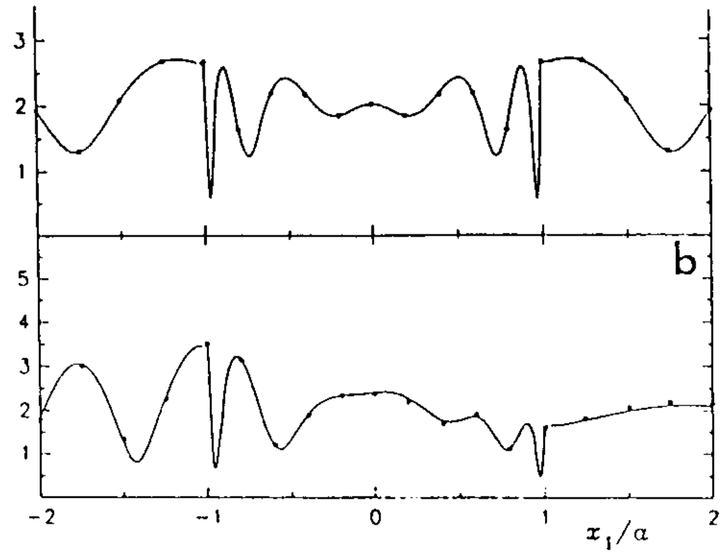
Trifunac (1973)

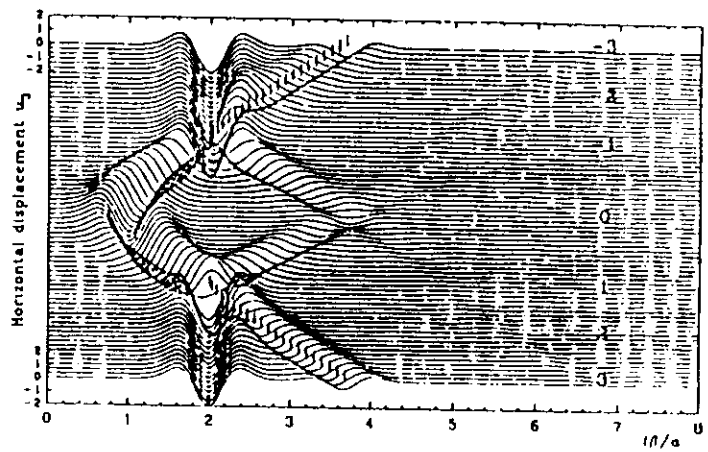


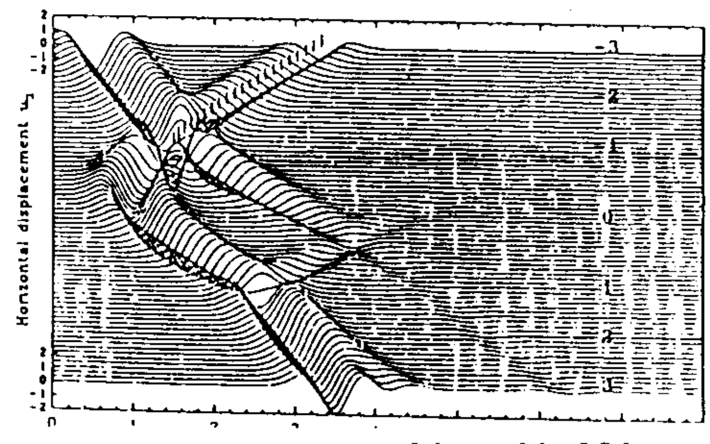

RAYLEIGH WAVE
O
X1
X2
HALF-SPACE(μ,α,β)
S or P WAVE
θ

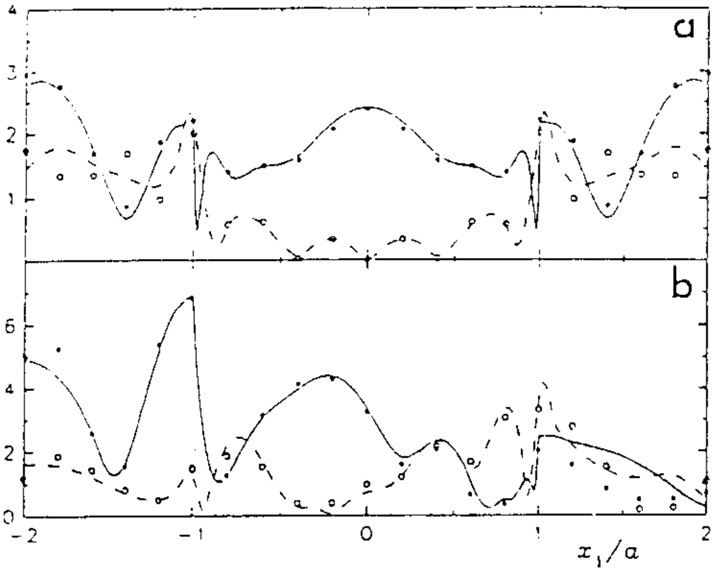

O
X1
X2
S = 1/2
S
n=n1, n2
(x1*, x2*)
N-th ELEMENT
(SOURCE)
(x1c, x2c)
S = -1/2
M-th ELEMENT
(RECEIVER)

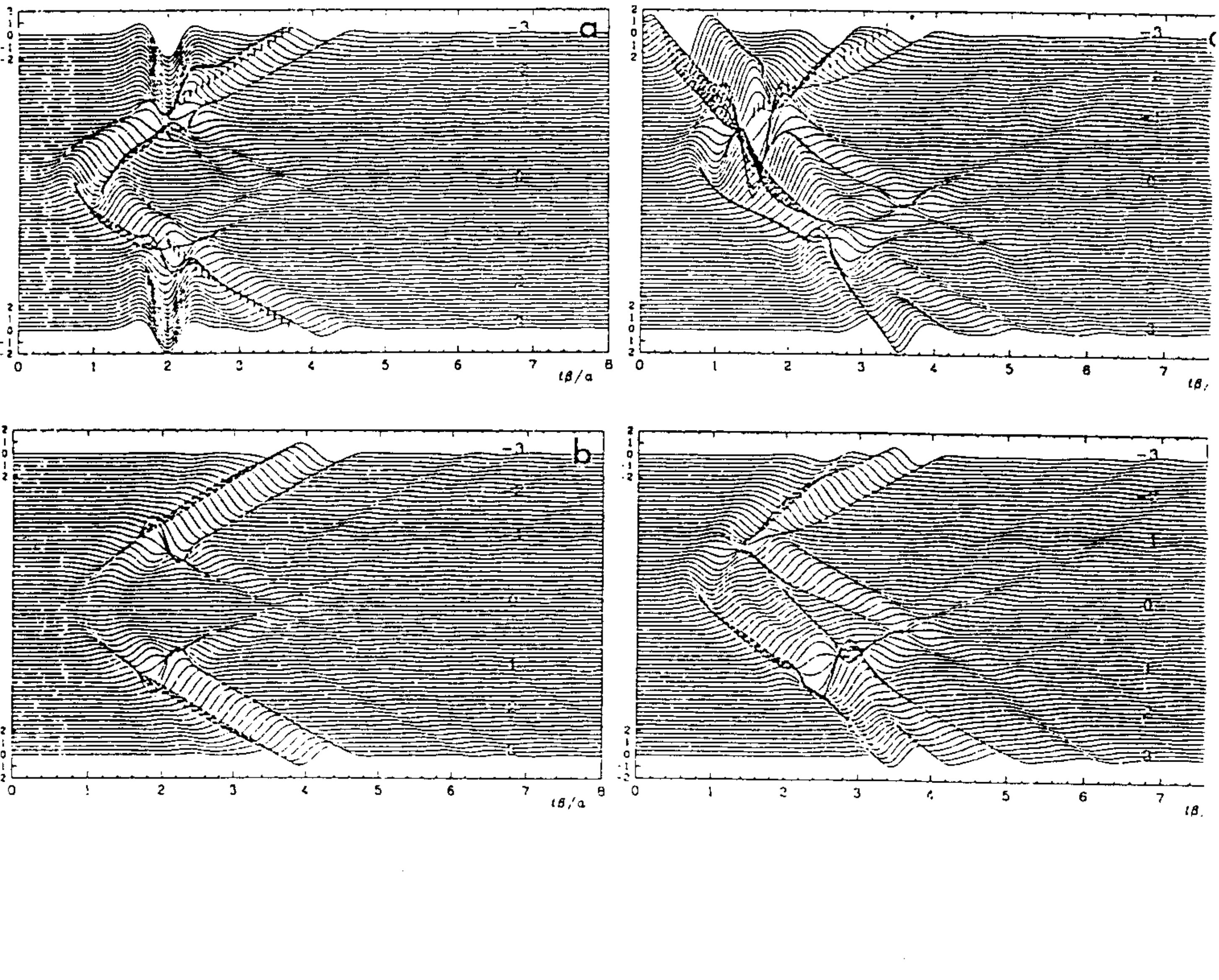

-3
-2
-1
1
2
3
20 OBSERVATION POINTS
20 OBSERVATION POINTS
a
O
31 BOUNDARY ELEMENTS

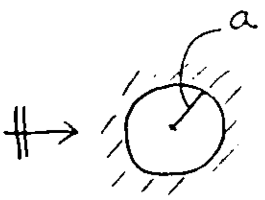

b
x1/a


Horizontal displacement


Horizontal displacement


a
b
-2
-1
0
1
2
x_1/a


a
b
0
1
2
3
4
5
6
7
8
tβ/a



$$w^{(inc)} = w_0\, e^{-ikx}\, e^{i\omega t}$$

$$w^{(d)} = w_0\, (-)\sum_{m=0}^{\infty} (-i)^m \varepsilon_m\, \frac{J_m'(ka)}{H_m^{(2)'}(ka)}\, H_m^{(2)}(kr) \cos m\theta$$

$$w^{(d)} = w_0 \sum_{m=0}^{\infty} (-i)^{m+1} \varepsilon_m \sin\gamma_m\, e^{-i\gamma_m}\, H_m^{(2)}(kr) \cos m\theta$$

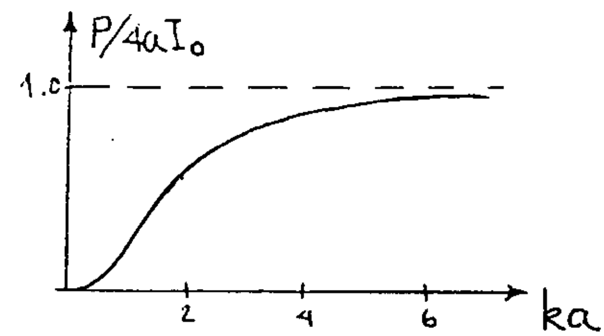
avec $\quad \cot\gamma_0 = \dfrac{Y_1(ka)}{J_1(ka)} \qquad \cot\gamma_m = \dfrac{Y_{m+1}(ka) - Y_{m-1}(ka)}{J_{m+1}(ka) - J_{m-1}(ka)}$

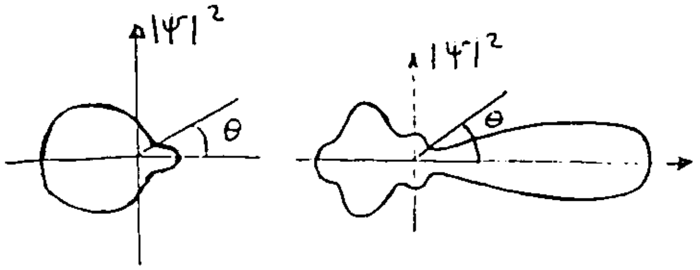
si $kr \gg 1 \qquad H_m^{(2)}(kr) \sim \sqrt{\dfrac{2}{\pi kr}}\; e^{-i\left(kr - \frac{2m+1}{4}\pi\right)}$

$$\Rightarrow \quad w^{(d)} = w_0 \sqrt{\frac{2}{\pi kr}}\; e^{i(\omega t - kr)} \left\{ \sum_{m=0}^{\infty} (-i)^{m+1} \varepsilon_m \sin\gamma_m\, e^{-i\left(\gamma_m - \frac{2m+1}{4}\pi\right)} \cos m\theta \right\}$$

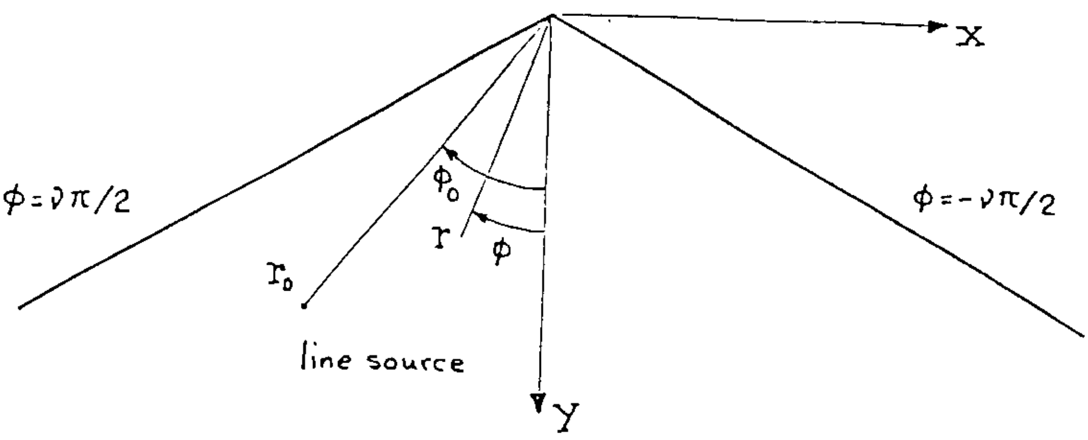
$$= w_0 \sqrt{\frac{2}{\pi kr}}\; e^{i(\omega t - kr)}\; \Psi(\theta)$$

$$P = \frac{2 I_0}{\pi k} \int_0^{2\pi} |\Psi(\theta)|^2\, d\theta$$





$ka = 1 \qquad\qquad ka = 5$



Wedge geometry

Incident waves $\quad w^{(i)} = w_0 \, H_0^{(1)}(kR)/H_0^{(1)}(kr_0) \quad e^{-i\omega t}$

$$k = \omega/\beta \; , \; \beta = \sqrt{\mu/\rho} \qquad R = \sqrt{r^2 + r_0^2 - 2 r r_0 \cos(\phi - \phi_0)}$$

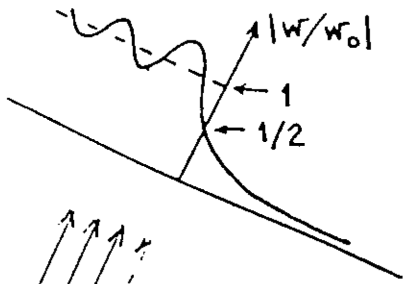
if $r_0 \to \infty \qquad w^{(i)} \to w_0 \exp[-ikr\cos(\phi - \phi_0)] \qquad$ *plane wave*

Total field

$$w = w_0 \, \frac{2}{\nu} \sum_{n=0}^{\infty} \varepsilon_n \cos \frac{n(\phi + \nu\pi/2)}{\nu} \cos \frac{n(\phi_0 + \nu\pi/2)}{\nu} \, S_{n/\nu}$$

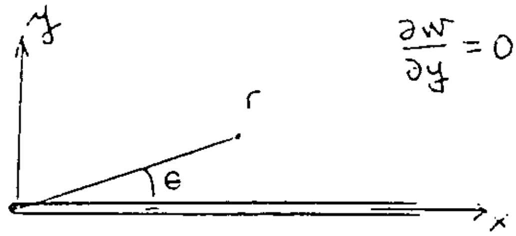
$$S_{n/\nu} = J_{n/\nu}(kr_<) H_{n/\nu}^{(1)}(kr_>)/H_0^{(1)}(kr_0) \qquad \begin{array}{l} r_< = \min(r, r_0) \\ r_> = \max(r, r_0) \end{array}$$

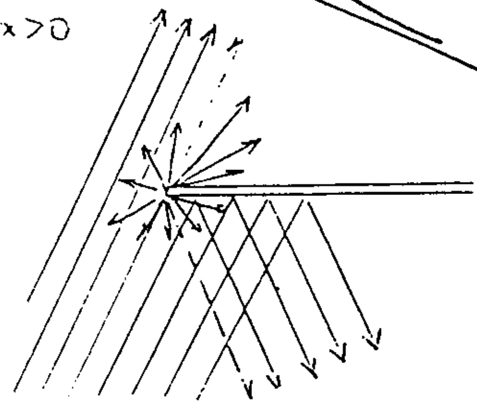
If $\quad r_0 \to \infty \qquad S_{n/\nu} \to e^{-i\frac{\pi n}{2\nu}} J_{n/\nu}(kr) \qquad$ Macdonald (1902)

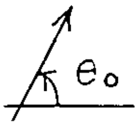


$$\nabla^2 w + k^2 w = 0$$

$$\frac{\partial w}{\partial y} = 0 \; ; \; y = 0, \, x > 0$$







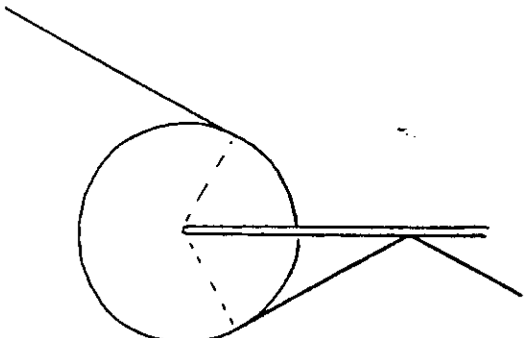
$$w^{inc} = w_0 \exp\left[ikr\cos(\theta - \theta_0)\right]$$

$$-\pi \leqslant \theta_0 \leqslant \pi \qquad 0 \leqslant \theta \leqslant 2\pi \qquad e^{-i\omega t}$$

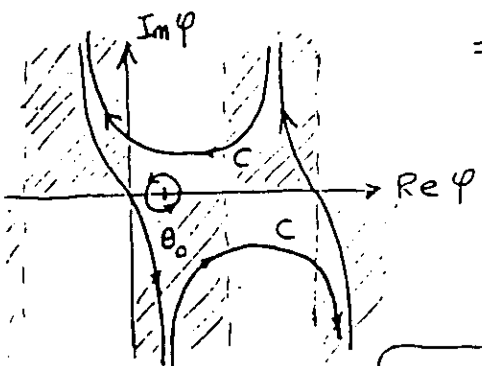
$$w^{inc} = w_0 \int A(\varphi) \exp\left[ikr\cos(\theta - \varphi)\right] d\varphi$$

$$A(\varphi) = \frac{1}{2\pi} \frac{e^{i\varphi}}{e^{i\varphi} - e^{i\theta_0}}$$

fonc avec residue $1/2\pi i$, periodique (2π)



$$= w_0 \oint \frac{1}{2\pi} \frac{e^{i\varphi}}{e^{i\varphi} - e^{i\theta_0}} \exp\left[ikr\cos(\theta - \varphi)\right] d\varphi$$



$$= w_0 \frac{1}{4\pi} \oint \underbrace{\frac{e^{i\varphi/2}}{e^{i\varphi/2} - e^{i\theta_0/2}}}_{\text{periodique } (4\pi)} e^{ikr\cos(\theta - \varphi)} d\varphi = W(r, \theta - \theta_0)$$

$$\boxed{w = W(r, \theta - \theta_0) + W(r, \theta + \theta_0)}$$

Méthode des images !

Sommerfeld (1896)

$$\boxed{w = w_0 \frac{1}{\sqrt{\pi}} e^{i(kr - \pi/4)} \left\{ F\left(\sqrt{2kr} \sin\frac{\theta_0 - \theta}{2}\right) + F\left(\sqrt{2kr} \sin\frac{\theta_0 + \theta}{2}\right) \right\}}$$

$$F(z) = \exp(-iz^2) \int_z^\infty \exp(i\zeta^2) d\zeta \;, \quad F(-z) = \sqrt{\pi}\, e^{-i(z^2 - \pi/4)} - F(z) \;, \quad F(0) = \frac{\sqrt{\pi}}{2} e^{i\frac{\pi}{4}}$$

pour $z > 0$ $\quad F(z) = i/2z + O(z^{-3})$ $\qquad$ pour $|z| \ll 1$ $\quad F(z) = F(0) - z + O(z^2)$

$$C(x) = \int_0^x \cos\left(\frac{\pi}{2}\zeta^2\right) d\zeta \qquad S(x) = \int_0^x \sin\left(\frac{\pi}{2}\zeta^2\right) d\zeta$$

Intégrales de Fresnel

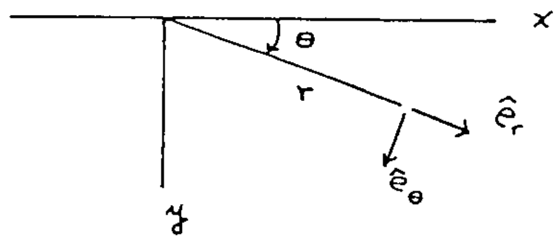
$$F(z) = \sqrt{\frac{\pi}{2}} \left\{ g\left(z\sqrt{\frac{2}{\pi}}\right) + i f\left(z\sqrt{\frac{2}{\pi}}\right) \right\}$$

$$g(x) \simeq (2 + 4.142x + 3.492x^2 + 6.67x^3)^{-1} + \varepsilon$$

$$f(x) \simeq (1 + 0.926x)(2 + 1.792x + 3.104x^2)^{-1} + \varepsilon$$

Cylindrical coordinates

(two-dimensional problems ; $u_z = 0$)



Equations of motion

$$\rho \frac{\partial^2 \hat{u}}{\partial t^2} = \rho \hat{F} + (\mathrm{div}\, \hat{\sigma}_r, \mathrm{div}\, \hat{\sigma}_\theta, 0)$$

$$+ \frac{1}{r}(-\sigma_{\theta\theta}, \sigma_{r\theta}, 0)$$

$$\mathrm{div}\, \hat{A} = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\theta}{\partial \theta}$$

the components of strain are

$$e_{rr} = \frac{\partial u_r}{\partial r}, \quad e_{\theta\theta} = \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{1}{r} u_r$$

$$e_{r\theta} = \frac{1}{2}\left[\frac{\partial u_\theta}{\partial r} - \frac{1}{r} u_\theta + \frac{1}{r}\frac{\partial u_r}{\partial \theta}\right]$$

and the stresses result

$$\sigma_{rr} = \lambda \Theta + 2\mu e_{rr}, \quad \sigma_{\theta\theta} = \lambda \Theta + 2\mu e_{\theta\theta}, \quad \sigma_{zz} = \lambda \Theta$$

$$\sigma_{r\theta} = 2\mu e_{r\theta} \qquad \text{with } \Theta = e_{rr} + e_{\theta\theta}$$

Navier's equation $\quad \rho \frac{\partial^2 \hat{u}}{\partial t^2} = \mu \nabla^2 \hat{u} + (\lambda + \mu) \nabla \nabla \cdot \hat{u}$

Independent solutions

P waves

$$u_r = \frac{\partial \phi_n}{\partial r}, \quad u_\theta = \frac{1}{r}\frac{\partial \phi_n}{\partial \theta}$$

$$\phi_n = \mathcal{E}_n(qr)\, e^{in\theta}, \quad q = \frac{\omega}{\alpha}$$

SV waves

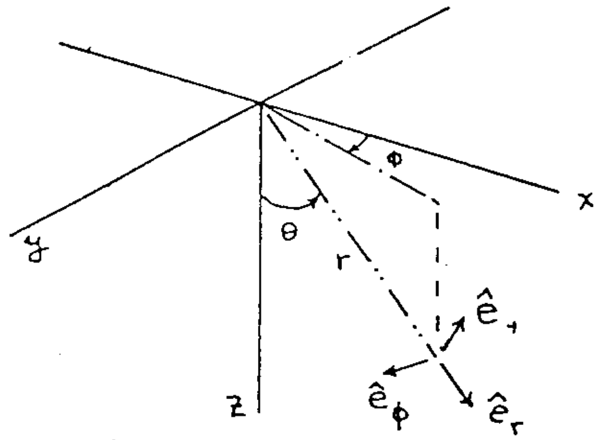
$$u_r = \frac{1}{r}\frac{\partial \psi_n}{\partial \theta}, \quad u_\theta = -\frac{\partial \psi_n}{\partial r}$$

$$\psi_n = \mathcal{E}_n(kr)\, e^{in\theta}, \quad k = \frac{\omega}{\beta}$$

$e^{i\omega t}$

$$\mathcal{E}_n(\cdot) = \begin{cases} J_n(\cdot) & \text{Bessel function of the first class and order } n\text{; bounded regions} \\ H_n^{(2)}(\cdot) & \text{Hankel function of the second class and order } n\text{; unbounded regions} \end{cases}$$

Spherical coordinates



Equations of motion

$$\rho \frac{\partial^2 \hat{u}}{\partial t^2} = \rho \hat{F} + \left(\operatorname{div} \hat{\sigma}_r, \operatorname{div} \hat{\sigma}_\theta, \operatorname{div} \hat{\sigma}_\phi\right)$$
$$+ \frac{1}{r}\left(-\sigma_{\theta\theta} - \sigma_{\phi\phi},\ \sigma_{r\theta} - \sigma_{\phi\phi}\cot\theta,\ \sigma_{\phi r} + \sigma_{\theta\phi}\cot\theta\right)$$

Components of strain $\qquad e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$

$$e_{rr} = \frac{\partial u_r}{\partial r}, \quad e_{\theta\theta} = \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{1}{r}u_r$$

$$e_{\phi\phi} = \frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial \phi} + \frac{1}{r}u_\theta\cot\theta + \frac{1}{r}u_r$$

$$e_{\theta\phi} = \frac{1}{2}\left[\frac{1}{r}\frac{\partial u_\phi}{\partial \theta} - \frac{1}{r}u_\phi\cot\theta + \frac{1}{r\sin\theta}\frac{\partial u_\theta}{\partial \phi}\right]$$

$$e_{\phi r} = \frac{1}{2}\left[\frac{1}{r\sin\theta}\frac{\partial u_r}{\partial \phi} + \frac{\partial u_\phi}{\partial r} - \frac{1}{r}u_\phi\right]$$

$$e_{r\theta} = \frac{1}{2}\left[\frac{\partial u_\theta}{\partial r} - \frac{1}{r}u_\theta + \frac{1}{r}\frac{\partial u_r}{\partial \theta}\right]$$

Elastic stress-strain relationships $\qquad \sigma_{ij} = \lambda e_{kk}\delta_{ij} + 2\mu e_{ij}$

$$\sigma_{rr} = \lambda H + 2\mu e_{rr}$$
$$\sigma_{\theta\theta} = \lambda H + 2\mu e_{\theta\theta}$$
$$\sigma_{\phi\phi} = \lambda H + 2\mu e_{\phi\phi}$$
$$\sigma_{\theta\phi} = 2\mu e_{\theta\phi}$$
$$\sigma_{\phi r} = 2\mu e_{\phi r}$$
$$\sigma_{r\theta} = 2\mu e_{r\theta}$$

$H = e_{rr} + e_{\theta\theta} + e_{\phi\phi}$

λ, μ = Lamé constants

Navier's equation

$$\rho \frac{\partial^2 \hat{u}}{\partial t^2} = \mu \nabla^2 \hat{u} + (\lambda+\mu) \nabla \nabla \cdot \hat{u}$$

Independent solutions

$$\hat{u}^T = \left\{ y_1^T(r)\, T_n^m(\theta,\phi) \right\} e^{i\omega t}$$
$$\hat{u}^P = \left\{ y_1^P(r)\, R_n^m(\theta,\phi) + y_3^P(r)\, S_n^m(\theta,\phi) \right\} e^{i\omega t}$$
$$\hat{u}^S = \left\{ y_1^S(r)\, R_n^m(\theta,\phi) + y_3^S(r)\, S_n^m(\theta,\phi) \right\} e^{i\omega t}$$

$$R_n^m(\theta,\phi) = Y_n^m(\theta,\phi)\, \hat{e}_r$$
$$S_n^m(\theta,\phi) = \frac{\partial Y_n^m}{\partial \theta} \hat{e}_\theta + \frac{1}{\sin\theta} \frac{\partial Y_n^m}{\partial \phi} \hat{e}_\phi$$
$$T_n^m(\theta,\phi) = \frac{1}{\sin\theta} \frac{\partial Y_n^m}{\partial \phi} \hat{e}_\theta - \frac{\partial Y_n^m}{\partial \theta} \hat{e}_\phi$$

Surface vector harmonics in spherical coordinates

$$Y_n^m(\theta,\phi) = P_n^m(\cos\theta)\, e^{im\phi}$$

$P_n^m(\cdot)$ = Legendre function

$$m = 0, \pm 1, \pm 2, \dots, \pm n$$

$$y_1^T(r) = \frac{1}{r}\left\{ kr\, \mathscr{E}_n(kr) \right\}$$
$$y_1^P(r) = \frac{1}{r}\left\{ n\mathscr{E}_n(qr) - qr\, \mathscr{E}_{n+1}(qr) \right\}$$
$$y_3^P(r) = \frac{1}{r}\left\{ \mathscr{E}_n(qr) \right\}$$
$$y_1^S(r) = \frac{1}{r}\left\{ -n(n+1)\, \mathscr{E}_n(kr) \right\}$$
$$y_3^S(r) = \frac{1}{r}\left\{ -(n+1)\, \mathscr{E}_n(kr) + kr\, \mathscr{E}_{n+1}(kr) \right\}$$

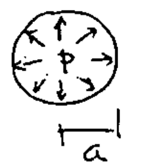
$k = \frac{\omega}{\beta}$ = S-wave number

$q = \frac{\omega}{\alpha}$ = P-wave number

$$\beta = \sqrt{\frac{\mu}{\rho}}$$
$$\alpha = \sqrt{\frac{\lambda+2\mu}{\rho}}$$

$$\mathscr{E}_n(\cdot) = \begin{cases} j_n(\cdot) & \text{Spherical Bessel function of order } n \text{ for a bounded region} \\ h_n^{(2)}(\cdot) & \text{spherical Hankel function of the second class and order } n\text{; for a unbounded region} \end{cases}$$

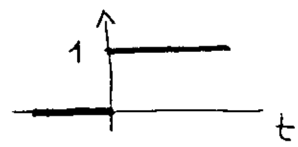
Cavité sphérique soumise à presion interne



$$\frac{\partial^2\phi}{\partial r^2}+\frac{2}{r}\frac{\partial\phi}{\partial r}=\frac{1}{\alpha^2}\frac{\partial^2\phi}{\partial t^2}$$

$$u_r=\frac{\partial\phi}{\partial r}\qquad \sigma_{rr}=(\lambda+2\mu)\frac{\partial^2\phi}{\partial r^2}+\frac{2\lambda}{r}\frac{\partial\phi}{\partial r}$$

$$\sigma_{rr}\Big|_{r=a}=-p\,H(t)$$

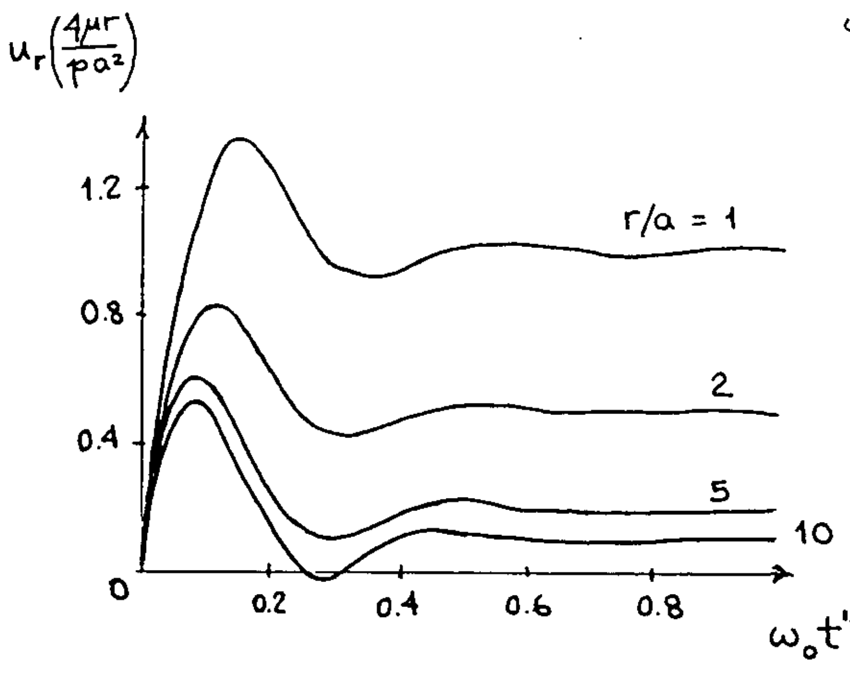


$$\phi=\frac{1}{r}f\left(t-\frac{r-a}{\alpha}\right)=\frac{1}{r}f(t')$$

$$\phi=\frac{pa^3}{4\mu r}\left[e^{-\omega_c K t'}(\cos\omega_c t'+K\sin\omega_c t')-1\right]H(t')$$

$$u_r=\frac{pa^3}{4\mu r}\left[\frac{1}{r}\left\{1-e^{-\omega_0 K t'}(\cos\omega_c t'+K\sin\omega_0 t')\right\}+\frac{(1+K^2)\omega_c}{\alpha}e^{-\omega_c K t'}\sin\omega_0 t'\right]H(t')$$

$$\omega_c=\frac{2\beta^2}{\alpha a K}\,,\quad K=\frac{1}{\sqrt{\alpha^2/\beta^2-1}}$$



$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial z^2}-\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2}=-2\pi\,\delta(x)\,\delta(z)\,e^{-i\omega t}=-\frac{\delta(r)}{r}\,e^{-i\omega t}$$

$$\phi=\bar\phi\,e^{-i\omega t}$$

$$\frac{\partial^2\bar\phi}{\partial x^2}+\frac{\partial^2\bar\phi}{\partial z^2}+q^2\bar\phi=-2\pi\,\delta(x)\,\delta(z)$$

$$\hat f(k)=\int_{-\infty}^{\infty}f(x)\,e^{-ikx}\,dx\quad\longleftrightarrow\quad f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\hat f(k)\,e^{ikx}\,dk$$

$$\tilde f(s)=\int_{-\infty}^{\infty}f(x)\,e^{-sz}\,dz\quad\longleftrightarrow\quad f(z)=\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\hat f(s)\,e^{sz}\,ds$$

$$\left(-k^2+s^2+q^2\right)\hat{\tilde\phi}=-2\pi$$

$$\hat{\tilde\phi}=\frac{-2\pi}{s^2-(k^2-q^2)}\longrightarrow\hat\phi=\pi\,\frac{\exp\left[-\sqrt{k^2-q^2}\,|z|\right]}{\sqrt{k^2-q^2}}$$

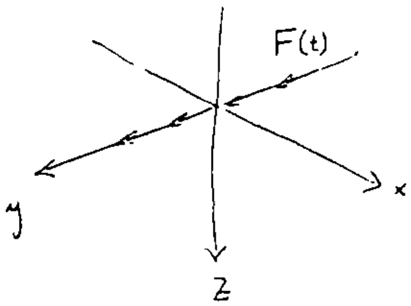
$$\bar\phi=\frac{1}{2}\int_{-\infty}^{\infty}\exp\left[-\sqrt{k^2-q^2}\,|z|+ikx\right]\frac{dk}{\sqrt{k^2-q^2}}=\int_0^{\infty}e^{-\sqrt{\,}|z|}\cos kx\,\frac{dk}{\sqrt{\,}}$$

$$\boxed{\bar\phi=i\,\frac{\pi}{2}\,H_0^{(1)}(qr)}$$

$$\delta(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-i\omega t}\,d\omega$$

$$\phi=\frac{i}{4}\int_{-\infty}^{\infty}H_0^{(1)}\left(\frac{\omega r}{c}\right)e^{-i\omega t}\,d\omega$$

$$\boxed{\phi=\frac{1}{\sqrt{t^2-r^2/c^2}}\,H\left(t-\frac{r}{c}\right)}$$



$$\mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial z^2}\right) - \rho\frac{\partial^2 v}{\partial t^2} = -F(t)\,\delta(x)\,\delta(z)$$

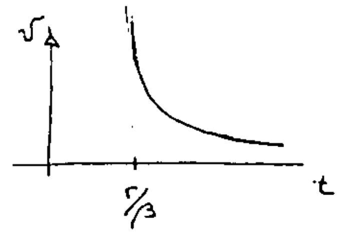
$$F(t) = F(\omega)\,e^{-i\omega t} \qquad v = v\,e^{-i\omega t}$$

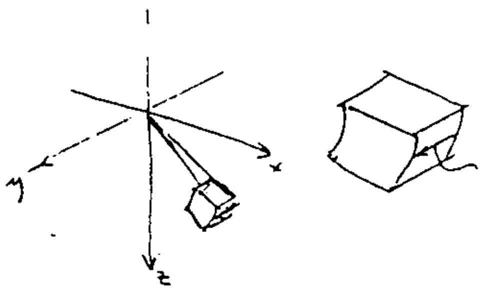
$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial z^2} + q^2 v = -\frac{F(\omega)}{\mu}\,\delta(x)\,\delta(z)$$

$$q = \frac{\omega}{\beta}$$

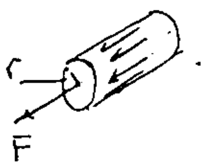
$$\boxed{v = \frac{i}{4}\,\frac{F(\omega)}{\mu}\,H_0^{(1)}(qr)\,e^{-i\omega t}}$$

$$\boxed{v = \frac{1}{2\pi\mu}\,\frac{1}{\sqrt{t^2 - r^2/\beta^2}}\,H\left(t - \frac{r}{\beta}\right)}$$





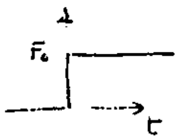
$$\sigma_{ry} = \mu\frac{\partial v}{\partial r} = \frac{i}{4}F(\omega)\,q\,H_0^{(1)\prime}(qr)$$



$$F + \lim_{r\to 0}\left(2\pi r\,\sigma_{rz}\right) = 0$$

$$F + \lim_{r\to 0}\left(2\pi r\,\frac{i}{4}F q\underbrace{\left[-H_1^{(1)}(qr)\right]}_{\frac{i}{\pi}\left(\frac{2}{qr}\right)}\right) = 0$$

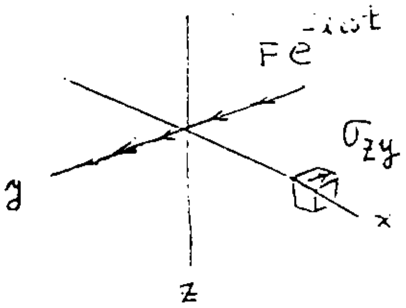
$$F(t) = F_0 H(t)$$



$$v = \frac{F_0}{2\pi\mu}\int_{r/\beta}^{t}\frac{dt}{\sqrt{t^2 - r^2/\beta^2}} = \frac{F_0}{2\pi\mu}\cosh^{-1}\left(\frac{t\beta}{r}\right)H(t - r/\beta) = \frac{F_0}{2\pi\mu}\ln\left[\frac{t\beta}{r} + \sqrt{\left(\frac{t\beta}{r}\right)^2 - 1}\right]H(t - r/\beta)$$

$$\sigma_{ry} = -\frac{F_0}{2\pi r}\,\frac{t}{\sqrt{t^2 - r^2/\beta^2}}\,H(t - r/\beta)$$

Representation d'une source SH avec ondes plans

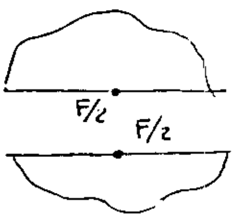


$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial z} + q^2 v = -\frac{F}{\mu}\,\delta(x)\,\delta(z)\,e^{-i\omega t}$$

$$e^{ikx + i\gamma|z|}$$

$$\gamma = \sqrt{q^2 - k^2}$$

$$\mathrm{Im}(\gamma) \geqslant 0$$



$$v = \int_{-\infty}^{\infty} A(k)\, e^{ikx + i\gamma|z|}\, dk$$

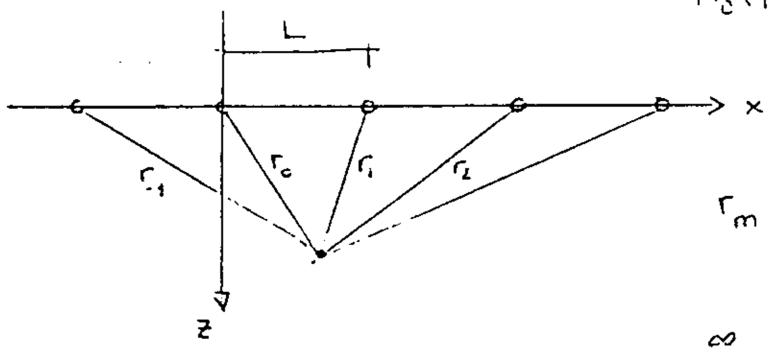
$$\sigma_{zy}\Big|_{z=0^+} = -\frac{F}{2}\,\delta(x) \qquad \sigma_{zy} = \mu\,\frac{\partial v}{\partial z} \qquad \delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\, dk$$

$$\mu i \int_{-\infty}^{\infty} A(k)\,\gamma\, e^{ikx}\, dk = -\frac{F}{4\pi}\int_{-\infty}^{\infty} e^{ikx}\, dk$$

$$A(k) = \frac{i}{4}\,\frac{F}{\mu}\,\frac{1}{\pi}\,\frac{1}{\gamma}$$

$$\boxed{v = \frac{F}{\mu}\,\frac{i}{4}\,\frac{1}{\pi}\int_{-\infty}^{\infty} e^{ikx + i\gamma|z|}\,\frac{dk}{\gamma}}$$

$$H_0^{(1)}(qr) = \frac{1}{\pi}\int_{-\infty}^{\infty} e^{ikx + i\gamma|z|}\,\frac{dk}{\gamma} \qquad \text{Lamb (1904)}$$



$$H_0^{(1)}(qr) = \frac{1}{\pi}\int_{-\infty}^{\infty} e^{ikx + i\gamma|z|}\,\frac{dk}{\gamma}$$

$$\gamma = \sqrt{q^2-k^2} \quad \mathrm{Im}(\gamma) \geqslant 0$$

$$r_m = \sqrt{z^2 + (x-mL)^2}$$

$$\sum_{m=-\infty}^{\infty} H_0^{(1)}(qr_m) = \frac{1}{\pi}\int_{-\infty}^{\infty}\sum_{m=-\infty}^{\infty} e^{ik(x-mL)+i\gamma|z|}\,\frac{dk}{\gamma}$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty}\left(\sum_{m=-\infty}^{\infty} e^{-ikmL}\right) e^{ikx+i\gamma|z|}\,\frac{dk}{\gamma}$$

mais

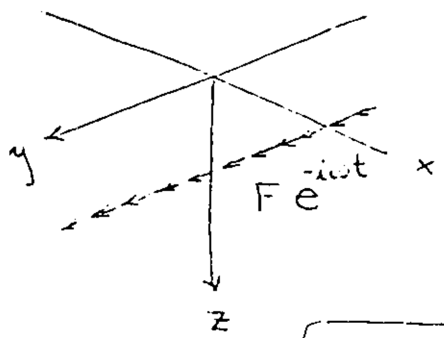
$$\sum_{m=-\infty}^{\infty} e^{-ikmL} \equiv \frac{2\pi}{L}\sum_{n=-\infty}^{\infty}\delta(k-k_n) \qquad \text{avec } k_n = \frac{2\pi}{L}\,n = \Delta k\, n$$

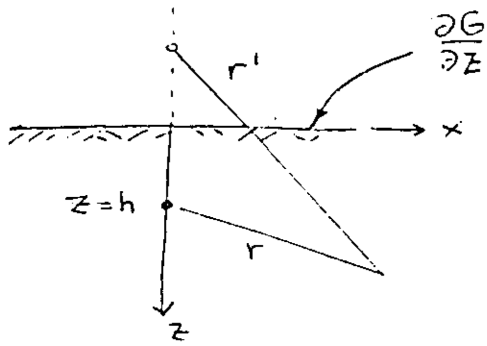
Théorème de Schwarz

$$\boxed{\sum_{m=-\infty}^{\infty} H_0^{(1)}(qr_m) = \frac{1}{\pi}\sum_{n=-\infty}^{\infty} e^{ik_n x + i\gamma_n|z|}\,\frac{\Delta k}{\gamma_n}} \qquad \gamma_n = \sqrt{q^2-k_n^2}$$

$$= \frac{2}{L}\sum_{n=0}^{\infty}\varepsilon_n \cos k_n x \; e^{i\gamma_n|z|}\,\frac{1}{\gamma_n} \qquad \varepsilon_n = \begin{cases}1 & n=0\\ 2 & n>0\end{cases}$$

Méthode des images — Source SH dans un demi space





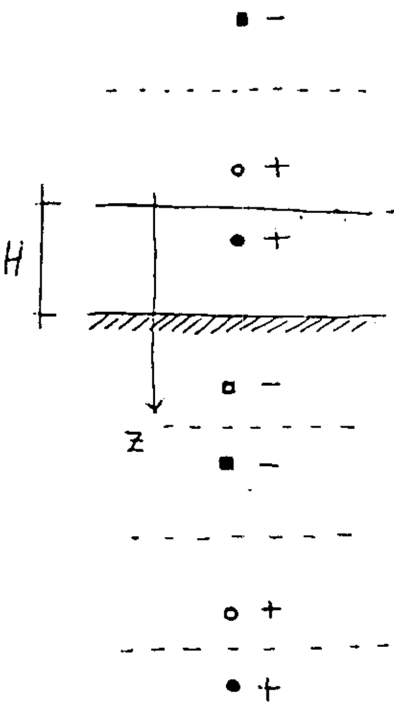
$$\left.\frac{\partial G}{\partial z}\right|_{z=0} = 0$$

$$\boxed{G = \frac{i}{4}\frac{F}{\mu}\left\{ H_0^{(1)}(qr) + H_0^{(1)}(qr')\right\} e^{-i\omega t}}$$

ici $q = \dfrac{\omega}{\beta}$

$$r = \sqrt{x^2 + (z-h)^2} \qquad r' = \sqrt{x^2 + (z+h)^2}$$

Source SH dans une couche à base rigide



$$G = \frac{iF}{4\mu}\left\{ H_0^{(1)}(qr) + \text{images}\right\}$$

$$= \frac{iF}{4\mu}\left\{ \sum_{m=-\infty}^{\infty} H_0^{(1)}\left(q\sqrt{(x-x_0)^2 + (z-[z_0+4mH])^2}\right)\right.$$

$$+ \sum_{m=-\infty}^{\infty} H_0^{(1)}\left(q\sqrt{(x-x_0)^2 + (z-[-z_0+4mH])^2}\right)$$

$$- \sum_{m=-\infty}^{\infty} H_0^{(1)}\left(q\sqrt{(x-x_0)^2 + (z-[2H+z_0+4mH])^2}\right)$$

$$\left. - \sum_{m=-\infty}^{\infty} H_0^{(1)}\left(q\sqrt{(x-x_0)^2 + (z-[2H-z_0+4mH])^2}\right)\right\}$$

$$= \frac{iF}{4\mu}\left\{ \frac{1}{2H}\sum_{n=0}^{\infty} \varepsilon_n \cos\frac{n\pi}{2H}(z-z_0)\, e^{i\gamma_n|x-x_0|}\frac{1}{\gamma_n}\right.$$

$$+ \frac{1}{2H}\sum_{n=0}^{\infty} \varepsilon_n \cos\frac{n\pi}{2H}(z+z_0)\, e^{i\gamma_n|x-x_0|}\frac{1}{\gamma_n}$$

$$- \frac{1}{2H}\sum_{n=0}^{\infty} \varepsilon_n \cos\frac{n\pi}{2H}(z-z_0-2H)\, e^{i\gamma_n|x-x_0|}\frac{1}{\gamma_n}$$

$$\left. - \frac{1}{2H}\sum_{n=0}^{\infty} \varepsilon_n \cos\frac{n\pi}{2H}(z+z_0-2H)\, e^{i\gamma_n|x-x_0|}\frac{1}{\gamma_n}\right\}$$

$(\cos\cos + \sin\sin)[1-(-1)^n]$
$+(\cos\cos - \sin\sin)[1-(-1)^n]$
$= 2\cos\cos[1-(-1)^n] = 4\cos\cos$, $n=1,3,5,\ldots$

$$\boxed{G = i\frac{F}{\mu}\sum_{n=1,3,5,\ldots}^{\infty} \cos\frac{n\pi z_0}{2H}\cos\frac{n\pi z}{2H}\,\frac{\exp(i\gamma_n|x-x_0|)}{\gamma_n H}}$$

$$\gamma_n = \sqrt{q^2 - \left(\frac{n\pi}{2H}\right)^2}$$

$$\operatorname{Im}(\gamma_n) \geq 0$$

$$(\lambda+2\mu)\nabla\nabla\cdot\bar{u} - \mu\nabla\times\nabla\times\bar{u} + \rho\omega^2\bar{u} = -\bar{e}\,F(\omega)\,\underbrace{\frac{\delta(r)}{2\pi r}}$$

$$\delta(x)\delta(z) = \frac{\nabla^2(\ln r)}{2\pi}$$

$$\bar{u} = \nabla(\nabla\cdot\bar{a}_p) - \nabla\times(\nabla\times\bar{a}_s)$$

$$-\bar{e}F\frac{\nabla^2(\ln r)}{2\pi} = \nabla^2\left[-\bar{e}\frac{F}{2\pi}\ln r\right] =$$

$$= \nabla\nabla\cdot\left(-\bar{e}\frac{F\ln r}{2\pi}\right) - \nabla\times\nabla\times\left(-\bar{e}\frac{F\ln r}{2\pi}\right)$$

$$\alpha^2\nabla\nabla\cdot\left[\nabla\nabla\cdot\bar{a}_p - \nabla\times\nabla\times\bar{a}_p + \overbrace{\frac{\omega^2}{\alpha^2}}^{q^2}\bar{a}_p + \bar{e}\frac{F\ln r}{2\pi\rho\alpha^2}\right]$$

$$+\beta^2\nabla\times\nabla\times\left[\nabla\times\nabla\times\bar{a}_s - \nabla\nabla\cdot\bar{a}_s - \underbrace{\frac{\omega^2}{\beta^2}}_{k^2}\bar{a}_s - \bar{e}\frac{F\ln r}{2\pi\rho\beta^2}\right] = 0$$

$$\nabla^2\bar{a}_p + q^2\bar{a}_p = -\bar{e}\frac{F\ln r}{2\pi\rho\alpha^2} \quad\rightarrow\quad \bar{a}_p = \frac{-\bar{e}F}{2\pi\rho\omega^2}\left[\ln r + \frac{i\pi}{2}H_0^{(1)}(qr)\right]$$

$$\nabla^2\bar{a}_s + k^2\bar{a}_s = -\bar{e}\frac{F\ln r}{2\pi\rho\beta^2} \quad\rightarrow\quad \bar{a}_s = \frac{-\bar{e}F}{2\pi\rho\omega^2}\left[\ln r + \frac{i\pi}{2}H_0^{(1)}(kr)\right]$$

$$\bar{u} = -\frac{iF}{4\rho\omega^2}\left\{\nabla\nabla\cdot\left[\bar{e}H_0^{(1)}(qr) - \bar{e}H_0^{(1)}(kr)\right] - \bar{e}k^2H_0^{(1)}(kr)\right\}$$

$$G_{ij} = -\frac{iF}{4\rho\omega^2}\left\{\frac{\partial^2}{\partial x_i\partial x_j}\left[H_0^{(1)}(qr) - H_0^{(1)}(kr)\right] - \delta_{ij}k^2H_0^{(1)}(kr)\right\}$$

$$G_{ij} = \frac{iF}{8\rho}\left\{\delta_{ij}A - \left(2\frac{\partial r}{\partial x_i}\frac{\partial r}{\partial x_j} - \delta_{ij}\right)B\right\}$$

$$A = \frac{H_0^{(1)}(qr)}{\alpha^2} + \frac{H_0^{(1)}(kr)}{\beta^2} \qquad B = \frac{H_2^{(1)}(qr)}{\alpha^2} - \frac{H_2^{(1)}(kr)}{\beta^2}$$

$$G_{ij} = \frac{iF}{4\rho\omega^2}\left\{ -\delta_{ij} k^2 H_0^{(2)}(kr) + \frac{\partial^2}{\partial x_i \partial x_\ell}\left[H_0^{(2)}(qr) - H_0^{(2)}(kr)\right]\delta_{\ell j} \right\} e^{i\omega t}$$

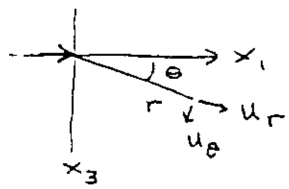
$$k = \frac{\omega}{\beta} \qquad q = \frac{\omega}{\alpha} \qquad \beta = \sqrt{\mu/\rho} \qquad \alpha = \sqrt{(\lambda+2\mu)/\rho}$$

$$G_{ij} = -\frac{iF}{8\rho}\left\{ \delta_{ij} A - \left(2\frac{\partial r}{\partial x_i}\frac{\partial r}{\partial x_j} - \delta_{ij}\right) B \right\} e^{i\omega t}$$

$$A = \frac{H_0^{(2)}(qr)}{\alpha^2} + \frac{H_0^{(2)}(kr)}{\beta^2} \qquad \lim_{\omega\to 0} A = -\frac{i2}{\pi}\left\{\frac{\ln qr}{\alpha^2} + \frac{\ln kr}{\beta^2}\right\} = -\frac{i2}{\pi}\left(\frac{1}{\alpha^2}+\frac{1}{\beta^2}\right)\ln r +$$

$$B = \frac{H_2^{(2)}(qr)}{\alpha^2} - \frac{H_2^{(2)}(kr)}{\beta^2} \qquad \lim_{\omega\to 0} B = \frac{i}{\pi}\left[\frac{4/(q^2r^2)}{\alpha^2} + \frac{1}{\alpha^2} - \frac{4/(k^2r^2)}{\beta^2} - \frac{1}{\beta^2}\right] = \frac{-i}{\pi}\left(\frac{1}{\beta^2} - \frac{1}{\alpha^2}\right)$$

If $j = 1$



$$u_r = \frac{iF}{8\rho}\{B - A\}\cos\theta$$

$$u_\theta = \frac{iF}{8\rho}\{B + A\}\sin\theta$$

$$\sigma_{rr} = \frac{iF}{4r}\left\{-2\beta^2 B + qr H_1^{(2)}(qr)\right\}\cos\theta \qquad \lim_{\omega\to 0} qr H_1^{(2)}(qr) = \frac{i2}{\pi}$$

$$\sigma_{r\theta} = \frac{iF}{4r}\left\{-2\beta^2 B - kr H_1^{(2)}(kr)\right\}\sin\theta$$

$$\sigma_{\theta\theta} = \frac{iF}{4r}\left\{2\beta^2 B + \left(1 - 2\frac{\beta^2}{\alpha^2}\right) qr H_1^{(2)}(qr)\right\}\cos\theta$$

$\omega \to 0$

$$G_{ij} = \frac{F}{2\pi\mu}\left\{\frac{1+\beta^2/\alpha^2}{2}\ln\frac{1}{r}\,\delta_{ij} + \frac{1-\beta^2/\alpha^2}{2}\frac{\partial r}{\partial x_i}\frac{\partial r}{\partial x_j}\right\} + (\text{const})\,\delta_{ij}$$

$$\sigma_{rr} = -\frac{F}{\pi}\frac{\cos\theta}{r}\left(1 - \frac{\beta^2}{2\alpha^2}\right)$$

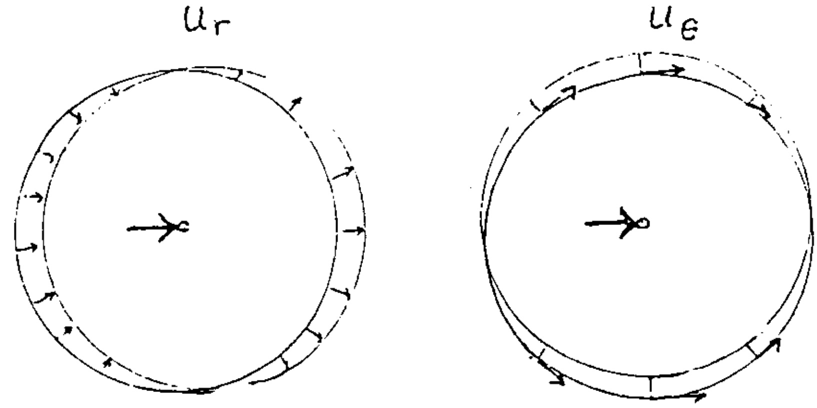
$$\sigma_{r\theta} = \frac{F}{\pi}\frac{\beta^2}{2\alpha^2}\frac{\sin\theta}{r}$$

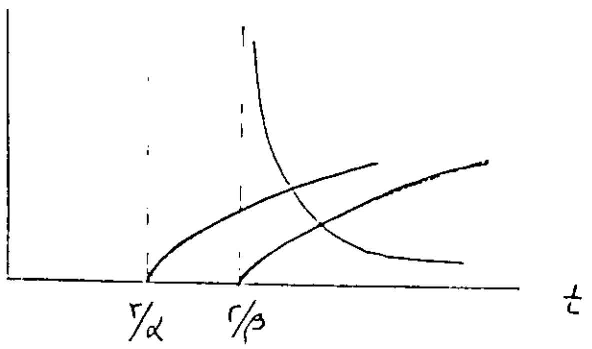
$$\sigma_{\theta\theta} = \frac{F}{\pi}\frac{\cos\theta}{r}\frac{\beta^2}{2\alpha^2}$$

$$\frac{1}{\cos\theta}\, u_r = \frac{F}{2\pi\rho}\left\{ \frac{H(t - r/\alpha)}{\alpha^2\sqrt{t^2 - r^2/\alpha^2}} + \frac{\sqrt{t^2 - r^2/\alpha^2}}{r^2}\, H(t - r/\alpha) - \frac{\sqrt{t^2 - r^2/\beta^2}}{r^2}\, H(t - r/\beta) \right\}$$

$$\frac{1}{\sin\theta}\, u_\theta = \frac{F}{2\pi\rho}\left\{ \frac{\sqrt{t^2 - r^2/\alpha^2}}{r^2}\, H(t - r/\alpha) - \frac{H(t - r/\beta)}{\beta^2\sqrt{t^2 - r^2/\beta^2}} - \frac{\sqrt{t^2 - r^2/\beta^2}}{r^2}\, H(t - r/\beta) \right\}$$

Eason *et al* (1956)





$$\nabla^2 \phi - \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} = -4\pi\, \delta(x)\,\delta(y)\,\delta(z)\, e^{-i\omega t}$$

$$= -2\,\frac{\delta(r)}{r}\, e^{-i\omega t}$$

$$= -\frac{\delta(R)}{R^2}\, e^{-i\omega t}$$

$$\phi = \frac{1}{R} e^{ikR}\, e^{-i\omega t} = ik\, h_0^{(1)}(kR)\, e^{-i\omega t}$$

$$k = \frac{\omega}{c}$$

$$e^{-i\omega t} \rightarrow \delta(t)$$

$$\phi = \frac{\delta(t - R/c)}{R}$$

$$\bar{u} = \nabla\phi(R,\alpha) = \bar{e}_R \frac{\partial \phi}{\partial R} \qquad u_i = \frac{\partial \phi}{\partial R}\gamma_i$$

$$\bar{u} = \bar{e}_\phi \frac{1}{R}\frac{\partial (R\Psi)}{\partial R}$$

$$\bar{u} = -\bar{e}_\theta \frac{1}{R}\frac{\partial (R\Psi)}{\partial R}$$

$$(\lambda+2\mu)\nabla\nabla\cdot\bar{u} - \mu\nabla\times\nabla\times\bar{u} - \rho\frac{\partial^2\bar{u}}{\partial t^2} = -\bar{e}_z f(t)\,\frac{\delta(R)}{R^2}$$

$$\bar{u} = \frac{-F}{4\pi\rho\omega^2}\left\{\nabla\frac{\partial}{\partial z}\left[\frac{e^{iqR}}{R} - \frac{e^{ikR}}{R}\right] - \bar{e}_z k^2\frac{e^{ikR}}{R}\right\}e^{-i\omega t}$$

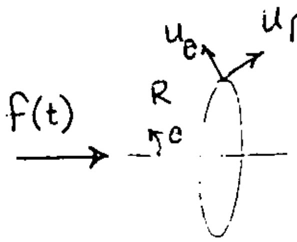
$$u_R = \frac{F\cos\theta}{4\pi\rho}\left\{\frac{e^{iqR}}{\alpha^2 R} + \frac{2i}{\omega R^2}\left(\frac{e^{iqR}}{\alpha} - \frac{e^{ikR}}{\beta}\right) - \frac{2}{\omega^2 R^3}\left(e^{iqR} - e^{ikR}\right)\right\}e^{-i\omega t}$$

$$u_\theta = \frac{F\sin\theta}{4\pi\rho}\left\{-\frac{e^{ikR}}{\beta^2 R} + \frac{i}{\omega R^2}\left(\frac{e^{iqR}}{\alpha} - \frac{e^{ikR}}{\beta}\right) - \frac{1}{\omega^2 R^3}\left(e^{iqR} - e^{ikR}\right)\right\}e^{-i\omega t}$$

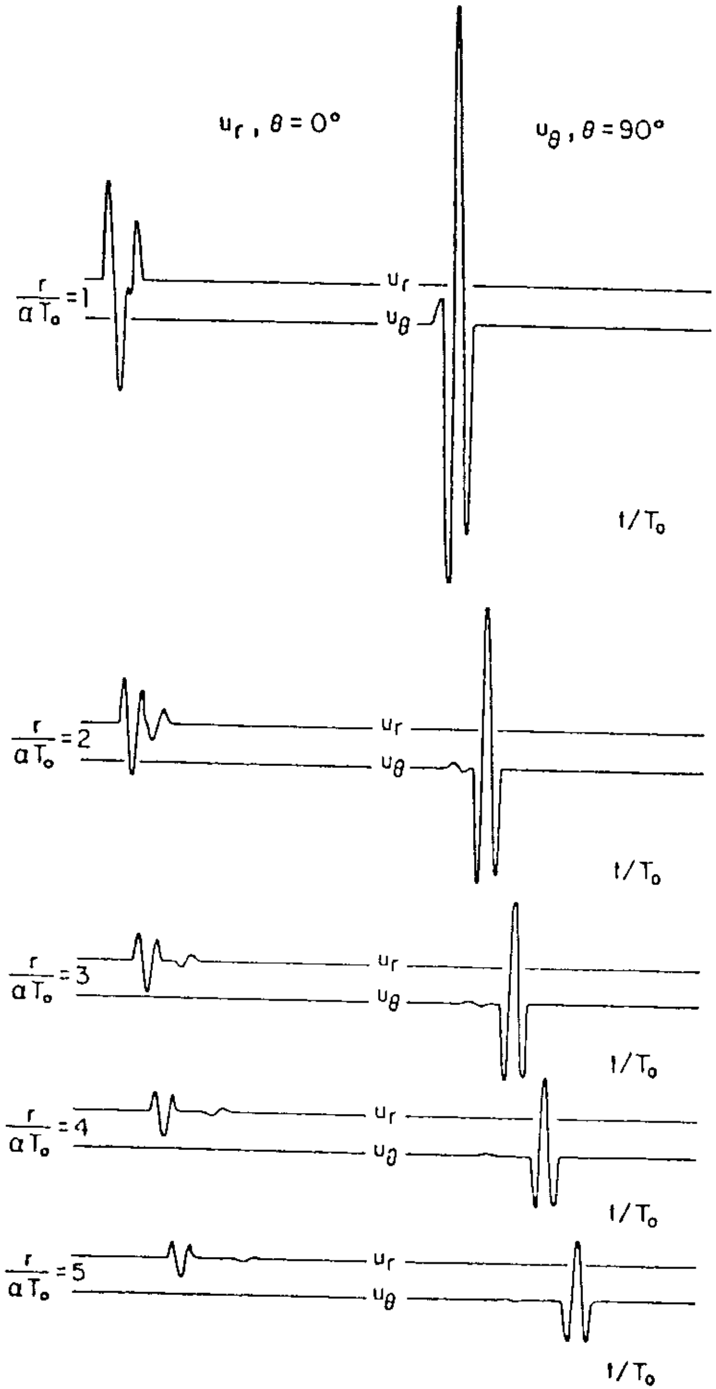
$$F \longleftrightarrow f \qquad -\frac{i}{\omega}F \longleftrightarrow \int f\,dt \qquad -\frac{1}{\omega^2}F \longleftrightarrow \iint f$$

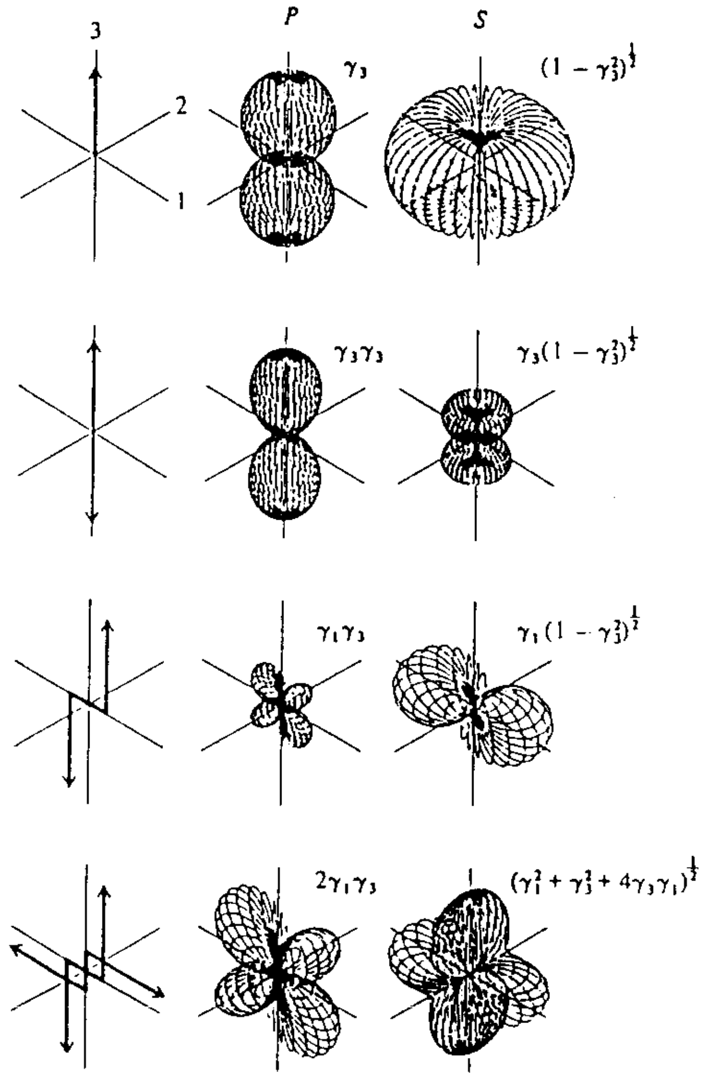
$$u_R = \frac{\cos\theta}{4\pi\rho\alpha^2 R}\left\{f(t-R/\alpha) + \frac{2\alpha}{R}\left(f_1(t-R/\alpha) - \frac{\alpha}{\beta}f_1(t-R/\beta)\right) + 2\frac{\alpha^2}{R^2}\left(f_2(t-R/\alpha) - f_2(t-R/\beta)\right)\right\}$$

$$u_\theta = \frac{\sin\theta}{4\pi\rho\beta^2 R}\left\{-f(t-R/\beta) + \frac{\beta}{R}\left(f_1(t-R/\alpha)\frac{\beta}{\alpha} - f_1(t-R/\beta)\right) + \frac{\beta^2}{R^2}\left(f_2(t-R/\alpha) - f_2(t-R/\beta)\right)\right\}$$

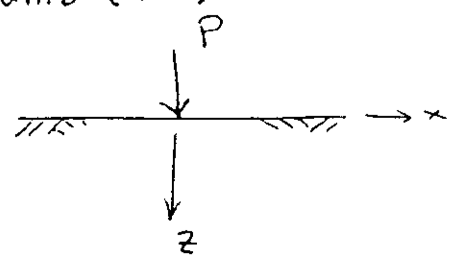


$$f_1(t) = \int^t f(\tau)\,d\tau$$
$$f_2(t) = \int^t f_1(\tau)\,d\tau$$


u_r, $\theta = 0°$
u_θ, $\theta = 90°$
$\frac{r}{\alpha T_0} = 1$
$\frac{r}{\alpha T_0} = 2$
$\frac{r}{\alpha T_0} = 3$
$\frac{r}{\alpha T_0} = 4$
$\frac{r}{\alpha T_0} = 5$
u_r
u_θ
t/T_0


P
S
3
2
1
γ_3
$(1 - \gamma_3^2)^{\frac{1}{2}}$
$\gamma_3\gamma_3$
$\gamma_3(1 - \gamma_3^2)^{\frac{1}{2}}$
$\gamma_1\gamma_3$
$\gamma_1(1 - \gamma_3^2)^{\frac{1}{2}}$
$2\gamma_1\gamma_3$
$(\gamma_1^2 + \gamma_3^2 + 4\gamma_3\gamma_1)^{\frac{1}{2}}$

Lamb (1904)



$$\phi = \int_{-\infty}^{\infty} A(k)\, e^{-ikx - \gamma z}\, dk$$

$$\psi = \int_{-\infty}^{\infty} B(k)\, e^{-ikx - \nu z}\, dk$$

$$u = \frac{\partial \phi}{\partial x} - \frac{\partial \psi}{\partial z} \qquad \nabla^2 \phi + \frac{\omega^2}{\alpha^2}\phi = 0 \qquad \gamma = \sqrt{k^2 - \omega^2/\alpha^2}$$

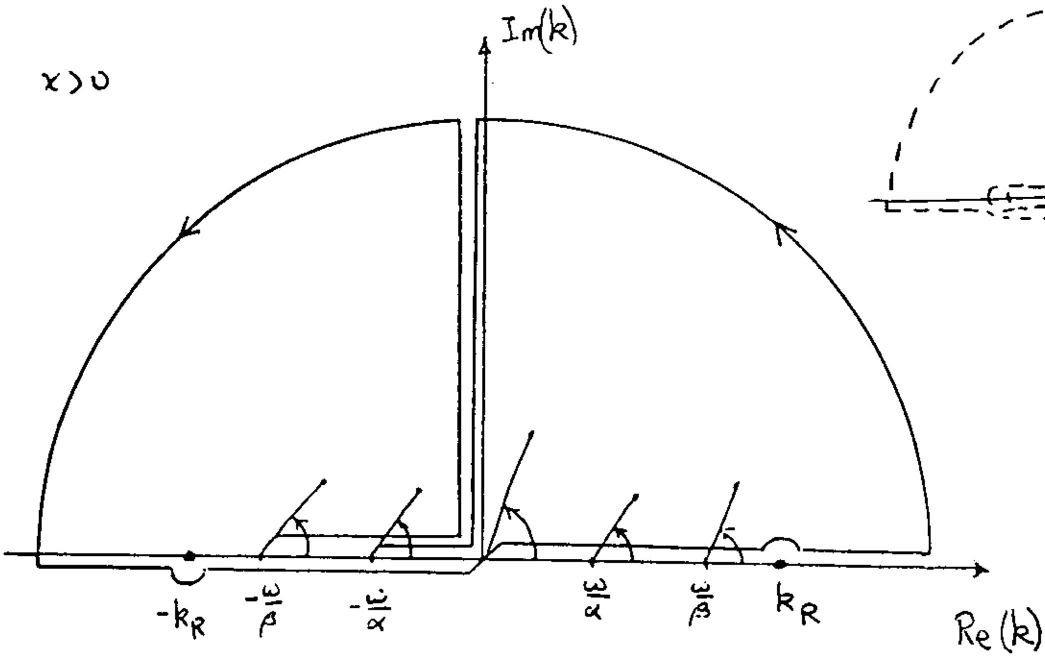
$$w = \frac{\partial \phi}{\partial z} + \frac{\partial \psi}{\partial x} \qquad \nabla^2 \psi + \frac{\omega^2}{\beta^2}\psi = 0 \qquad \nu = \sqrt{k^2 - \omega^2/\beta^2}$$

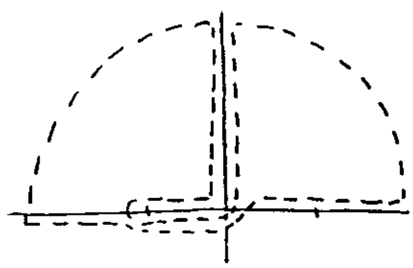
$$\left.\begin{aligned} u &= \int_{-\infty}^{\infty} \left[A\cdot(-ik)\, e^{-\gamma z} - B\cdot(-\nu)\, e^{-\nu z} \right] e^{-ikx}\, dk \\ w &= \int_{-\infty}^{\infty} \left[A\cdot(-\gamma)\, e^{-\gamma z} + B\cdot(-ik)\, e^{-\nu z} \right] e^{-ikx}\, dk \end{aligned}\right\} \qquad \begin{aligned} \sigma_{zx}\big|_{z=0^+} &= 0 \\ \sigma_{zz}\big|_{z=0^+} &= -P\delta(x) \end{aligned}$$

$$u = \frac{P}{i\mu} \int_{-\infty}^{\infty} k \left\{ 2\gamma\nu\, e^{-\nu z} + \left(\frac{\omega^2}{\beta^2} - 2k^2 \right) e^{-\gamma z} \right\} e^{-ikx}\, \frac{dk}{F(k)}$$

$$w = \frac{P}{\mu} \int_{-\infty}^{\infty} \gamma \left\{ 2k^2\, e^{-\nu z} + \left(\frac{\omega^2}{\beta^2} - 2k^2 \right) e^{-\gamma z} \right\} e^{-ikx}\, \frac{dk}{F(k)}$$

$$F(k) = \left(2k^2 - \omega^2/\beta^2 \right)^2 - 4k^2 \gamma \nu$$

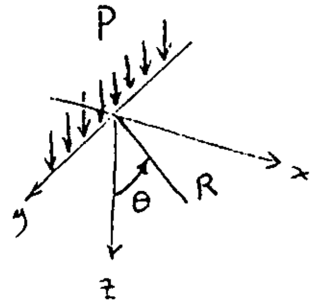




Miller & Pursey (1954)

$$u \sim \frac{P}{2\mu} \cos\theta \, e^{i\pi/4} \sqrt{\frac{2}{\pi q R}} \left\{ -\left(\frac{\alpha}{\beta}\right)^{5/2} \frac{\sin 2\theta \left(\frac{\alpha^2}{\beta^2}\sin^2\theta - 1\right)^{1/2}}{F_0\left(\frac{\alpha}{\beta}\sin\theta\right)} e^{-ikR} + \right.$$

$$\left. + \frac{i \sin\theta \left(\frac{\alpha^2}{\beta^2} - 2\sin^2\theta\right)}{F_0(\quad \sin\theta)} e^{-iqR} \right\}$$



$$w \sim \frac{P}{2\mu} \cos\theta \, e^{i\pi/4} \sqrt{\frac{2}{\pi q R}} \left\{ 2\left(\frac{\alpha}{\beta}\right)^{5/2} \frac{\sin^2\theta \left(\frac{\alpha^2}{\beta^2}\sin^2\theta - 1\right)^{1/2}}{F_0\left(\frac{\alpha}{\beta}\sin\theta\right)} e^{-ikR} + \right.$$

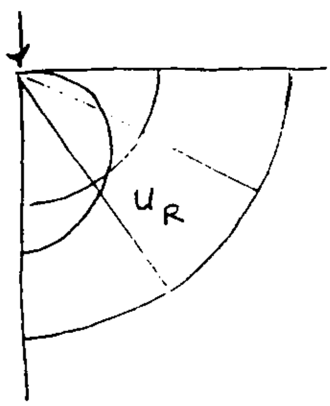
$$\left. + \frac{i \cos\theta \left(\frac{\alpha^2}{\beta^2} - 2\sin^2\theta\right)}{F_0(\quad \sin\theta)} e^{-iqR} \right\}$$

$$\boxed{\begin{aligned} u_R &\sim \frac{P}{2\mu}\sqrt{\frac{2}{\pi q R}} \exp\left[i\left(\frac{3\pi}{4} - qR\right)\right] \frac{\cos\theta\left(\frac{\alpha^2}{\beta^2} - 2\sin^2\theta\right)}{F_0(\sin\theta)} \\ u_\theta &\sim \frac{P}{2\mu}\sqrt{\frac{2}{\pi k R}} \exp\left[i\left(\frac{5\pi}{4} - kR\right)\right] \left(\frac{\alpha}{\beta}\right)^3 \frac{\sin 2\theta\left(\frac{\alpha^2}{\beta^2}\sin^2\theta - 1\right)^{1/2}}{F_0\left(\frac{\alpha}{\beta}\sin\theta\right)} \end{aligned}}$$

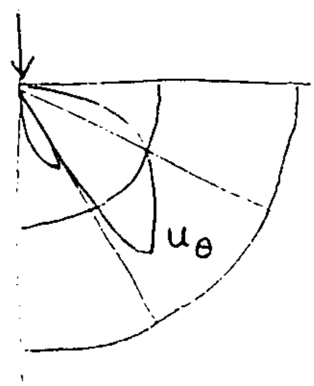
ici $k = \frac{\omega}{\beta}$ $\quad q = \frac{\omega}{\alpha}$ $\quad R = \sqrt{x^2 + z^2}$ $\quad z = R\cos\theta$

$x = R\sin\theta$

$$F_0(\xi) = \left(2\xi^2 - \frac{\alpha^2}{\beta^2}\right)^2 - 4\xi^2(\xi^2 - 1)^{1/2}\left(\xi^2 - \frac{\alpha^2}{\beta^2}\right)^{1/2}$$



$\nu = 1/3$



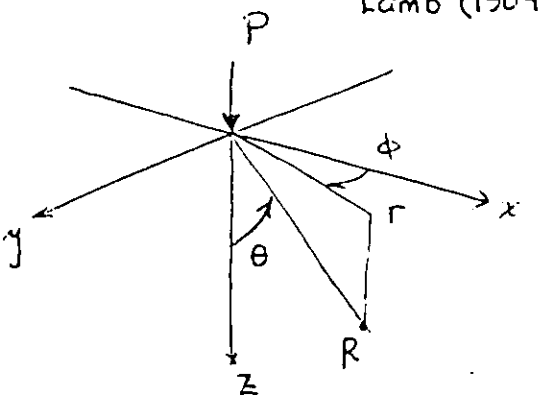
Ondes de Rayleigh (pour $x >>$)

$$\left.\begin{aligned} u(x,0) &\sim 0.125 \frac{P}{\mu} \exp(-i\, 1.884\, q x) \\ w(x,0) &\sim 0.184 \frac{P}{\mu}\, i \exp(-i\, 1.884\, q x) \end{aligned}\right\} \nu = 1/4$$

$$\left.\begin{aligned} u(x,0) &\sim 0.099 \frac{P}{\mu} \exp(-i\, 2.145\, q x) \\ w(x,0) &\sim 0.156 \frac{P}{\mu}\, i \exp(-i\, 2.145\, q x) \end{aligned}\right\} \nu = 1/3$$

3D

Lamb (1904)



Miller & Pursey (pour $R >>$)

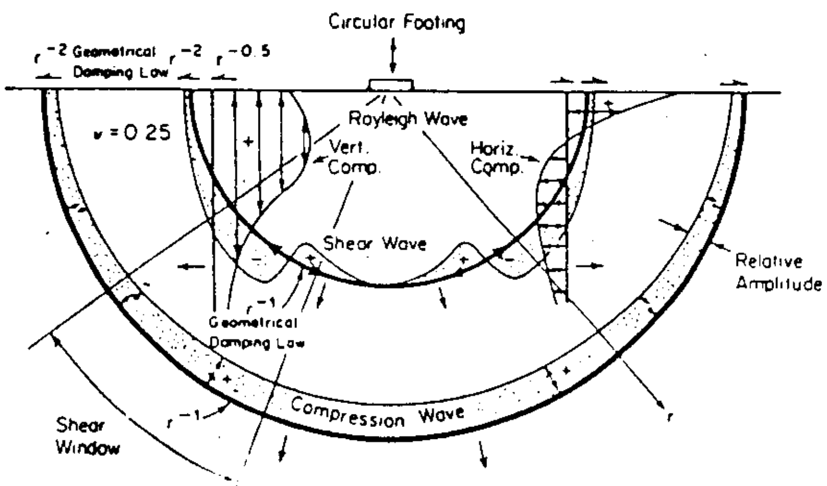
$$\boxed{\begin{aligned} u_R &\sim -\frac{P}{2\pi\mu} \frac{e^{-iqR}}{qR} \frac{\cos\theta\left(\frac{\alpha^2}{\beta^2} - 2\sin^2\theta\right)}{F_0(\sin\theta)} \\ u_\theta &\sim i \frac{P}{2\pi\mu}\left(\frac{\alpha}{\beta}\right)^3 \frac{e^{-ikR}}{qR} \frac{\sin 2\theta\left(\frac{\alpha^2}{\beta^2}\sin^2\theta - 1\right)}{F_0(\sin\theta)} \end{aligned}}$$

Ondes de Rayleigh

0.182 ($\nu = 1/3$)

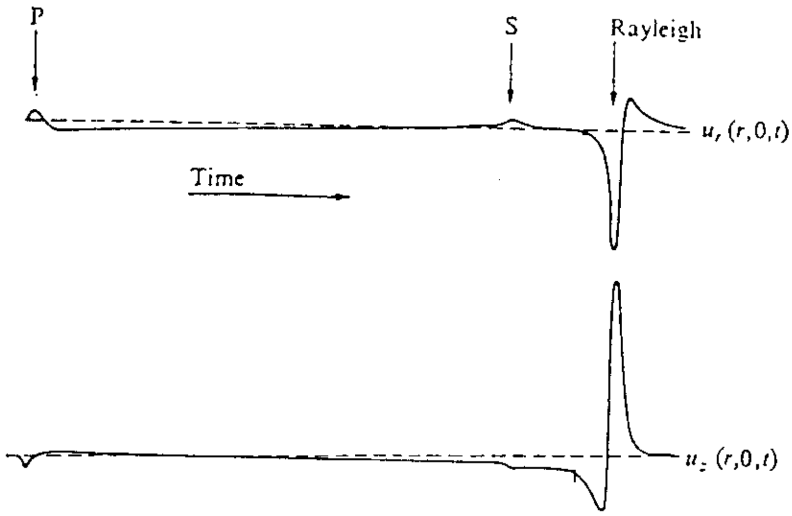
$$\left.\begin{aligned} u_r(r,0) &\sim 0.215 \frac{P}{\pi\mu} \frac{\exp(i\pi/4)}{\sqrt{qr}} \exp(-i\, 1.884\, q r) \\ u_z(r,0) &\sim -0.316 \frac{P}{\pi\mu} \frac{\exp(-i\pi/4)}{\sqrt{qr}} \exp(-i\, 1.884\, q r) \end{aligned}\right\} \nu = 1/4$$

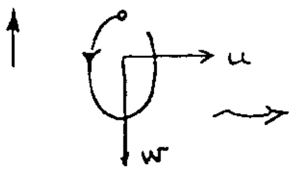
0.286 ($\nu = 1/3$)

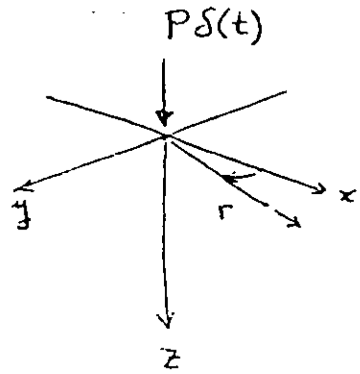


Onde	% Energie
Rayleigh	67
S	26
P	7

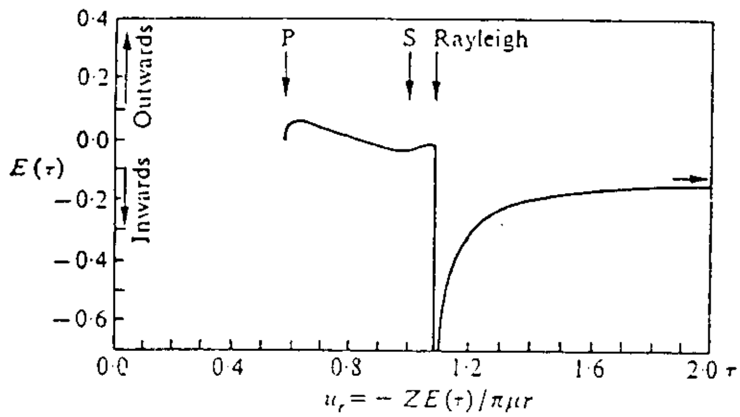
← Woods (1968)



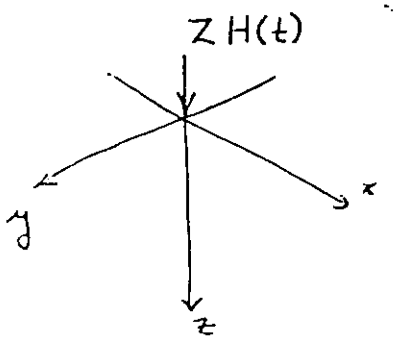




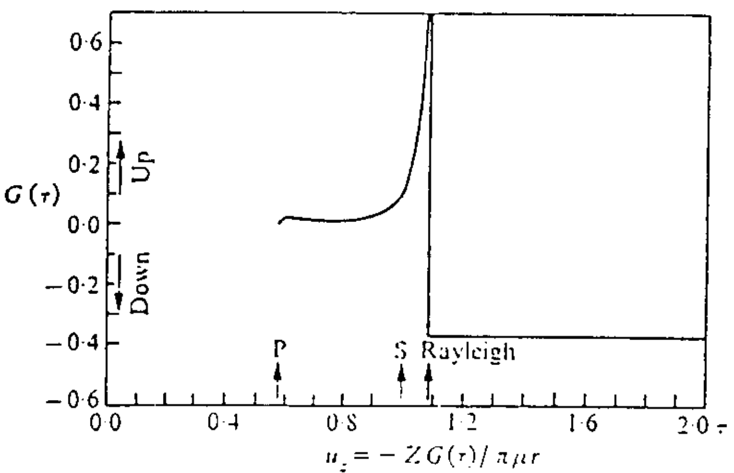
Lamb (1904)



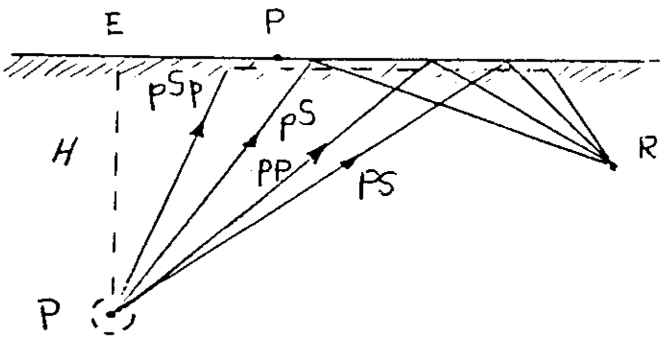
$$\tau = \frac{\beta t}{r}$$

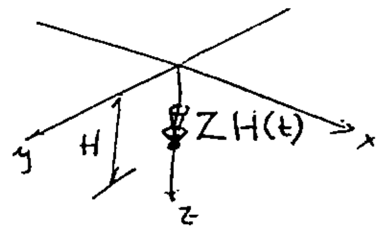


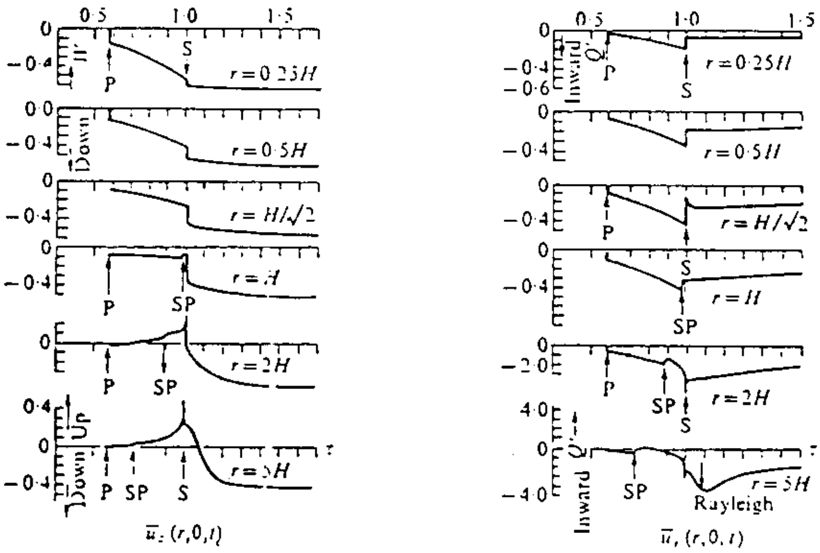
Pekeris (1955)



$$\frac{EP}{c_R} = \frac{\sqrt{H^2+EP^2}}{\alpha}$$



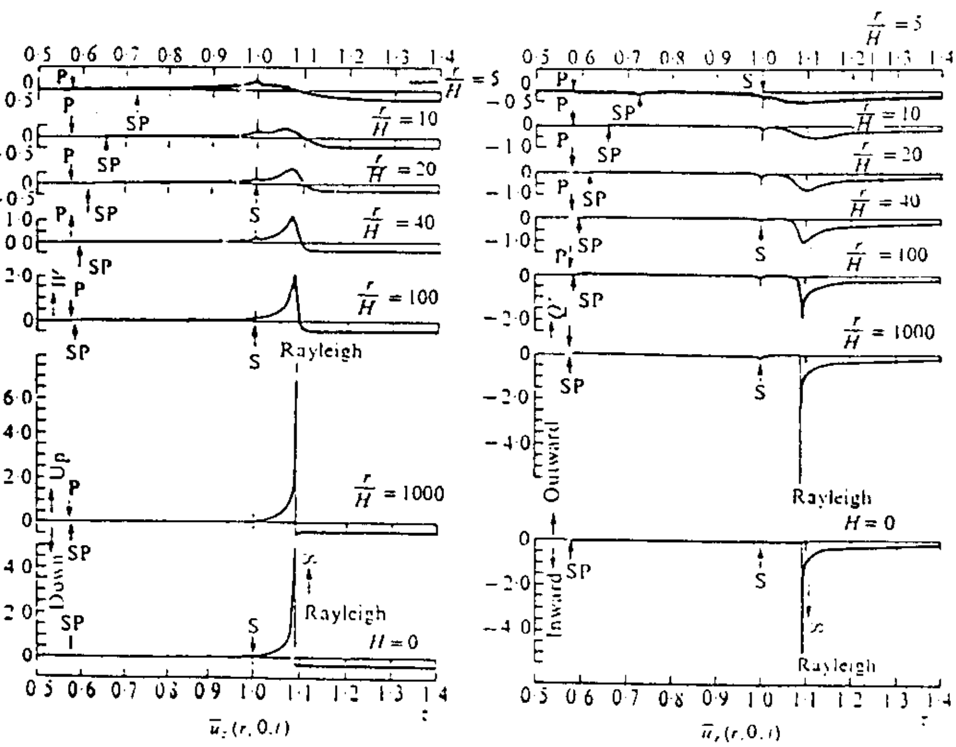




$$u_z = \frac{3Z}{\pi^2 \mu R}\,\bar{u}_z$$

$$u_r = \frac{-3Z}{\pi^2 \mu R}\,\bar{u}_r$$

Pekeris & Lifson (1957)



CHAPTER 4

Elastodynamics

Shoichi KOBAYASHI

School of Civil Engineering
Kyoto University
Kyoto 606, Japan

Boundary Element Methods in Mechanics
Edited by D.E. Beskos

Introduction

The boundary integral equation method (BIEM) or boundary element method (BEM) is one of the most effective numerical methods for elastodynamics. It easily manages unbounded domains and reduces the number of unknowns drastically compared with domain-type techniques. Moreover, it usually provides very accurate results even if it is applied to rather complicated boundary shapes.

The use of integral equation formulations for initial-boundary-value problems has a long history. In relation to elasticity, about a century ago Betti derived the reciprocal theorem (1872), Somigliana formulated the integral representation theorem (Somigliana formula) (1885, 1886) and Kirchhoff derived an expression called the Helmholtz–Kirchhoff integral formula by the use of the retarded time potential (1883) [75].

The theory of integral equations was first studied by Fredholm [39] at the beginning of this century. Since then a great many theoretical contributions have been made by applied mathematicians, specifically of the Tbilisi school. Among them, Kupradze [70, 71] extensively discussed the integral equation method for elastodynamics by the aid of potentials.

Classical works done in the field of elastodynamics are discussed in Achenbach [2], Aki and Richards [3], Ben-Menahem and Singh [8], Eringen and Suhubi [37], Love [75], Pao and Mow [101], Pao and Varatharajulu [102] and Wheeler and Sternberg [131], among others.

Although the basic integral equation formulations for wave problems have been known for a long time, their applications in obtaining numerical solutions of initial-boundary-value problems were begun only recently. Some of the early developments in elastic wave scattering were done by Banaugh [6], Banaugh and Goldsmith [7] and Kupradze [70]. For scattering problems of the horizontally polarized shear (SH) scalar wave, similar integral formulations as in acoustics are used. Readers are advised to see Chapter 9 for its developments.

Cruse and Rizzo [23] and Cruse [24] derived the boundary integral equation formulation in conjunction with the Laplace transform in order to solve the transient wave propagation in a half-plane. A modified version of their approach was developed by Manolis [77] and Manolis and Beskos [79, 80].

The steady-state solutions and the transient responses reconstructed by the Fourier transform synthesis were obtained by Kobayashi and Nishimura [64–67], Niwa et al. [96, 97] and Rizzo et al. [106, 107].

Eigenvalue problems in elastodynamics were studied by Kitahara [57] and Niwa et al. [98]. For SH-wave problems, or for acoustics, see references in Chapter 9.

Integral formulations in the time–space domain were studied by Cole et al. [20], Friedman and Shaw [40] and Mansur and Brebbia [81] for scalar wave problems, by Manolis [77], Mansur and Brebbia [82], Niwa et al. [91] and Shaw [113] for two-dimensional elastic waves and by Karabalis and Beskos [51–53] and Karabalis et al. [54] for three-dimensional elastic waves. Recent developments in the boundary element methods for elastodynamics are also discussed in some articles [27, 61, 106].

In this chapter, first we shall discuss formulations of integral equations both in the time–space domain and the Fourier-transformed domain and then describe numerical implementation and some remarks inherent to elastodynamic problems. Finally, we shall show some application examples, which demonstrate the effectiveness and versatility of boundary integral equation methods.

1. Elastodynamics problems

1.1. Field equations

Let D be the domain in three- or two-dimensional space $\mathbb{R}^N (N = 3 \text{ or } 2)$ occupied by an elastic body, and ∂D be its boundary. The body is asumed to be in a state of motion relative to an inertial frame of reference. Also assume that the density field $\rho(\boldsymbol{x}, t)$ and the velocity field $\boldsymbol{v}(\boldsymbol{x}, t)$ are defined throughout the space–time cylinder $(\boldsymbol{x}, t) \in \mathbb{R}^N \times T$, $T = (-\infty, \infty)$.

The principle of linear momentum with the conservation of mass lead to the Euler equation of motion

$$\nabla \cdot \boldsymbol{\sigma}(\boldsymbol{x}, t) + \rho \boldsymbol{b}(\boldsymbol{x}, t) = \rho \frac{\mathrm{D}}{\mathrm{D}t} \boldsymbol{v}(\boldsymbol{x}, t) , \tag{1.1}$$

where $\boldsymbol{\sigma}$, $\boldsymbol{b}$ stand for the stress tensor and body force vector, respectively, ∇ is the del operator, $\nabla \cdot (\ )$ implies the divergence of $(\ )$, and $\mathrm{D}(\ )/\mathrm{D}t$ denotes the material derivative. The principle of the moment of momentum provides, in addition to this, the symmetry of stress $\boldsymbol{\sigma}$.

Assuming infinitesimal deformations, i.e.

$$\boldsymbol{v} = \frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = \frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{v} \cdot \nabla \boldsymbol{u} \doteq \frac{\partial \boldsymbol{u}}{\partial t} , \qquad \frac{\mathrm{D}\boldsymbol{v}}{\mathrm{D}t} = \frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v} \cdot \nabla \boldsymbol{v} \doteq \frac{\partial \boldsymbol{v}}{\partial t} , \tag{1.2}$$

where $\boldsymbol{u}$ stands for displacement, Eq. (1.1) reduces to the Cauchy equation of motion

$$\nabla \cdot \boldsymbol{\sigma}(\boldsymbol{x}, t) + \rho \boldsymbol{b}(\boldsymbol{x}, t) = \rho \ddot{\boldsymbol{u}}(\boldsymbol{x}, t) , \tag{1.3}$$

where $(\dot{\ }) = \partial(\ )/\partial t$, i.e. the dot denotes differentiation with respect to time.

The kinematical relation for the infinitesimal deformations is given by

$$\boldsymbol{\varepsilon}(\boldsymbol{x}, t) = \tfrac{1}{2}\{\nabla \boldsymbol{u}(\boldsymbol{x}, t) + \boldsymbol{u}(\boldsymbol{x}, t)\nabla\} \tag{1.4}$$

and the generalized Hooke's law for the linear elastic medium is

$$\boldsymbol{\sigma}(\boldsymbol{x}, t) = \mathbf{C} : \boldsymbol{\varepsilon}(\boldsymbol{x}, t) , \tag{1.5a}$$

where $\boldsymbol{\varepsilon}$, $\mathbf{C}$ stand for the strain tensor and elastic tensor, respectively, and the double dot implies the double inner product. When the medium is isotropic, we have

$$\boldsymbol{\sigma}(\boldsymbol{x}, t) = \lambda(\boldsymbol{x}) \operatorname{tr} \boldsymbol{\varepsilon}(\boldsymbol{x}, t)\mathbf{1} + 2\mu(\boldsymbol{x})\boldsymbol{\varepsilon}(\boldsymbol{x}, t) , \tag{1.5b}$$

where $\operatorname{tr} \boldsymbol{\varepsilon}$ means the trace of $\boldsymbol{\varepsilon}$, $\mathbf{1}$ denotes the unit tensor, and λ and μ are the Lamé constants.

Substitution of Eq. (1.4) into Eq. (1.5a) or (1.5b) leads to

$$\boldsymbol{\sigma} = \mathbf{C} : \nabla \boldsymbol{u} \tag{1.6a}$$

or

$$\boldsymbol{\sigma} = \lambda \nabla \cdot \boldsymbol{u}\mathbf{1} + \mu(\nabla \boldsymbol{u} + \boldsymbol{u}\nabla) . \tag{1.6b}$$

Substituting further Eq. (1.6a) into Eq. (1.3), we have the Navier–Cauchy equation of motion

$$\nabla \cdot (\mathbf{C} : \nabla \boldsymbol{u}) + \rho \boldsymbol{b} = \rho \ddot{\boldsymbol{u}} . \tag{1.7a}$$

For the isotropic medium, we have

$$\mu \nabla^2 \boldsymbol{u} + (\lambda + \mu)\nabla\nabla \cdot \boldsymbol{u} + (\nabla\lambda)\nabla \cdot \boldsymbol{u} + \nabla\mu \cdot (\nabla \boldsymbol{u} + \boldsymbol{u}\nabla) + \rho \boldsymbol{b} = \rho \ddot{\boldsymbol{u}} . \tag{1.7b}$$

In addition, when the medium is homogeneous, Eq. (1.7b) reduces to

$$\mu \nabla^2 \boldsymbol{u} + (\lambda + \mu)\nabla\nabla \cdot \boldsymbol{u} + \rho \boldsymbol{b} = \mathbf{E}\boldsymbol{u} + \rho \boldsymbol{b} = \rho \ddot{\boldsymbol{u}} \tag{1.8a}$$

or, equivalently

$$(\lambda + 2\mu)\nabla\nabla \cdot \boldsymbol{u} - \mu \nabla \times \nabla \times \boldsymbol{u} + \rho \boldsymbol{b} = \rho \ddot{\boldsymbol{u}} , \tag{1.8b}$$

where use was made of

$$\nabla^2 \boldsymbol{u} = \nabla\nabla \cdot \boldsymbol{u} - \nabla \times \nabla \times \boldsymbol{u} .$$

In what follows, we mainly consider the homogeneous and isotropic elastic **medium and** sometimes disregard the body force, for simplicity. We also use the **indicial** notation referred to the Cartesian coordinate system $(0; x_1, x_2, x_3)$ interchangeably. In that case, the summation convention rule is implied over repeated indices. For example, $\boldsymbol{u} = u_k \boldsymbol{i}_k$, $\nabla(\ ) = (\ )_{,k}\boldsymbol{i}_k = \partial_k(\ )\boldsymbol{i}_k = \partial(\ )/\partial x_k \boldsymbol{i}_k$ and $\nabla \cdot \boldsymbol{\sigma}$, $\mathbf{C} : \boldsymbol{\varepsilon}$ imply $\sigma_{ij,i}$, $C_{ijkl}\varepsilon_{kl}$ in component form, where $\boldsymbol{i}_k$ is the unit base vector.

Now making use of the Stokes–Helmholtz decomposition of the displacement vector

$$\boldsymbol{u} = \boldsymbol{u}^{L} + \boldsymbol{u}^{T} , \qquad \boldsymbol{u}^{L} = \nabla\phi , \qquad \boldsymbol{u}^{T} = \nabla \times \boldsymbol{\psi} \tag{1.9}$$

where ϕ and $\boldsymbol{\psi}$ are scalar and vector potentials (called Lamé potentials), Eq. (1.8b) is satisfied if the following equations are fulfilled:

$$c_L^2 \nabla^2 \phi = \ddot{\phi} , \tag{1.10}$$

$$c_T^2 \nabla^2 \boldsymbol{\psi} = \ddot{\boldsymbol{\psi}} \tag{1.11}$$

where

$$c_L = \left(\frac{\lambda + 2\mu}{\rho}\right)^{1/2} , \qquad c_T = \left(\frac{\mu}{\rho}\right)^{1/2} . \tag{1.12}$$

Such solutions are complete [117].

By taking the divergence and curl of Eq. (1.8b), we know that the Navier–Cauchy equation of motion is satisfied whenever the following equations are assured:

$$c_L^2 \nabla^2 \boldsymbol{u}^{L} = \ddot{\boldsymbol{u}}^{L} , \qquad \nabla \times \boldsymbol{u}^{L} = \mathbf{0} \tag{1.13}$$

and

$$c_T^2 \nabla^2 \boldsymbol{u}^{T} = \ddot{\boldsymbol{u}}^{T} , \qquad \nabla \cdot \boldsymbol{u}^{T} = 0 . \tag{1.14}$$

These equations imply that $\boldsymbol{u}^{L}$ and $\boldsymbol{u}^{T}$ define the dilatational (longitudinal) and the distortional (equivolumetric, or shear) waves, respectively, and hence c_L and c_T stand for the phase speeds of the respective waves.

In the two-dimensional case, when the displacement is assumed in the form

$$\boldsymbol{u}(\boldsymbol{x}, t) = u_\alpha(x_1, x_2, t)\boldsymbol{i}_\alpha + u_3(x_1, x_2, t)\boldsymbol{i}_3 , \qquad (\alpha = 1, 2) \tag{1.15}$$

with $\boldsymbol{i}_\alpha$ being the unit base vector, we have the following equations of motion:

$$\mu u_{\alpha,\beta\beta} + (\lambda + \mu)u_{\beta,\beta\alpha} = \rho \ddot{u}_\alpha , \qquad (\alpha, \beta = 1, 2) , \tag{1.16}$$

$$\mu u_{3,\alpha\alpha} = \rho \ddot{u}_3 , \qquad (\alpha = 1, 2) . \tag{1.17}$$

Equation (1.16) expresses the in-plane (x_1x_2 plane) motion and Eq. (1.17) implies the anti-plane ($\boldsymbol{i}_3$ direction) motion. The corresponding stress states are given as follows:

$$\sigma_{\alpha\beta} = \lambda u_{\gamma,\gamma}\delta_{\alpha\beta} + \mu(u_{\alpha,\beta} + u_{\beta,\alpha}), \qquad \sigma_{33} = \lambda u_{\alpha,\alpha}, \qquad (\alpha, \beta, \gamma = 1, 2), \tag{1.18}$$

$$\sigma_{3\alpha} = \mu u_{3,\alpha}, \qquad (\alpha = 1, 2), \tag{1.19}$$

where $\delta_{\alpha\beta}$ denotes the Kronecker delta.

Making use of the Stokes–Helmholtz decomposition

$$\boldsymbol{u} = \boldsymbol{u}^L + \boldsymbol{u}^T + u_3\boldsymbol{i}_3,$$

we have the following wave equations from Eqs. (1.16) and (1.17):

$$\left.\begin{aligned} c_L^2\nabla^2\boldsymbol{u}^L &= \ddot{\boldsymbol{u}}^L, \qquad \nabla\times\boldsymbol{u}^L = \boldsymbol{0}, \\ c_T^2\nabla^2\boldsymbol{u}^T &= \ddot{\boldsymbol{u}}^T, \qquad \nabla\cdot\boldsymbol{u}^T = 0, \end{aligned}\right\} \tag{1.20}$$

$$c_T^2\nabla^2 u_3 = \ddot{u}_3. \tag{1.21}$$

Equations (1.20) imply that the in-plane motion obeys the system of equations of the dilatational and distorsional waves, whereas Eq. (1.21) shows that the anti-plane motion is governed only by the distorsional wave motion. The former distorsional wave is often called the SV wave (vertically polarized shear wave), while the latter is termed the SH wave (horizontally polarized shear wave).

1.2. Elastodynamic problems

The general elastodynamic problem is defined such as to find solutions of Eq. (1.8) with initial conditions

$$\boldsymbol{u}(\boldsymbol{x}, 0) = \boldsymbol{u}_0(\boldsymbol{x}), \qquad \dot{\boldsymbol{u}}(\boldsymbol{x}, 0) = \dot{\boldsymbol{u}}_0(\boldsymbol{x}) \tag{1.22}$$

and boundary conditions

$$\begin{aligned} \boldsymbol{u}(\boldsymbol{x}, t) &= \bar{\boldsymbol{u}}(\boldsymbol{x}, t), && (\boldsymbol{x}, t) \in \partial D_1 \times T, \\ \boldsymbol{t}(\boldsymbol{x}, t) &= \boldsymbol{n}\cdot\boldsymbol{\sigma} = \overset{n}{\mathbf{T}}\boldsymbol{u}(\boldsymbol{x}, t) && (\boldsymbol{x}, t) \in \partial D_2 \times T, \\ &= \bar{\boldsymbol{t}}(\boldsymbol{x}, t), && \partial D = \partial D_1 \cup \partial D_2, \end{aligned} \tag{1.23}$$

where $\boldsymbol{u}_0$, $\dot{\boldsymbol{u}}_0$ are given initial displacement and velocity, $\bar{\boldsymbol{u}}$, $\bar{\boldsymbol{t}}$ are prescribed displacement on ∂D_1 and traction on ∂D_2, and $\boldsymbol{n}$ denotes the unit outward normal

vector to ∂D. The operator $\overset{n}{\mathbf{T}}$ is defined as

$$\begin{aligned} \overset{n}{\mathbf{T}}\boldsymbol{u} &= \boldsymbol{n}\cdot\mathbf{C} : \nabla\boldsymbol{u} \\ &= \lambda\boldsymbol{n}(\nabla\cdot\boldsymbol{u})\mathbf{I} + \mu\boldsymbol{n}\cdot(\nabla\boldsymbol{u} + \boldsymbol{u}\nabla) \\ &= \{\lambda\boldsymbol{n}\nabla\cdot + 2\mu(\boldsymbol{n}\cdot\nabla)\mathbf{I} + \mu\boldsymbol{n}\times\nabla\times\}\boldsymbol{u}. \end{aligned} \tag{1.24}$$

In the two-dimensional problem, we have to solve Eq. (1.16) or (1.17) instead of Eq. (1.8) with two-dimensional initial and boundary conditions, in which the tractions are given for the inplane case by

$$t_\beta = \overset{n}{T}_{\beta\alpha}u_\alpha = \lambda n_\beta u_{\gamma,\gamma} + \mu n_\alpha(u_{\alpha,\beta} + u_{\beta,\alpha}), \qquad (\alpha, \beta, \gamma = 1, 2) \tag{1.25}$$

and for the anti-plane case by

$$t_3 = \overset{n}{T}u_3 = \mu n_\alpha u_{3,\alpha} = \mu\frac{\partial u_3}{\partial n}, \qquad (\alpha = 1, 2). \tag{1.26}$$

For the exterior problem which includes the boundary extended to infinity, it is reasonable to require physically that the displacement at infinity must be bounded and that no wave can be reflected back from infinity. These conditions, called radiation conditions, are expressed as follows [37, 70, 114]:

$$\begin{aligned} \frac{\partial\boldsymbol{u}^L}{\partial r} + \frac{1}{c_L}\frac{\partial\boldsymbol{u}^L}{\partial t} &= o(r^{-(N-1)/2}), \qquad \boldsymbol{u}^L = o(r^{-(N-3)/2}) \\ \frac{\partial\boldsymbol{u}^T}{\partial r} + \frac{1}{c_T}\frac{\partial\boldsymbol{u}^T}{\partial t} &= o(r^{-(N-1)/2}), \qquad \boldsymbol{u}^T = o(r^{-(N-3)/2}) \end{aligned} \tag{1.27}$$

as $r = |\boldsymbol{x}| \to \infty$. In these expressions, N implies the number of dimensions ($N = 2$ or 3), and o denotes Landau's symbol, that is, $f(x) = o[g(x)]$ implies that $\lim f(x)/g(x) \to 0$ as $x \to a$, where a is a fixed value.

Radiation conditions for elastodynamics may be alternatively expressed as [28, 37, 70]:

$$\begin{aligned} &\overset{n}{\mathbf{T}}\boldsymbol{u}^L + \rho c_L\dot{\boldsymbol{u}}^L = o(r^{-(N-1)/2}), \qquad \overset{n}{\mathbf{T}}\boldsymbol{u}^T + \rho c_T\dot{\boldsymbol{u}}^T = o(r^{-(N-1)/2}), \\ &\dot{u}^L_t = o(r^{-(N-1)/2}), \qquad \dot{u}^T_t = o(r^{-(N-1)/2}) \\ &\dot{u}^L_n = o(r^{-(N-3)/2}), \qquad \dot{u}^T_n = o(r^{-(N-3)/2}), \end{aligned} \tag{1.28}$$

where subscripts n and t mean the normal and tangential components of the velocity vector, respectively, on the infinite sphere when r approaches infinity.

The first expression, for instance for $N = 3$, may be rewritten as [28, 37]

$$\lim_{R\to\infty}\int_{S_R}|\overset{n}{\mathbf{T}}\boldsymbol{u}^{\mathrm{L}}+\rho c_{\mathrm{L}}\dot{\boldsymbol{u}}^{\mathrm{L}}|^2\,\mathrm{d}S=0$$

and thus

$$\lim_{R\to\infty}\int_{S_R}\overset{n}{\mathbf{T}}\boldsymbol{u}^{\mathrm{L}}\cdot\dot{\boldsymbol{u}}^{\mathrm{L}}\,\mathrm{d}S=-\lim_{R\to\infty}\left(\frac{1}{2\rho c_{\mathrm{L}}}\right)\int_{S_R}\{|\overset{n}{\mathbf{T}}\boldsymbol{u}^{\mathrm{L}}|^2+\rho^2c_{\mathrm{L}}^2|\dot{\boldsymbol{u}}^{\mathrm{L}}|^2\}\,\mathrm{d}S<0\,.$$

Hence the efflux of energy per unit time across the surface S

$$P=-\lim_{R\to\infty}\int_{S_R}\{\overset{n}{\mathbf{T}}\boldsymbol{u}^{\mathrm{L}}\cdot\dot{\boldsymbol{u}}^{\mathrm{L}}+\overset{n}{\mathbf{T}}\boldsymbol{u}^{\mathrm{T}}\cdot\dot{\boldsymbol{u}}^{\mathrm{T}}\}\,\mathrm{d}S$$

is proved to be positive. Therefore, the radiation conditions of elastodynamics guarantee that an outward flow of energy is only allowed at infinity, which implies no reflection of waves from infinity.

The radiation conditions are crucial in determing the unique solution in unbounded domains, particularly in time-harmonic problems.

1.3. *Fourier-transformed elastodynamic problem*

It is sometimes convenient to solve elastodynamic problems with the aid of the Fourier transform with respect to time [61, 64, 66, 70]. Introducing the Fourier transform defined by

$$\hat{f}(\boldsymbol{x};\omega)=\int_{-\infty}^{\infty}f(\boldsymbol{x},t)\,\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{d}t\,,\tag{1.29}$$

we have three-dimensional elastodynamic problems expressed in the transformed domain as follows (body force is disregarded for simplicity):

$$\begin{aligned}&\mu\nabla^2\hat{\boldsymbol{u}}(\boldsymbol{x};\omega)+(\lambda+\mu)\nabla\nabla\cdot\hat{\boldsymbol{u}}(\boldsymbol{x};\omega)+\rho\omega^2\hat{\boldsymbol{u}}(\boldsymbol{x};\omega)\\&\quad=(\mathbf{E}+\rho\omega^2\mathbf{1})\hat{\boldsymbol{u}}(\boldsymbol{x};\omega)=\mathbf{L}\hat{\boldsymbol{u}}(\boldsymbol{x};\omega)=\mathbf{0}\,,\end{aligned}\tag{1.30}$$

$$\left.\begin{aligned}&\hat{\boldsymbol{u}}(\boldsymbol{x};\omega)=\hat{\bar{\boldsymbol{u}}}(\boldsymbol{x};\omega)\,, && \boldsymbol{x}\in\partial D_1\,,\\&\hat{\boldsymbol{t}}(\boldsymbol{x};\omega)=\overset{n}{\mathbf{T}}\hat{\boldsymbol{u}}(\boldsymbol{x};\omega)=\hat{\bar{\boldsymbol{t}}}(\boldsymbol{x};\omega)\,, && \boldsymbol{x}\in\partial D_2\,.\end{aligned}\right\}\tag{1.31}$$

Similar expressions are obtained for the two-dimensional problems.

The transformed problems are of a more convenient form since they are of elliptic type and similar solution techniques as for elastostatics can be available. Solutions of the original problem can be obtained by inverting the solutions obtained in the transformed domain, i.e. by

$$\boldsymbol{u}(\boldsymbol{x},t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{\boldsymbol{u}}(\boldsymbol{x};\omega)\,\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}\omega\,.\tag{1.32}$$

As for the unbounded domain, the Fourier-transformed radiation conditions must be satisfied [65, 70], i.e. for example,

$$\begin{aligned}&\frac{\partial\hat{\boldsymbol{u}}^{\mathrm{L}}}{\partial r}-\frac{\mathrm{i}\omega}{c_{\mathrm{L}}}\hat{\boldsymbol{u}}^{\mathrm{L}}=o(r^{-(N-1)/2})\,, &&\hat{\boldsymbol{u}}^{\mathrm{L}}=o(r^{-(N-3)/2})\,,\\&\frac{\partial\hat{\boldsymbol{u}}^{\mathrm{T}}}{\partial r}-\frac{\mathrm{i}\omega}{c_{\mathrm{T}}}\hat{\boldsymbol{u}}^{\mathrm{T}}=o(r^{-(N-1)/2})\,, &&\hat{\boldsymbol{u}}^{\mathrm{T}}=o(r^{-(N-3)/2})\,.\end{aligned}\tag{1.33}$$

1.4. *Fundamental solutions*

1.4.1. *General concept*

Fundamental solutions are of basic importance in the formulation of integral equations. The fundamental solution, which is sometimes called the free-space Green's function or spectral Green's tensor, is defined as the displacement field at $\boldsymbol{x}$ and at time t in the infinite domain due to an unit impulse applied at $\boldsymbol{x}=\boldsymbol{y}$ and $t=\tau$. If we denote the ith component of the displacement at $\boldsymbol{x}$ and at time t due to the unit impulse in the k-direction by $U_{ik}(\boldsymbol{x},t;\boldsymbol{y},\tau)$, it satisfies the equation

$$\frac{\partial}{\partial x_j}\left(C_{ijmn}\frac{\partial}{\partial x_n}U_{mk}\right)+\delta_{ik}\delta(\boldsymbol{x}-\boldsymbol{y})\delta(t-\tau)=\rho\ddot{U}_{ik}\tag{1.34}$$

for the general case and

$$\mu\frac{\partial^2}{\partial x_j\partial x_j}U_{ik}+(\lambda+\mu)\frac{\partial^2}{\partial x_i\partial x_j}U_{jk}+\delta_{ik}\delta(\boldsymbol{x}-\boldsymbol{y})\delta(t-\tau)=\rho\ddot{U}_{ik}\tag{1.35}$$

for the isotropic homogeneous case, where $\delta(\ )$ means the Dirac delta function. The same definition is also made for the two-dimensional case.

Hereafter we assume the validity of the causality requirement such that $\boldsymbol{U}(\boldsymbol{x},t;\boldsymbol{y},\tau)$ and $\partial\boldsymbol{U}(\boldsymbol{x},t;\boldsymbol{y},\tau)/\partial t$ must be zero for $t\leq\tau$ and $\boldsymbol{x}\neq\boldsymbol{y}$. It is easily recognized from Eq. (1.34) or (1.35) that the time origin can be shifted at will and hence

$$\boldsymbol{U}(\boldsymbol{x},t;\boldsymbol{y},\tau)=\boldsymbol{U}(\boldsymbol{x},t-\tau;\boldsymbol{y},0)=\boldsymbol{U}(\boldsymbol{x},-\tau;\boldsymbol{y},-t)\,,\tag{1.36}$$

which is a reciprocal relation for source and observer times. Using the elastodynamic reciprocal theorem, (see Eq. (2.2) in section 2.1), we also have the relation

$$U_{ik}(\boldsymbol{y}_2,\tau+\tau_2;\boldsymbol{y}_1,\tau_1)=U_{ki}(\boldsymbol{y}_1,\tau-\tau_1;\boldsymbol{y}_2,-\tau_2)\,.\tag{1.37a}$$

which includes, as a special case, purely spatial reciprocity

$$U_{ik}(y_2, \tau; y_1, 0) = U_{ki}(y_1, \tau; y_2, 0) \tag{1.37b}$$

and space-time reciprocity

$$U_{ik}(y_2, \tau_2; y_1, \tau_1) = U_{ki}(y_1, -\tau_1; y_2, -\tau_2)\,. \tag{1.37c}$$

The fundamental solution also satisfies the translation identity for time and space, i.e.

$$\left.\begin{aligned} U_{ik}(x, t; y, \tau) &= U_{ik}(x-y, t; 0, \tau)\,,\\ U_{ik}(x, t; y, \tau) &= U_{ik}(x, t+t_0; y, \tau+t_0)\,. \end{aligned}\right\} \tag{1.37d}$$

Although we know the physical meaning and properties of the fundamental solution, it is very difficult to find out general elastodynamic fundamental solutions. Hence we restrict ourselves to the homogeneous and isotropic case.

1.4.2. Fundamental solutions in the Fourier-transformed domain

Taking the Fourier transform of Eq. (1.35) with respect to time, we have for $\tau = 0$

$$\mu \frac{\partial^2}{\partial x_j \partial x_j} \bar{U}_{ik} + (\lambda + \mu) \frac{\partial^2}{\partial x_i \partial x_j} \bar{U}_{jk} + \rho\omega^2 \bar{U}_{ik} = -\delta_{ik}\delta(x-y)\,, \tag{1.38}$$

where $\bar{U}_{ik}$ is the Fourier-transformed fundamental solution.

The solution $\bar{U}_{ik}$ may be obtained as follows by the aid of the Fourier transform with respect to space variables [35, 63]: The transformed solution $\hat{\bar{U}}_{ik}$ is expressed as

$$\hat{\bar{U}}_{ik} = -\frac{1}{\mu} \frac{(k_t^2 - \xi^2)\delta_{ik} + (1 - k_t^2/k_l^2)\xi_i\xi_k}{(k_t^2 - \xi^2)(k_l^2 - \xi^2)}\, e^{i\xi\cdot y}\,, \tag{1.39a}$$

where ξ is the parameter of the Fourier transform and

$$k_t = \omega/c_t\,, \qquad k_l = \omega/c_l\,, \qquad \xi^2 = \boldsymbol{\xi}\cdot\boldsymbol{\xi} = \xi_1^2 + \xi_2^2 + \xi_3^2\,. \tag{1.40}$$

Equation (1.39a) can be also expressed as

$$\hat{\bar{U}}_{ik} = \frac{1}{\mu}\left\{\frac{\xi_i\xi_k}{k_t^2}\left(\frac{1}{\xi^2 - k_l^2} - \frac{1}{\xi^2 - k_t^2}\right) + \frac{\delta_{ik}}{\xi^2 - k_t^2}\right\} e^{i\xi\cdot y}\,. \tag{1.39b}$$

The inverse Fourier transform

$$I_\alpha = \frac{1}{(2\pi)^3}\int_V \frac{\exp[-i\boldsymbol{\xi}\cdot(\boldsymbol{x}-\boldsymbol{y})]}{\xi^2 - k_\alpha^2}\, d\boldsymbol{\xi} \qquad (\alpha = \mathrm{L, T})$$

can be evaluated by using a transformation to spherical coordinates (ρ, θ, ϕ), where the azimuthal axis is chosen in the direction of $x - y$, and ϕ is the azimuthal angle. Hence, $\xi = \rho$ and $\boldsymbol{\xi}\cdot(\boldsymbol{x}-\boldsymbol{y}) = \rho r\cos\phi$, where $r^2 = (x_i - y_i)(x_i - y_i)$ and $d\boldsymbol{\xi} = \rho^2 \sin\phi\, d\rho\, d\theta\, d\phi$. We have

$$I_\alpha = \frac{1}{(2\pi)^3}\int_0^\infty\int_0^\pi\int_0^{2\pi} \frac{\rho^2\sin\phi}{\rho^2 - k_\alpha^2}\exp(-i\rho r\cos\phi)\, d\theta\, d\phi\, d\rho$$

and the integration can be reduced to

$$I_\alpha = -\frac{i}{4\pi^2 r}\int_{-\infty}^{\infty} \frac{\zeta e^{i\zeta r}}{\zeta^2 - k_\alpha^2}\, d\zeta = \frac{1}{4\pi r} e^{ik_\alpha r}\,,$$

where the radiation conditions are taken into account for time dependence of the form $\exp(-i\omega t)$ in evaluating the last contour integral, that is, the contour is taken in the upper half of the ζ-plane so as to contain the pole $\zeta = k_\alpha$ in it.

Using this result together with the rule that the factor $-i\xi_k$ in the ξ-space corresponds to $\partial/\partial x_k$ in the x-space, we have from Eq. (1.39b)

$$\bar{U}_{ik} = \frac{1}{4\pi\mu}\left[\frac{e^{ik_t r}}{r}\delta_{ik} + \frac{\partial^2}{\partial x_i\partial x_k}\left\{\frac{1}{k_t^2}\left(\frac{e^{ik_t r}}{r} - \frac{e^{ik_l r}}{r}\right)\right\}\right].$$

For two-dimensional problems, we can easily derive fundamental solutions, which are the wave fields generated by a line load. Since a line load can be considered as a superposition of point loads, the fundamental solutions are expressed as a superposition integral of $\bar{U}_{ik}$ along, say, the x_3-axis from $-\infty$ to ∞.

Fundamental solutions in the Fourier-transformed domain are summarized as follows [61, 70, 71]:

$$\bar{U}_{ik}(x, y; \omega) = A(\bar{U}_1\delta_{ik} - \bar{U}_2 r_{,i} r_{,k})\,, \tag{1.40a}$$

(i) for the three-dimensional case:

$$\left.\begin{aligned} A &= \frac{1}{4\pi\mu}\,,\\ \bar{U}_1 &= \frac{e^{ik_t r}}{r} + \left\{\frac{i}{k_t r} - \frac{1}{(k_t r)^2}\right\}\frac{e^{ik_t r}}{r} - \left(\frac{k_t}{k_l}\right)^2\left\{\frac{i}{k_l r} - \frac{1}{(k_l r)^2}\right\}\frac{e^{ik_l r}}{r}\,,\\ \bar{U}_2 &= \left\{1 + \frac{3i}{k_t r} - \frac{3}{(k_t r)^2}\right\}\frac{e^{ik_t r}}{r} - \left(\frac{k_t}{k_l}\right)^2\left\{1 + \frac{3i}{k_l r} - \frac{3}{(k_l r)^2}\right\}\frac{e^{ik_l r}}{r}\,. \end{aligned}\right\} \tag{1.40b}$$

(ii) for the plane-strain case:

$$A = \frac{\mathrm{i}}{4\mu},$$

$$\hat{U}_1 = H_0^{(1)}(k_1 r) - \frac{1}{k_1 r} H_1^{(1)}(k_1 r) + \left(\frac{k_t}{k_1}\right)^2 \frac{1}{k_t r} H_1^{(1)}(k_t r),$$

$$\hat{U}_2 = -H_2^{(1)}(k_1 r) + \left(\frac{k_t}{k_1}\right)^2 H_2^{(1)}(k_t r), \qquad (1.40c)$$

(iii) for the anti-plane case:

$$\hat{U}_{33}(x, y; \omega) = \frac{\mathrm{i}}{4\mu} H_0^{(1)}(k_t r), \qquad (1.41)$$

where $H_n^{(1)}(\ )$ is the Hankel function of the nth order of the first kind, and $r_{,i} = \partial r/\partial x_i$.

Here we introduce a new entity $\hat{\Sigma}$, which is obtained from $\hat{U}$ in the same manner as we obtain the stress σ from the displacement u. For the isotropic homogeneous case, it is defined as

$$\begin{aligned}
\hat{\Sigma}_{ijk}(\hat{U}) &= \lambda \hat{U}_{mk,m}\delta_{ij} + \mu(\hat{U}_{ik,j} + \hat{U}_{jk,i}) \\
&= \rho[(c_1^2 - 2c_t^2)\hat{U}_{mk,m}\delta_{ij} + c_t^2(\hat{U}_{ik,j} + \hat{U}_{jk,i})] \\
&= -\rho[(c_1^2 - 2c_t^2)\hat{U}_{mk,m}\delta_{ij} + c_t^2(\hat{U}_{ik,j} + \hat{U}_{jk,i})],
\end{aligned} \qquad (1.42)$$

where $(\ )_{,i} = \partial(\ )/\partial x_i$ and $(\ )_{,\bar{i}} = \partial(\ )/\partial y_i = -\partial(\ )/\partial x_i$. Obviously, $\hat{\Sigma}_{ijk} = \hat{\Sigma}_{jik}$.

We now define the second fundamental solution by

$$\hat{W}_{ik}(x, y; \omega) = n_j(y)\hat{\Sigma}_{ijk} = \overset{n_y}{T_{ki}}\hat{U}_{ij}(x, y; \omega), \qquad (1.43)$$

which is explicity written as follows:

$$\begin{aligned}
\hat{W}_{ik}(x, y; \omega) = \mu A \Bigg[&\left\{\left(\delta_{ik}\frac{\partial r}{\partial n} + n_i r_{,k}\right) + \frac{\lambda}{\mu} n_k r_{,i}\right\} \frac{\mathrm{d}\hat{U}_1}{\mathrm{d}r} \\
&- \left\{\left(\delta_{ik}\frac{\partial r}{\partial n} + n_i r_{,k}\right) + 2\left(n_k r_{,i} - 2r_{,i} r_{,k}\frac{\partial r}{\partial n}\right) + \alpha \frac{\lambda}{\mu} n_k r_{,i}\right\} \frac{\hat{U}_2}{r} \\
&- \left\{2r_{,i}r_{,k}\frac{\partial r}{\partial n} + \frac{\lambda}{\mu} n_k r_{,i}\right\} \frac{\mathrm{d}\hat{U}_2}{\mathrm{d}r} \Bigg],
\end{aligned} \qquad (1.44a)$$

where

(i) for the three-dimensional case:

$$\alpha = 2 \text{ and } A, \hat{U}_1, \hat{U}_2 \text{ are given in Eq. (1.40b)}, \qquad (1.44b)$$

(ii) for the plane-strain state:

$$\alpha = 1 \text{ and } A, \hat{U}_1, \hat{U}_2 \text{ are given in Eq. (1.40c)}, \qquad (1.44c)$$

(iii) for the anti-plane case:

$$\hat{W}_{33}(x, y; \omega) = -\frac{\mathrm{i}k_t}{4} H_1^{(1)}(k_t r)\frac{\partial r}{\partial n}, \qquad (1.45)$$

with $\partial r/\partial n = (\partial r/\partial y_i)n_i$.

If we use the Laplace transform with respect to time, the corresponding fundamental solutions are easily derived by simply converting the parameter ω to $\mathrm{i}s$, where s is the parameter of the Laplace transform [4, 23, 24, 77–80].

1.4.3. *Fundamental solutions in the time–space domain*

The time–space fundamental solutions, i.e. $U_{ik}(x, t; y, \tau)$ satisfying Eq. (1.34), can be derived by inverting the Fourier-transformed solutions $\hat{U}_{ik}(x, y; \omega)$ with the aid of the convolution theorem. According to this theorem, if $\hat{f}(\omega)$ and $\hat{g}(\omega)$ are Fourier transforms of $f(t)$ and $g(t)$, then $\hat{f}(\omega)\hat{g}(\omega)$ is the Fourier transform of the convolution integral

$$\int_0^t f(\tau)g(t-\tau)\,\mathrm{d}\tau \qquad \text{or } f(t) * g(t)$$

provided that $g(t) = 0$ for $t < 0$. Hence, for example, $\omega^{-2}\hat{f}(\omega)$ is the Fourier transform of

$$\int_0^t f(\tau)(t-\tau)\,\mathrm{d}\tau$$

since $\hat{g}(\omega) = \omega^{-2}$ so that $g(t) = tH(t)$.

The inverse Fourier transform of $\hat{g}(\omega)\exp(-\mathrm{i}\omega r/c_\alpha)$ is $g(t - r/c_\alpha)$. Therefore, from Eqs. (1.40) and (1.44) we have the following expressions for $\tau = 0$ [2, 3, 8, 37, 61, 130, 131]:

$U_{jk}(\boldsymbol{x}, t; \boldsymbol{y}, 0)$

$$= \frac{1}{4\pi\rho}\left\{\frac{t}{r^2}\left(\frac{3r_jr_k}{r^3} - \frac{\delta_{jk}}{r}\right)\left[H\left(t-\frac{r}{c_L}\right) - H\left(t-\frac{r}{c_T}\right)\right]\right.$$
$$\left. + \frac{r_jr_k}{r^3}\left[\frac{1}{c_L^2}\,\delta\left(t-\frac{r}{c_L}\right) - \frac{1}{c_T^2}\,\delta\left(t-\frac{r}{c_T}\right)\right] + \frac{\delta_{jk}}{rc_T^2}\,\delta\left(t-\frac{r}{c_T}\right)\right\}$$
$$= \frac{t}{4\pi\rho}\left\{\left(\frac{1}{r}\right)_{,jk}\left[H\left(t-\frac{r}{c_L}\right) - H\left(t-\frac{r}{c_T}\right)\right]\right.$$
$$\left. - r_{,j}\left(\frac{1}{r}\right)_{,k}\left[\frac{1}{c_L}\,\delta\left(t-\frac{r}{c_L}\right) - \frac{1}{c_T}\,\delta\left(t-\frac{r}{c_T}\right)\right] + \frac{\delta_{jk}}{r^2c_T}\,\delta\left(t-\frac{r}{c_T}\right)\right\}$$
$$= \frac{t}{4\pi\rho}\left\{\left(\frac{1}{r}\right)_{,jk}\left[H\left(t-\frac{r}{c_L}\right) - H\left(t-\frac{r}{c_T}\right)\right]\right.$$
$$\left. - r_{,j}\left(\frac{1}{r}\right)_{,k}\left[\frac{1}{c_L}\,\delta\left(t-\frac{r}{c_L}\right) - \frac{1}{c_T}\,\delta\left(t-\frac{r}{c_T}\right)\right] + \frac{\delta_{jk}}{r^2c_T}\,\delta\left(t-\frac{r}{c_T}\right)\right\}. \tag{1.46}$$

$$W_{jk}(\boldsymbol{x}, t; \boldsymbol{y}, 0) = n_i(\boldsymbol{y})\Sigma_{ijk}(\boldsymbol{x}, t; \boldsymbol{y}, 0)\,. \tag{1.47}$$

$\Sigma_{ijk}(\boldsymbol{x}, t; \boldsymbol{y}, 0)$

$$= \frac{1}{4\pi}\left\{-6c_T^2\left[5\,\frac{r_ir_jr_k}{r^5} - \frac{\delta_{ij}r_k + \delta_{ik}r_j + \delta_{jk}r_i}{r^3}\right]\frac{t}{r^2}\left[H\left(t-\frac{r}{c_L}\right) - H\left(t-\frac{r}{c_T}\right)\right]\right.$$
$$+ 2\left[6\,\frac{r_ir_jr_k}{r^5} - \frac{\delta_{ij}r_k + \delta_{ik}r_j + \delta_{jk}r_i}{r^3}\right]\left[\delta\left(t-\frac{r}{c_T}\right) - \left(\frac{c_T}{c_L}\right)^2\delta\left(t-\frac{r}{c_L}\right)\right]$$
$$+ 2\,\frac{r_ir_jr_k}{r^4c_T}\left[\dot\delta\left(t-\frac{r}{c_T}\right) - \left(\frac{c_T}{c_L}\right)^3\dot\delta\left(t-\frac{r}{c_L}\right)\right]$$
$$- \frac{r_k\delta_{ij}}{r^3}\left\{1-2\left(\frac{c_T}{c_L}\right)^2\right\}\left[\delta\left(t-\frac{r}{c_L}\right) + \frac{r}{c_L}\,\dot\delta\left(t-\frac{r}{c_L}\right)\right]$$
$$\left. - \frac{\delta_{ik}r_j + \delta_{jk}r_i}{r^3}\left[\delta\left(t-\frac{r}{c_T}\right) + \frac{r}{c_T}\,\dot\delta\left(t-\frac{r}{c_T}\right)\right]\right\}. \tag{1.48}$$

For the two-dimensional case, we have

$$V_{jk}(x_\alpha, t; y_\beta, 0) \equiv \int_{-\infty}^{\infty} U_{jk}(\boldsymbol{x}, t; \boldsymbol{y}, 0)\,\mathrm{d}x_3$$
$$S_{jk}(x_\alpha, t; y_\beta, 0) \equiv \int_{-\infty}^{\infty} W_{jk}(\boldsymbol{x}, t; \boldsymbol{y}, 0)\,\mathrm{d}x_3 \tag{1.49}$$

and hence in detail [37],

$$V_{\alpha\beta} = \frac{1}{2\pi\rho}\left\{\left[\frac{(2t^2 - r^2/c_L^2)}{(t^2 - r^2/c_L^2)^{1/2}}\,H\left(t-\frac{r}{c_L}\right) - \frac{(2t^2 - r^2/c_T^2)}{(t^2 - r^2/c_T^2)^{1/2}}\,H\left(t-\frac{r}{c_T}\right)\right]\frac{r_\alpha r_\beta}{r^4}\right.$$
$$- \left[H\left(t-\frac{r}{c_L}\right)(t^2 - r^2/c_L^2)^{1/2} - H(t-r/c_T)(t^2 - r^2/c_T^2)^{1/2}\right]\frac{\delta_{\alpha\beta}}{r^2}$$
$$\left. + \frac{H(t-r/c_T)}{c_T^2(t^2 - r^2/c_T^2)^{1/2}}\,\delta_{\alpha\beta}\right\}, \tag{1.50}$$

$$V_{3\alpha} = 0\,,$$

$$V_{33} = \frac{H(t-r/c_T)}{2\pi\rho c_T^2(t^2 - r^2/c_T^2)^{1/2}}$$

and

$$\Sigma_{\alpha\beta\gamma} = -\rho(c_L^2 - 2c_T^2)V_{\kappa\gamma,\kappa}\delta_{\alpha\beta} - \rho c_T^2(V_{\alpha\gamma,\beta} + V_{\beta\gamma,\alpha})\,,$$
$$\Sigma_{\alpha\beta3} = \Sigma_{\alpha3\gamma} = 0\,,$$
$$\Sigma_{\alpha33} = -\rho c_T^2 V_{33,\alpha}\,, \qquad \Sigma_{33\alpha} = -\rho(c_L^2 - 2c_T^2)V_{\alpha\beta,\beta} \tag{1.51}$$
$$\Sigma_{333} = 0\,.$$

$$S_{\alpha\beta} = n_\gamma(\boldsymbol{y})\Sigma_{\gamma\alpha\beta}\,, \qquad S_{3\alpha} = S_{\alpha3} = 0\,, \qquad S_{33} = n_\alpha(\boldsymbol{y})\Sigma_{\alpha33}\,. \tag{1.52}$$

2. Representation theorems

2.1. *Betti's reciprocal theorem*

Let $\boldsymbol{u}$ and $\boldsymbol{v}$ be two single-valued, twice continuously differentiable vector fields defined in a domain D bounded by a regular surface ∂D in the sense of Kellogg [55]. Then we have for the operator defined in Eqs. (1.8a) and (1.24),

$$\int_D [\boldsymbol{v}\cdot\mathbf{E}\boldsymbol{u} - \mathbf{E}^*\boldsymbol{v}\cdot\boldsymbol{u}]\,\mathrm{d}V = \int_{\partial D}[\boldsymbol{v}\cdot\overset{n}{\mathbf{T}}\boldsymbol{u} - \overset{n}{\mathbf{T}}{}^*\boldsymbol{v}\cdot\boldsymbol{u}]\,\mathrm{d}S\,, \tag{2.1a}$$

where $\mathbf{E}^*$ and $\mathbf{T}^*$ are adjoint of $\mathbf{E}$ and $\mathbf{T}$, respectively, and moreover $\mathbf{E}^* = \mathbf{E}$, $\mathbf{T}^* = \mathbf{T}$ in elasticity. This identity is known as Green's second formula.

Using the indicial notation, the general form is given as

$$\int_D \{v_i\partial_j(C_{ijkl}\partial_l u_k) - u_k\partial_l(\partial_j v_i C_{ijkl})\}\,\mathrm{d}V$$
$$= \int_{\partial D}\{n_j(v_iC_{ijkl}\partial_l u_k) - n_l(\partial_j v_i C_{ijkl}u_k)\}\,\mathrm{d}S\,. \tag{2.1b}$$

Since $\boldsymbol{u}$, $\boldsymbol{v}$ satisfy Eq. (1.8a), i.e.

$$\mathbf{E}\boldsymbol{u} = \rho\ddot{\boldsymbol{u}} - \rho\boldsymbol{b}\,, \qquad \mathbf{E}\boldsymbol{v} = \rho\ddot{\boldsymbol{v}} - \rho\boldsymbol{f}$$

we also have

$$\int_D \{\boldsymbol{v}\cdot(\ddot{\boldsymbol{u}} - \boldsymbol{b}) - (\ddot{\boldsymbol{v}} - \boldsymbol{f})\cdot\boldsymbol{u}\}\rho\,\mathrm{d}V = \int_{\partial D} (\boldsymbol{v}\cdot\overset{n}{\mathbf{T}}\boldsymbol{u} - \overset{n}{\mathbf{T}}\boldsymbol{v}\cdot\boldsymbol{u})\,\mathrm{d}S \tag{2.2a}$$

This is known as Betti's reciprocal relation.

Betti's relation does not involve initial values of $\boldsymbol{u}$ and $\boldsymbol{v}$. However, it is also valid even if $\boldsymbol{u}$ and $\boldsymbol{v}$ are respectively evaluated at different times. If we evaluate $\boldsymbol{u}$ at $t_1 = \tau$, and $\boldsymbol{v}$ at $t_2 = t - \tau$ and integrate over the temporal range 0 to t, we have

$$\int_0^t \{\ddot{\boldsymbol{u}}(\boldsymbol{x},\tau)\cdot\boldsymbol{v}(\boldsymbol{x},t-\tau) - \boldsymbol{u}(\boldsymbol{x},\tau)\cdot\ddot{\boldsymbol{v}}(\boldsymbol{x},t-\tau)\}\rho\,\mathrm{d}\tau$$

$$= \rho\int_0^t \frac{\partial}{\partial\tau}\{\dot{\boldsymbol{u}}(\boldsymbol{x},\tau)\cdot\boldsymbol{v}(\boldsymbol{x},t-\tau) + \boldsymbol{u}(\boldsymbol{x},\tau)\cdot\dot{\boldsymbol{v}}(\boldsymbol{x},t-\tau)\}\,\mathrm{d}\tau$$

$$= \rho\{\dot{\boldsymbol{u}}(\boldsymbol{x},\tau)\cdot\boldsymbol{v}(\boldsymbol{x},t-\tau) + \boldsymbol{u}(\boldsymbol{x},\tau)\cdot\dot{\boldsymbol{v}}(\boldsymbol{x},t-\tau)\}\big|_0^t\,.$$

Denoting here the convolution integral by

$$\boldsymbol{a}(\boldsymbol{x},t) * \boldsymbol{b}(\boldsymbol{x},t) = \begin{cases} \displaystyle\int_0^t \boldsymbol{a}(\boldsymbol{x},\tau)\cdot\boldsymbol{b}(\boldsymbol{x},t-\tau)\,\mathrm{d}\tau\,, & t > 0 \\ 0\,, & t < 0 \end{cases} \tag{2.3}$$

we have, from the Betti relation (2.2a)

$$\int_D \{(\ddot{\boldsymbol{u}} - \boldsymbol{b}) * \boldsymbol{v} - \boldsymbol{u} * (\ddot{\boldsymbol{v}} - \boldsymbol{f})\}\rho\,\mathrm{d}V = \int_{\partial D} (\overset{n}{\mathbf{T}}\boldsymbol{u} * \boldsymbol{v} - \boldsymbol{u} * \overset{n}{\mathbf{T}}\boldsymbol{v})\,\mathrm{d}S \tag{2.2b}$$

and hence [44]

$$\int_D \boldsymbol{u} * \boldsymbol{f}\rho\,\mathrm{d}V + \int_{\partial D} \boldsymbol{u} * \overset{n}{\mathbf{T}}\boldsymbol{v}\,\mathrm{d}S + \int_D \{\dot{\boldsymbol{u}}(\boldsymbol{x},t)\cdot\boldsymbol{v}_0(\boldsymbol{x}) + \boldsymbol{u}(\boldsymbol{x},t)\cdot\dot{\boldsymbol{v}}_0(\boldsymbol{x})\}\rho\,\mathrm{d}V$$

$$= \int_D \boldsymbol{b} * \boldsymbol{v}\rho\,\mathrm{d}V + \int_{\partial D} \mathbf{T}_n\boldsymbol{u} * \boldsymbol{v}\,\mathrm{d}S + \int_D \{\dot{\boldsymbol{v}}(\boldsymbol{x},t)\cdot\boldsymbol{u}_0(\boldsymbol{x}) + \boldsymbol{v}(\boldsymbol{x},t)\cdot\dot{\boldsymbol{u}}_0(\boldsymbol{x})\}\rho\,\mathrm{d}V\,. \tag{2.4}$$

where $\boldsymbol{u}_0$, $\boldsymbol{v}_0$, $\dot{\boldsymbol{u}}_0$, $\dot{\boldsymbol{v}}_0$ are initial displacements and velocities, respectively.

In the special case of zero initial conditions, the third integral of both sides of Eq. (2.4) disappears. Furthermore, when the boundary conditions are homogeneous, we have

$$\int_D \boldsymbol{u} * \boldsymbol{f}\rho\,\mathrm{d}V = \int_D \boldsymbol{b} * \boldsymbol{v}\rho\,\mathrm{d}V\,. \tag{2.5}$$

In the Fourier-transformed domain, we have

$$\int_D [\hat{\boldsymbol{v}}\cdot\mathbf{L}\hat{\boldsymbol{u}} - \mathbf{L}^*\hat{\boldsymbol{v}}\cdot\hat{\boldsymbol{u}}]\,\mathrm{d}V = \int_{\partial D} [\hat{\boldsymbol{v}}\cdot\overset{n}{\mathbf{T}}\hat{\boldsymbol{u}} - \overset{n}{\mathbf{T}}{}^*\hat{\boldsymbol{v}}\cdot\hat{\boldsymbol{u}}]\,\mathrm{d}S\,, \tag{2.6}$$

where $\mathbf{L}\hat{\boldsymbol{u}}$ is defined by Eq. (1.30). This counterpart of Eq. (2.2b) is the second Green's identity and also the Fourier transform of the general Betti relation (2.2a)

2.2. *Representation theorems*

Making use of the Green's identity or the general Betti relation and the properties of the fundamental solution, we can have integral representations of the elastodynamic solution.

If we use the fundamental solution $\hat{U}_{ik}$ in place of v_i in Eq. (2.6), for example, and consider the property of $\hat{U}_{ik}$, that is

$$L^*_{ij}\hat{U}_{ik} = -\delta(\boldsymbol{x} - \boldsymbol{y})\delta_{jk} \tag{2.7}$$

and hence

$$\int_D L^*_{ij}\hat{U}_{ik}\hat{u}_i\,\mathrm{d}V(\boldsymbol{x}) = -\int_D \hat{u}_i\delta(\boldsymbol{x} - \boldsymbol{y})\delta_{ik}\,\mathrm{d}V(\boldsymbol{x}) = \begin{cases} -\hat{u}_k(\boldsymbol{y})\,, & \boldsymbol{y}\in D \\ 0\,, & \boldsymbol{y}\in D_c\,, \end{cases} \tag{2.8}$$

where D_c is the complement of D, we have

$$\varepsilon(\boldsymbol{y})\hat{u}_k(\boldsymbol{y};\omega) = \int_{\partial D} [\hat{U}_{ik}(\boldsymbol{x},\boldsymbol{y};\omega)\hat{t}_i(\boldsymbol{x};\omega) - \hat{W}_{ik}(\boldsymbol{x},\boldsymbol{y};\omega)\hat{u}_i(\boldsymbol{x};\omega)]\,\mathrm{d}S(\boldsymbol{x})$$

$$+ \int_D \rho\hat{b}_i(\boldsymbol{x};\omega)\hat{U}_{ik}(\boldsymbol{x},\boldsymbol{y};\omega)\,\mathrm{d}V(\boldsymbol{x})\,. \tag{2.9}$$

where

$$\varepsilon(\boldsymbol{y}) = \begin{cases} 1\,, & \boldsymbol{y}\in D\,, \\ 0\,, & \boldsymbol{y}\in D_c\,, \end{cases}$$

and

$$\hat{t}_i(\boldsymbol{x};\omega) = \overset{n}{T}_{ij}\hat{u}_j(\boldsymbol{x};\omega)\,, \qquad \hat{W}_{ik}(\boldsymbol{x},\boldsymbol{y};\omega) = \overset{n}{T}_{ij}\hat{U}_{jk}(\boldsymbol{x},\boldsymbol{y};\omega)\,.$$

This expression is called the representation theorem or Somigliana formula. Sometimes it may be more convenient to change integration variable from $\boldsymbol{x}$ to $\boldsymbol{y}$ and to write down Eq. (2.9) using the spatial reciprocity (see Eq. (1.37b))

$$\varepsilon(\boldsymbol{x})\hat{u}_k(\boldsymbol{x};\omega) = \int_{\partial D} [\hat{U}_{ki}(\boldsymbol{x},\boldsymbol{y};\omega)\hat{t}_i(\boldsymbol{y};\omega) - \hat{W}_{ki}(\boldsymbol{x},\boldsymbol{y};\omega)\hat{u}_i(\boldsymbol{y};\omega)]\,\mathrm{d}S(\boldsymbol{y})$$
$$+ \int_D \rho\hat{b}_i(\boldsymbol{y};\omega)\hat{U}_{ki}(\boldsymbol{x},\boldsymbol{y};\omega)\,\mathrm{d}V(\boldsymbol{y})\,, \tag{2.10}$$

where

$$\varepsilon(\boldsymbol{x}) = \begin{cases} 1\,, & \boldsymbol{x}\in D\,,\\ 0\,, & \boldsymbol{x}\in D_c\,,\end{cases}$$

$$\hat{t}_i(\boldsymbol{y};\omega) = \overset{n_y}{T}_{ij}\hat{u}_j(\boldsymbol{y};\omega)\,, \qquad \hat{W}_{ki}(\boldsymbol{x},\boldsymbol{y};\omega) = \overset{n_y}{T}_{ij}\hat{U}_{kj}(\boldsymbol{x},\boldsymbol{y};\omega)$$

and note that $\overset{n_y}{\boldsymbol{T}}$ operates at $\boldsymbol{y}\in\partial D$ on the boundary. This expression means that the displacement $\hat{\boldsymbol{u}}(\boldsymbol{x};\omega)$ at any point $\boldsymbol{x}$ in D can be evaluated whenever boundary displacement $\hat{\boldsymbol{u}}(\boldsymbol{y};\omega)$ on ∂D_1, and traction $\hat{\boldsymbol{t}}(\boldsymbol{y};\omega)$ on ∂D_2 are given. However, part of them, i.e. $\hat{\boldsymbol{u}}(\boldsymbol{y};\omega)$ on ∂D_2 and $\hat{\boldsymbol{t}}(\boldsymbol{y};\omega)$ on ∂D_1 are not prescribed as boundary conditions, and they must be determined beforehand in order to make use of Eq. (2.10). It is also noted here that boundary displacements and tractions must satisfy the constraint equation, that is the right-hand side of Eq. (2.10) must be zero for $\boldsymbol{x}\in D_c$.

The time-domain representation theorem is similarly obtained from Eq. (2.4) by substituting $\delta_{ik}\delta(\boldsymbol{x}-\boldsymbol{y})\delta(t)$ for $\rho f_i(\boldsymbol{x},t)$ and $U_{ik}(\boldsymbol{x},t;\boldsymbol{y},0)$ for $v_i(\boldsymbol{x},t)$ and reads

$$\varepsilon(\boldsymbol{y})u_k(\boldsymbol{y},t) = \int_{\partial D} [t_i(\boldsymbol{x},t)*U_{ik}(\boldsymbol{x},t;\boldsymbol{y},0)$$
$$- u_i(\boldsymbol{x},t)*W_{ik}(\boldsymbol{x},t;\boldsymbol{y},0)]\,\mathrm{d}S(\boldsymbol{x})$$
$$+ \int_D \rho b_i(\boldsymbol{x},t)*U_{ik}(\boldsymbol{x},t;\boldsymbol{y},0)\,\mathrm{d}V(\boldsymbol{x})$$
$$+ \int_D \rho[\dot{U}_{ik}(\boldsymbol{x},t;\boldsymbol{y},0)u_{0i}(\boldsymbol{x}) + U_{ik}(\boldsymbol{x},t;\boldsymbol{y},0)\dot{u}_{0i}(\boldsymbol{x})]\,\mathrm{d}V(\boldsymbol{x})\,, \tag{2.11}$$

where

$$\varepsilon(\boldsymbol{y}) = \begin{cases} 1\,, & \boldsymbol{y}\in D\\ 0\,, & \boldsymbol{y}\in D_c\end{cases}$$

and

$$t_i(\boldsymbol{x},t) = \overset{n}{T}_{ij}u_j(\boldsymbol{x},t)\,, \qquad W_{ik}(\boldsymbol{x},t;\boldsymbol{y},0) = \overset{n}{T}_{ij}U_{jk}(\boldsymbol{x},t;\boldsymbol{y},0)\,.$$

It is helpful to rewrite Eq. (2.11) by interchanging the symbols $\boldsymbol{x}$ and $\boldsymbol{y}$, and considering the spatial reciprocity as

$$\varepsilon(\boldsymbol{x})u_k(\boldsymbol{x},t) = \int_{\partial D} [t_i(\boldsymbol{y},t)*U_{ki}(\boldsymbol{x},t;\boldsymbol{y},0) - u_i(\boldsymbol{y},t)*W_{ki}(\boldsymbol{x},t;\boldsymbol{y},0)]\,\mathrm{d}S(\boldsymbol{y})$$
$$+ \int_D \rho b_i(\boldsymbol{y},t)*U_{ki}(\boldsymbol{x},t;\boldsymbol{y},0)\,\mathrm{d}V(\boldsymbol{y})$$
$$+ \int_D \rho[\dot{U}_{ki}(\boldsymbol{x},t;\boldsymbol{y},0)u_{0i}(\boldsymbol{y}) + U_{ki}(\boldsymbol{x},t;\boldsymbol{y},0)\dot{u}_{0i}(\boldsymbol{y})]\,\mathrm{d}V(\boldsymbol{y})\,, \tag{2.12}$$

where

$$\varepsilon(\boldsymbol{x}) = \begin{cases} 1\,, & \boldsymbol{x}\in D\\ 0\,, & \boldsymbol{x}\in D_c\end{cases}$$

and

$$t_i(\boldsymbol{y},t) = \overset{n_y}{T}_{ij}u_j(\boldsymbol{y},t)\,, \qquad W_{ki}(\boldsymbol{x},t;\boldsymbol{y},0) = \overset{n_y}{T}_{ij}U_{kj}(\boldsymbol{x},t;\boldsymbol{y},0)\,.$$

Similar expressions are easily deduced for the two-dimensional case. In the time–space domain, for example, we have

$$\varepsilon(\boldsymbol{x})u_\alpha(\boldsymbol{x},t) = \int_{\partial D} [t_\beta(\boldsymbol{y},t)*V_{\alpha\beta}(\boldsymbol{x},t;\boldsymbol{y},0)$$
$$- u_\beta(\boldsymbol{y},t)*S_{\alpha\beta}(\boldsymbol{x},t;\boldsymbol{y},0)]\,\mathrm{d}C(\boldsymbol{y})$$
$$+ \int_D \rho b_\beta(\boldsymbol{y},t)*V_{\alpha\beta}(\boldsymbol{x},t;\boldsymbol{y},0)\,\mathrm{d}S(\boldsymbol{y})$$
$$+ \int_D \rho[\dot{V}_{\alpha\beta}(\boldsymbol{x},t;\boldsymbol{y},0)u_{0\beta}(\boldsymbol{y}) + V_{\alpha\beta}(\boldsymbol{x},t;\boldsymbol{y},0)\dot{u}_{0\beta}(\boldsymbol{y})]\,\mathrm{d}S(\boldsymbol{y})\,. \tag{2.13}$$

$$\epsilon(\boldsymbol{x})u_3(\boldsymbol{x},t) = \int_{\partial D} [t_3(\boldsymbol{y},t) * V_{33}(\boldsymbol{x},t;\boldsymbol{y},0)$$

$$- u_3(\boldsymbol{y},t) * S_{33}(\boldsymbol{x},t;\boldsymbol{y},0)]\,\mathrm{d}C(\boldsymbol{y})$$

$$+ \int_D \rho b_3(\boldsymbol{y},t) * V_{33}(\boldsymbol{x},t;\boldsymbol{y},0)\,\mathrm{d}S(\boldsymbol{y})$$

$$+ \int_D \rho[\dot{V}_{33}(\boldsymbol{x},t;\boldsymbol{y},0)u_{03}(\boldsymbol{y}) + V_{33}(\boldsymbol{x},t;\boldsymbol{y},0)u_{03}(\boldsymbol{y})]\,\mathrm{d}S(\boldsymbol{y})\,. \tag{2.14}$$

where

$$\epsilon(x) = \begin{cases} 1\,, & \boldsymbol{x} \in D \\ 0\,, & \boldsymbol{x} \in D_c \end{cases}$$

and

$$t_\alpha(\boldsymbol{y},t) = \overset{n_y}{T}_{\alpha\beta}u_\beta(\boldsymbol{y},t)\,, \qquad S_{\alpha\beta}(\boldsymbol{x},t;\boldsymbol{y},0) = \overset{n_y}{T}_{\beta\gamma}U_{\alpha\gamma}(\boldsymbol{x},t;\boldsymbol{y},0)\,,$$

$$\boldsymbol{x} = (x_1, x_2)\,, \qquad \boldsymbol{y} = (y_1, y_2)\,.$$

Stresses are obtained by differentiating $\boldsymbol{u}$ with respect to x and substituting into Eq. (1.6b). For instance, from Eq. (2.10), we have

$$\hat{\sigma}_{kl}(\boldsymbol{x};\omega) = \int_{\partial D} [\hat{t}_i(\boldsymbol{y};\omega)\bar{\hat{U}}_{kil}(\boldsymbol{x},\boldsymbol{y};\omega) - \hat{u}_i(\boldsymbol{y};\omega)\hat{W}_{kil}(\boldsymbol{x},\boldsymbol{y};\omega)\,\mathrm{d}S(\boldsymbol{y})$$

$$+ \int_D \rho b_i(\boldsymbol{y};\omega)\bar{\hat{U}}_{kil}(\boldsymbol{x},\boldsymbol{y};\omega)\,\mathrm{d}V(\boldsymbol{y})\,, \tag{2.15}$$

where

$$\{\bar{\hat{U}}_{kil}, \bar{\hat{W}}_{kil}\} = \rho(c_1^2 - 2c_T^2)\{\hat{U}_{kmm}, \hat{W}_{kmm}\}$$

$$- \rho c_T^2\{\hat{U}_{k..} - \hat{U}_{k..}\,,\ \hat{W}_{k..} - \hat{W}_{k..}\}\,,$$

$$\hat{U}_{kil} = \hat{U}_{k..}\,, \qquad \hat{W}_{kil} = \hat{W}_{k..}\,.$$

Similar expressions are easily obtained for the time–space domain formulations.

3. Integral equations

3.1. *Integral equations in the Fourier-transformed domain*

We know two ways of approach to time-dependent problems such as those of elastodynamics. One is to solve the problems directly in the time–space domain and the other is to treat them in conjunction with integral transforms. In what follows, we formulate integral equations of elastodynamic problems first in the Fourier-transformed domain and then in the time–space domain.

Knowing that

$$\hat{U}(\boldsymbol{x},\boldsymbol{y};\omega) = \mathrm{O}(1/r)\,, \qquad \hat{W}(\boldsymbol{x},\boldsymbol{y};\omega) = \mathrm{O}(1/r^2) \qquad \text{for 3D}\,,$$

and

$$\hat{U}(\boldsymbol{x},\boldsymbol{y};\omega) = \mathrm{O}(\ln 1/r)\,, \qquad \hat{W}(\boldsymbol{x},\boldsymbol{y};\omega) = \mathrm{O}(1/r) \qquad \text{for 2D}$$

as $r = |\boldsymbol{x} - \boldsymbol{y}| \to 0$, we can evaluate the limit value of the surface integral on the part of a sphere (or circle) $s(\boldsymbol{x}_0;\epsilon)$ with radius ϵ and centered at $\boldsymbol{x}_0$ included in D, when $\epsilon \to 0$ as shown in Fig. 1 and obtain

$$\lim_{\epsilon\to 0} \int_{s(\boldsymbol{x}_0;\epsilon)\cap D} \hat{U}_{ki}(\boldsymbol{x}_0,\boldsymbol{y};\omega)h_i(\boldsymbol{y})\,\mathrm{d}S(\boldsymbol{y}) = 0$$

$$\lim_{\epsilon\to 0} \int_{s(\boldsymbol{x}_0;\epsilon)\cap D} \hat{W}_{ki}(\boldsymbol{x}_0,\boldsymbol{y};\omega)g_i(\boldsymbol{y})\,\mathrm{d}S(\boldsymbol{y}) = C^e_{ki}(\boldsymbol{x}_0)g_i(\boldsymbol{x}_0) \tag{3.1}$$

when $\boldsymbol{h}$ and $\boldsymbol{g}$ are regular in D. In these expressions, $C^e_{ki}(\boldsymbol{x}_0)$ is δ_{ki} for $\boldsymbol{x}_0 \in D$, and $\delta_{ki}/2$ for $\boldsymbol{x}_0 \in \partial D$ when the boundary is smooth in the neighborhood of $\boldsymbol{x}_0$. However, at a corner point on ∂D, $C^e_{ki}(\boldsymbol{x}_0)$ must be evaluated with complicated computations. As will be seen later (see section 4.3.2), since the order of singularities of $\hat{U}$ and $\hat{W}$ as $r \to 0$ is the same as that of the fundamental solutions for elastostatics, we can make use of C^e_{ki} for elastostatics [46, 106, 107].

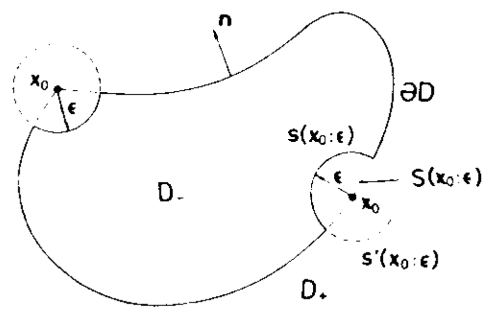


Fig. 1. Domain, boundary, normal and boundary point.

Taking the above results into consideration, we have the following integral equation as $x \in D_c \to x_0 \in \partial D$

$$C^c_{ki}(x_0)\hat{u}_i(x_0;\omega) = \int_{\partial D} \hat{U}_{ki}(x_0, y;\omega)\hat{t}_i(y;\omega)\,dS(y) - \fint_{\partial D} \hat{W}_{ki}(x_0, y;\omega)\hat{u}_i(y;\omega)\,dS(y) + \int_D \rho\hat{U}_{ki}(x_0, y;\omega)b_i(y;\omega)\,dV(y)\,, \tag{3.2}$$

where $\fint(\;)\,dS(y)$ implies the principal-value integral in Cauchy's sense.

For the two-dimensional case, we have the same expressions with the two-dimensional kernels.

It is noted that the body force is often disregarded in elastodynamics and that Eq. (3.2) then reduces to a boundary integral equation.

For unbounded exterior problems, integral equations are formulated for the scattered field $\hat{u}^S(x;\omega) = \hat{u}(x;\omega) - \hat{u}^I(x;\omega)$, where $\hat{u}^I(x;\omega)$ is the incident wave field.

However, the alternative formulation for the total field $\hat{u}(x;\omega)$ is also possible [64, 66]. Since for the scattered field Eq. (2.10) is valid for the exterior domain D_c of the scatterer boundary ∂D

$$\fint_{\partial D} \hat{W}(x, y;\omega)\cdot\hat{u}^S(y;\omega)\,dS(y) - \int_{\partial D} \hat{U}(x, y;\omega)\cdot\hat{t}^S(y;\omega)\,dS(y) = \begin{cases} \hat{u}^S(x;\omega)\,, & x \in D_c\,,\\ \mathbf{0}\,, & x \in D_i\,, \end{cases} \tag{3.3}$$

where the normal on ∂D is taken positive as pointing into the exterior domain, and the incident wave field Eq. (2.10) is again true for the interior domain D_i of the scatterer

$$\int_{\partial D} \hat{U}(x, y;\omega)\cdot\hat{t}^I(y;\omega)\,dS(y) - \fint_{\partial D} \hat{W}(x, y;\omega)\cdot\hat{u}^I(y;\omega)\,dS(y) = \begin{cases} \hat{u}^I(x;\omega)\,, & x \in D_i\,,\\ 0\,, & x \in D_c\,, \end{cases} \tag{3.4}$$

we have the following integral representation by substracting them:

$$\fint_{\partial D} \hat{W}(x, y;\omega)\cdot\hat{u}(y;\omega)\,dS(y) - \int_{\partial D} \hat{U}(x, y;\omega)\cdot\hat{t}(y;\omega)\,dS(y) = \begin{cases} \hat{u}(x;\omega) - \hat{u}^I(x;\omega)\,, & x \in D_c\,,\\ -\hat{u}^I(x;\omega)\,, & x \in D_i\,, \end{cases} \tag{3.5}$$

Hence we have the following boundary integral equation, noting that the incident wave field is regular:

$$\fint_{\partial D} \hat{W}(x, y;\omega)\cdot\hat{u}(y;\omega)\,dS(y) - \int_{\partial D} \hat{U}(x, y;\omega)\cdot\hat{t}(y;\omega)\,dS(y) + \hat{u}^I(x_0;\omega) - C^c(x_0)\cdot\hat{u}(x_0;\omega)\,, \quad x_0 \in \partial D\,. \tag{3.6}$$

The boundary integral equation (3.6) thus obtained is easily applied to boundary-value-problems for multiply connected regions and for multi-foldedly connected regions as it is the case of a lined tunnel in an infinite medium. In the latter case, boundary integral equations are derived separately for each subdomain and then they are combined using the continuity conditions on the common interface of subdomains.

Sometimes, for simplicity, the Born approximation is used to find the far-field behavior in the exterior domain. This approximation is derived by substituting the incident field for the total field in the integrals on the left-hand side of Eq. (3.5) [16–18, 47].

3.2. *Integral equations in time-space domain*

Taking the limit of $x \in D_c \to x_0 \in \partial D$ in Eqs. (2.12)–(2.14), we have integral equations expressed in the time–space domain as

$$C^c_{ki}(x_0)u_i(x_0, t) = \int_{\partial D} U_{ki}(x_0, t; y, 0) * t_i(y, t)\,dS(y) - \fint_{\partial D} W_{ki}(x_0, t; y, 0) * u_i(y, t)\,dS(y) + \int_D \rho U_{ki}(x_0, t; y, 0) * b_i(y, t)\,dV(y) + \int_D \rho[\dot{U}_{ki}(x_0, t; y, 0)u_{0i}(y) + U_{ki}(x_0, t; y, 0)\dot{u}_{0i}(y)]\,dV(y)\,, \tag{3.7}$$

$$C^c_{\alpha\beta}(x_0)u_\beta(x_0, t) = \int_{\partial D} V_{\alpha\beta}(x_0, t; y, 0) * t_\beta(y, t)\,dC(y) - \fint_{\partial D} S_{\alpha\beta}(x_0, t; y, 0) * u_\beta(y, t)\,dC(y) + \int_D \rho V_{\alpha\beta}(x_0, t; y, 0) * b_\beta(y, t)\,dS(y) + \int_D \rho[\dot{V}_{\alpha\beta}(x_0, t; y, 0)u_{0\beta}(y) + V_{\alpha\beta}(x_0, t; y, 0)\dot{u}_{0\beta}(y)]\,dS(y)\,, \tag{3.8}$$

$$C^{*}(x_0)u_3(x_0,t) = \int_{\partial D} V_{33}(x_0,t;y,0) * t_3(y,t)\,\mathrm{d}C(y)$$

$$\fint_{\partial D} S_{33}(x_0,t;y,0) * u_3(y,t)\,\mathrm{d}C(y) + \int_D \rho V_{33}(x_0,t;y,0) * b_3(y,t)\,\mathrm{d}S(y)$$

$$+ \int_D \rho[\dot{V}_{33}(x_0,t;y,0)u_{03}(y) + V_{33}(x_0,t;y,0)\dot{u}_{03}(y)]\,\mathrm{d}S(y)\,, \tag{3.9}$$

where

$$C^{*}_{ki}(x_0)g_i(x_0,t) = \lim_{\epsilon\to 0} \int_{\partial(\Omega_{0,\epsilon})-D} W_{ki}(x_0,t;y,0) * g_i(y,t)\,\mathrm{d}S(y) \tag{3.10}$$

is similar to the previously defined Eq. $(3.1)_2$ and $\fint(\ )\,\mathrm{d}S(y)$ implies improper integral [130, 131].

3.3. *Remarks on formulations*

3.3.1. *The indirect formulation*

In section 3.1 we have formulated integral equations by the use of the Green's identity. In the formulation, unknowns are displacements and tractions on the boundary. Hence, the formulation is called "direct", in the sense that unknowns are direct physical quantities. Alternatively, we can formulate integral equations with the aid of potential theory [48, 70, 71, 98]. In this formulation, unknowns are densities of the simple and double layer potentials, and physical quantities such as displacements and tractions can be obtained indirectly by the use of predetermined densities. Therefore, the method is sometimes called "indirect".

We denote the simple and double layer potentials by

$$S_i(x) = \int_{\partial D} U_{ik}(x,y)\phi_k(y)\,\mathrm{d}S(y)\,, \tag{3.11}$$

$$D_i(x) = \int_{\partial D} U_{ik,l}(x,y)\psi_{kl}(y)\,\mathrm{d}S(y)\,, \tag{3.12}$$

where ϕ_k and ψ_{kl} are densities distributed over the boundary ∂D. For later use, we here summarize properties of the potentials for a sufficiently smooth boundary and for continuous densities [48, 55, 57, 70, 71].

For the kernel function (fundamental solution) for elastostatics $\hat{U}(x,y)$ defined by

$$E_{ij}(\partial)\hat{U}_{jk}(x,y) \equiv C_{ijmn}\hat{U}_{jk,mn}(x,y) = -\delta_{ik}\delta(x-y)\,, \tag{3.13}$$

section 4.3.2 for $\hat{U}_{jk}(x,y)$ for the isotropic case), the potentials have the following properties [70, 90]:

(i) Simple layer potential is continuous throughout $\mathbb{R}^N$ and

$$(\partial_i S_j)^{\pm}(x) = \mp \tfrac{1}{2}n_i(E_{jk})^{-1}(n)\phi_k(x) + \fint_{\partial D} \partial_i\hat{U}_{jk}(x,y)\phi_k(y)\,\mathrm{d}S(y)\,, \qquad x \in D$$

$$E_{ij}^{-1}(n) = \frac{1}{\mu}\left(\delta_{ij} - \frac{\lambda+\mu}{\lambda+2\mu}\,n_i n_j\right) \tag{3.14}$$

for the isotropic case on ∂D, where $E_{ij}^{-1}(n)$ is the inverse operator of E_{ij}. In particular,

$$(\overset{n}{\mathbf{T}}S)^{\pm}(x) = \mp\tfrac{1}{2}\,\phi(x)\mathbf{1}(x) + \fint_{\partial D} \overset{n_x}{\mathbf{T}}\hat{U}(x,y)\cdot\phi(y)\,\mathrm{d}S(y)\,. \tag{3.15}$$

(ii) Double layer potential satisfies

$$D_i^{\pm}(x) = \pm\tfrac{1}{2}\,n_j E_{jk}^{-1}(n)\psi_{kl}(x) + \fint_{\partial D} \hat{U}_{ik,l}(x,y)\psi_{kl}(y)\,\mathrm{d}S(y)\,, \tag{3.16}$$

where $(\ )_{,l}$ implies differentiation with respect to y, and its derivative is continuous throughout $\mathbb{R}^N$.

Making use of these properties and noting that the order of the singularities of the fundamental solutions for elastodynamics is the same as that of elastostatics, we can formulate indirect boundary integral equations with ease. As an example, we formulate the ones for the second exterior problem. If we assume that the displacement can be expressed in terms of the simple layer potential, we immediately have for a smooth boundary

$$-\tfrac{1}{2}\mathbf{1}(x_0)\cdot\hat{\phi}(x_0;\omega) + \fint_{\partial D} \overset{n_x}{\mathbf{T}}\hat{U}(x_0,y;\omega)\cdot\hat{\phi}(y;\omega)\,\mathrm{d}S(y) = \hat{\bar{t}}(x_0;\omega)\,. \tag{3.17}$$

If we assume that the displacement can be expressed in terms of the double layer potential, we simply have

$$\hat{\bar{t}}(x_0;\omega) = \overset{n_x}{\mathbf{T}}\int_{\partial D} \overset{n_y}{\mathbf{T}}\hat{U}(x,y;\omega)\cdot\psi(y;\omega)\,\mathrm{d}S(y)\,, \tag{3.18}$$

although the kernel of this integral equation of the first kind is strongly singular and the integral may be evaluated by regularizing the kernel. For other forms of indirect formulations, see references [57, 98].

The indirect formulations are often conveniently used for the first and second boundary value problems.

3.3.2. *Integral equations for inhomogeneous media* [92–95]

If the field is inhomogeneous and the elastic moduli and mass density vary as a function of position, we have to modify the treatment, since we do not know the fundamental solutions for such fields. In what follows, we formulate domain-type integral equations in the Fourier-transformed domain.

In the Fourier-transformed domain, the Navier–Cauchy equation (1.7a) is expressed as:

$$\{C_{ijkl}(x)\hat{u}_{k,l}(x;\omega)\}_{,j} + \rho(x)\hat{b}_i(x;\omega) + \rho(x)\omega^2\hat{u}_i(x;\omega) = 0\,, \tag{3.19}$$

where C_{ijkl} is, in general, the anisotropic elastic tensor.

Here we introduce for later convenience the differentiated volume potential

$$V_i(x) = \int_D \hat{U}_{jk,l}(x,y)\nu_{kl}(y)\,\mathrm{d}V(y)\,. \tag{3.20}$$

This potential is continuous througout $\mathbb{R}^N$ and its derivative can be written as

$$\partial_i V_j(x) = \begin{cases} v_{ijkl}\nu_{kl}(x) + \displaystyle\fint_D \partial_i\hat{U}_{jk,l}(x,y)\nu_{kl}(y)\,\mathrm{d}V(y)\,, & x\in D \\ \displaystyle\int_D \partial_i V_{jk,l}(x,y)\nu_{kl}(y)\,\mathrm{d}V(y)\,, & x\in D_c \end{cases} \tag{3.21}$$

where v_{ijkl} is a tensor defined by

$$v_{ijkl} = \frac{1}{|S^N|}\int_{S^N} \xi_i E_{jk}^{-1}(\xi)\xi_l\,\mathrm{d}S(\xi)$$
$$= \frac{\{N(\lambda+2\mu)+(\lambda+3\mu)\}\delta_{il}\delta_{jk} - (\lambda+\mu)(\delta_{ij}\delta_{kl}+\delta_{ik}\delta_{jl})}{\mu(\lambda+2\mu)N(N+2)} \tag{3.22}$$

for the isotropic case, where S^N is the N-dimensional unit sphere and $|S^N|$ is the area of S^N. The derivative shows a discontinuity across ∂D, and

$$(\partial_i V_j)^{\pm}(x) = \tfrac{1}{2}\, v_{ijkl}\nu_{kl}(x) \mp \tfrac{1}{2} n_i E_{jk}^{-1}(n) n_l \nu_{kl}(x)$$
$$+ \fint_D \partial_i\hat{U}_{jk,l}(x,y)\nu_{kl}(y)\,\mathrm{d}V(y)\,, \quad x\in\partial D\,, \tag{3.23}$$

where

$$\fint (\ )(x,y)\,\mathrm{d}V(y) = \lim_{\epsilon\to 0}\int_{D-B(x,\epsilon)} (\ )(x,y)\,\mathrm{d}V(y)$$

is defined in the principal-value sense.

If the density ν_{ij} is a constant over D, we have

$$\int_D \hat{U}_{ij,k}(x,y)\nu_{jk}(y)\,\mathrm{d}V(y) = \int_{\partial D} \hat{U}_{ij}(x,y)n_k\nu_{jk}(y)\,\mathrm{d}S(y)\,. \tag{3.24}$$

Now, we are ready to formulate integral equations. Multiplying Eq. (3.19) by $\hat{U}_{im}(x,y)$, and intergrating over the volume domain and utilizing Gauss's divergence theorem and the reciprocity of $\hat{U}$, i.e. $\hat{U}_{im}(x,y) = \hat{U}_{mi}(y,x)$, we have the following expression after some manipulation:

$$\int_{\partial D} \{\hat{U}_{mi}(x,y)t_i(y) - \hat{W}_{mi}(x,y)u_i(y)\}\,\mathrm{d}S(y)$$
$$+ \int_D \{\hat{U}_{mi,j}(x,y)C_{ijkl}(y)\}_{,l}u_k(y)\,\mathrm{d}V(y)$$
$$+ \int_D \hat{U}_{mi}(x,y)\{\rho(y)b_i(y) + \rho(y)\omega^2 u_i(y)\}\,\mathrm{d}V(y) = 0\,, \tag{3.25}$$

where

$$\hat{W}_{mi}(x,y) = n_l(y)C_{kpil}(y)\hat{U}_{mk,p}(x,y)$$

and $(\ )_{,j} = \partial(\ )/\partial y_j$, and C_{ijkl} is assumed continuous.

Further, making use of the properties of the aforementioned potentials, we have

$$\epsilon_{mk}(x)u_k(x) = \int_{\partial D} \{\hat{U}_{mi}(x,y)t_i(y) - \hat{W}_{mi}(x,y)u_i(y)\}\,\mathrm{d}S(y)$$
$$+ \fint_D \hat{U}_{mi,jl}(x,y)C_{ijkl}(y)u_k(y)\,\mathrm{d}V(y)$$
$$+ \int_D \hat{U}_{mi,j}(x,y)C_{ijkl,l}(y)u_k(y)\,\mathrm{d}V(y)$$
$$+ \int_D \hat{U}_{mi}(x,y)\{\rho(y)b_i(y) + \rho(y)\omega^2 u_i(y)\}\,\mathrm{d}V(y)\,, \tag{3.26}$$

where

$$\varepsilon_{mk}(x) = \begin{cases} v_{lmij} C_{ijkl}(x) , & x \in D , \\ 0 , & x \in D_c , \\ \frac{1}{2} V_{lmij} C_{ijkl}(x) , & x \in \partial D . \end{cases}$$

If C_{ijkl} is not continuous, we have to modify the result. We assume a jump in C_{ijkl} when crossing ∂S such that

$$\Delta C_{ijkl}(x) = \lim_{x^- \to x} C_{ijkl}(x) - \lim_{x^+ \to x} C_{ijkl}(x) \tag{3.27}$$

where x^- and x^+ imply the minus and plus sides of ∂S. The integral equation for the domain in which a discontinuity surface in C_{ijkl} is included can be easily obtained by adding two equations, each of which is formulated for each domain of either side of ∂S, i.e.

$$\begin{aligned} &\int_{\partial D} \{\hat{U}_{mi}(x, y) t_i(y) - \hat{W}_{mi}(x, y) u_i(y)\} \, dS(y) - \int_{\partial S} \Delta \hat{W}_{mi}(x, y) u_i(y) \, dS(y) \\ &+ \int_D \{\hat{U}_{mi,j}(x, y) C_{ijkl}(y)\}_{,l} u_k(y) \, dV(y) \\ &+ \int_D \hat{U}_{mi}(x, y)\{\rho(y) b_i(y) + \rho(y)\omega^2 u_i(y)\} \, dV(y) \\ &= \begin{cases} v_{lmij} C_{ijkl}(x) u_k(x) , & x \in D - \partial S \\ \frac{1}{2} v_{lmij}\{C_{ijkl}(x) + C'_{ijkl}(x)\} u_k(x) , & x \in \partial S \\ \frac{1}{2} v_{lmij} C_{ijkl}(x) u_k(x) , & x \in \partial D \\ 0 , & x \in D_c , \end{cases} \end{aligned} \tag{3.28}$$

where

$$\Delta \hat{W}_{mi}(x, y) = n_l(y) \Delta C_{ijkl}(y) \hat{U}_{km,j}(x, y) , \qquad y \in \partial S .$$

In the special case, when the elastic medium is homogeneous and isotropic, Eq. (3.26) reduces to the integral equation for elastostatics with body force $\rho \boldsymbol{b} + \rho\omega^2 \boldsymbol{u}$. When the body force $\boldsymbol{b}$ is disregarded, as it is often the case in elastodynamics, Eq. (3.26) shows that time-harmonic elastodynamic problems can also be solved by the use of domain-type integral equations with virtual body forces of the d'Alembert type. Synthesizing time-harmonic solutions, we have of course transient solutions.

The domain integrals, however, sometimes can be converted to boundary integrals by the technique which will be mentioned in section 4.4.4. A formulation in the time–space domain of such a type was studied by Nardini and Brebbia [88, 89].

Two-dimensional problems can be also similarly formulated

3.3.3. *Eigenfrequency problems*

Eigenfrequencies can be obtained as the real and non-negative parameters ω (circular frequencies) which assure the existence of non-trivial solutions of the homogeneous boundary integral equations derived by just deleting all the terms relevant to the prescribed boundary values. This leads to the well known condition that the determinant of the coefficient matrix of the discretized equations of the system of boundary integral equations is equal to zero. In the numerical treatment, it is hard to obtain the true roots of the determinant, since ω is implicitly contained in the fundamental solutions $\hat{U}(x, y; \omega)$ and $\hat{W}(x, y; \omega)$. However, if the frequency ω is very close to one of the eigenfrequencies, the value of the determinant becomes very close to zero. This phenomenon can easily be detected during the triangular (L – U) decomposition process by the Crout method [66]. Hence, roots of the determinant are searched using the (L – U) decomposition by direct substitution of successively increased values of ω into the fundamental solutions [57, 98].

When we use the domain-type formulation such as Eq. (3.26), we may obtain eigenfrequencies more easily, since the parameter ω is explicitly expressed.

It is noted here that

(i) for the boundary integral equation of the second kind, the integral equations for the first (second) exterior problems determine the eigenfrequencies of the second (first) interior problems, and

(ii) for the boundary integral equations of the first kind, the integral equations for the first (second) exterior problems determine the eigenfrequencies of the first (second) interior problems [57, 61, 98].

Eigenmodes are determined from the discretized (boundary) equations by substituting the eigenfrequencies [57, 98].

Free vibration problems of thin plates have been also extensively studied [57, 88, 89, 98, 126].

3.3.4. *Fictitious eigenfrequencies*

It is well known that the Fourier-transformed domain boundary integral formulation for the exterior problems in acoustics fails to yield a unique solution at certain wave-numbers which correspond to the eigenfrequencies of the interior region of the closed surface [22, 73, 112, 125]. A similar fact is also known in elastodynamics [66, 70, 106]. These special wavenumbers are called "eigen-" or "characteristic" wavenumbers and the corresponding eigenfrequencies are termed "fictitious" eigenfrequencies. When we solve transient exterior problems, we must pay special attention to the treatment of the fictitious eigenfrequencies.

In order to circumvent the fictitious eigenfrequencies, the following four types of techniques have been proposed:

(i) To use the interior representation, say Eq. (3.5) for $x \in D$, that is, with null displacements in the interior, [22, 70, 120], or use it as a supplementary condition for the boundary integral equation [69, 104, 106, 112].

(ii) To utilize two integral equations, for example (a) to solve simultaneously both the first and the second kind integral equations for the same problem, (b) to solve the combined equation of them with a suitably selected coupling constant [12, 85], or (c) to use a mixed potential [45].

(iii) To formulate a boundary integral equation using modified fundamental solutions which satisfy a certain vanishing condition inside the interior domain [10, 50, 65, 125].

(iv) To use interpolation techniques and numerically approximate behaviors corresponding to the fictitious eigenfrequencies by using several values of ω other than the eigenfrequencies of the interior domain [66].

3.3.5. *Peripheral stresses* [66]

The evaluation of the peripheral stresses may be the most important problem. The boundary integral equation method is suited for this purpose.

(i) For the three-dimensional case, expressing the boundary surface ∂D by a curvilinear coordinate system $\xi^K(K = 1, 2)$ and ξ^3 in the positive normal direction $\boldsymbol{n}$ to ∂D, the peripheral stresses have the form

$$\left.\begin{aligned} \sigma_{KL} &= \lambda a_{KL} \frac{2\mu u_{K|L} a^{KL} + t_3}{\lambda + 2\mu} + \mu(u_{K|L} + u_{L|K}) \\ \sigma_{K3} &= t_K , \qquad \sigma_{33} = t_3 , \end{aligned}\right\} \tag{3.29}$$

where $u_{K|L}$ indicates the (three-dimensional) covariant derivative defined as

$$u_{K|L} = \frac{\partial u_K}{\partial \xi^L} - u_M \Gamma^M_{KL} - u_3 b_{KL} \tag{3.30}$$

and a_{KL}, b_{KL}, Γ^M_{KL} stand for the first and second fundamental forms of the surface and the Christoffel symbol calculated from a_{KL}. They are defined as

$$\left.\begin{aligned} ds^2 &= dx^i\, dx^i = a_{KL}\, d\xi^K\, d\xi^L , \\ b_{KL} &= \frac{1}{\sqrt{a}}\, \varepsilon_{ijk} \frac{\partial^2 x^i}{\partial \xi^K \partial \chi^L} \frac{\partial x^j}{\partial \xi^1} \frac{\partial x^k}{\partial \xi^2} , \\ \Gamma^M_{KL} &= \tfrac{1}{2} a^{MN}\left(\frac{\partial a_{KN}}{\partial \xi^L} + \frac{\partial a_{LN}}{\partial \xi^K} - \frac{\partial a_{KL}}{\partial \xi^N}\right) , \\ a^{KL} a_{LM} &= \delta^K_M . \end{aligned}\right| \tag{3.31}$$

for the rectangular Cartesian coordinate system x^i, where ds is a line element, ε_{ijk} means the permutation symbol and $a = \det(a_{KL})$.

(ii) For the two-dimensional case, we can write down the result in physical components as

$$\left.\begin{aligned} \sigma_{\xi\xi} &= \frac{4\mu(\lambda + \mu)}{(\lambda + 2\mu)a} \frac{\partial u_\alpha}{\partial \xi} \frac{\partial x_\alpha}{\partial \xi} + \frac{\lambda n_\alpha t_\alpha}{\lambda + 2\mu} \\ \sigma_{\xi n} &= \frac{1}{\sqrt{a}}\, t_\alpha \frac{\partial x_\alpha}{\partial \xi} , \qquad \sigma_{nn} = n_\alpha t_\alpha , \end{aligned}\right\} \tag{3.32}$$

where ξ is the parameter which describes the boundary curve $x_\alpha(\xi)$ and

$$a = \frac{\partial x_\alpha}{\partial \xi} \frac{\partial x_\alpha}{\partial \xi} .$$

Note that the vectors on the right-hand sides of Eqs. (3.32) are referred to the Cartesian coordinate system.

3.3.6. *Use of Green's functions*

As a kernel function of integral equations, it is advisable to use a Green's function, if available, in place of the fundamental solution. Even if it satisfies only a certain amount of the boundary conditions of the problem, it may be helpful to reduce the dimension of the discretized equations and to improve the solutions at the same time.

Green's functions for the half-space are available in [11, 49] for the time domain and in [13, 60, 76] for the frequency domain. A semi-infinite element for Rayleigh waves has also been devised [67].

For the anti-plane problem, the Green function for a half-space is easily obtained by superposing the fundamental solution of the infinite space to its mirror image with respect to the traction free boundary.

4. Numerical procedures

4.1. *Discretization of integral equations*

The integral equation thus obtained is usually solved numerically by discretizing it into a system of algebraic equations with the aid of a scheme similar to that used in the finite element method.

The boundary integral is evaluated for each boundary element numerically, say by the use of the Gaussian quadrature formula. Usually quadratic isoparametric elements are used for it, although constant elements may be also applicable. For the evaluation of domain integrals, a similar technique is adopted.

In what follows, we show a discretization procedure of the integral equation in conjunction with the collocation technique [72, 128]. In an isoparametric element, the local coordinate ξ_i in the natural coordinate system is related to the global coordinate x_i in the rectangular Cartesian coordinate system by the expression

$$x_i(\xi) = N^\alpha(\xi)x_i^\alpha , \tag{4.1}$$

where x_i^α is the coordinate of the nodal point $\alpha(\alpha = 1, 2, 3)$ and N^α implies the shape function with respect to the node. The shape of the boundary element is approximated by Eq. (4.1) and functions on the element can be similarly expressed by

$$u_i(\xi) = N^\alpha(\xi)u_i^\alpha , \qquad t_i(\xi) = N^\alpha(\xi)t_i^\alpha . \tag{4.2}$$

The function $b(x)$ in the domain is also similarly approximated by

$$b_i(\xi) = M^\gamma(\xi)b_i^\gamma , \tag{4.3}$$

where the shape function M^γ may not be in general the same as the N^α.

By discretizing now the boundary into M segments with N nodes and the domain into P elements having Q nodes, and using the above-mentioned approximation, we have a system of algebraic equation. Equation (3.2), for example, becomes in discretized form

$$\begin{aligned} &C_{ik}^{\circ}(x^a)\hat{u}_i(x^a;\omega) \\ &\quad + \sum_{\beta=1}^{M}\sum_{\alpha=1}^{N} \hat{u}_i(x^{k(\beta,\alpha)};\omega) \int_{\partial D^\beta} N^\alpha(\xi)\hat{W}_{ki}^{\beta}(x^a, y(\xi);\omega)J_S(\xi)\,dS(\xi) \\ &= \sum_{\beta=1}^{M}\sum_{\alpha=1}^{N} \hat{t}_i(x^{k(\beta,\alpha)};\omega) \int_{\partial D^\beta} N^\alpha(\xi)\hat{U}_{ki}^{\beta}(x^a; y(\xi);\omega)J_S(\xi)\,dS(\xi) \\ &\quad + \sum_{\delta=1}^{P}\sum_{\gamma=1}^{Q} \hat{b}_i(x^{h(\delta,\gamma)};\omega) \int_{D^\delta} \rho M^\gamma(\xi)\hat{U}_{ki}^{\delta}(x^a, y(\xi);\omega)J_V(\xi)\,dV(\xi) , \end{aligned} \tag{4.4}$$

where we denote the number k of the boundary node α of the element β in the global coordinates by $k(\beta, \alpha)$ and the number h of the node γ in the domain element δ in the global coodinates by $h(\delta, \gamma)$, and $J_S(\xi)$ and $J_V(\xi)$ imply the Jacobians with respect to the surface and the volume integrals, respectively. It is also noted that the summation convention rule is implied over the repeated index i.

This equation can be simply rewritten in matrix form for the unknown nodal values $\hat{u}$ and $\hat{t}$ as

$$\mathbf{W}\hat{u} - \mathbf{U}\hat{t} = F . \tag{4.5}$$

where F contains values due to prescribed boundary conditions and the last integral in Eq. (4.4).

4.2. *Time-marching process*

Making use of the space and time interpolation (basis) functions N and ϕ, respectively, we can express the time–space function $f(x, t)$ in the form [81]

$$f_i(\xi, \tau) = N^\alpha(\xi)\phi_\beta(\tau)f_i(x^\alpha, t_\beta) . \tag{4.6}$$

If the boundary is discretized into M elements having N nodes and the time interval is divided into K steps having L nodes, substitution of Eq. (4.6) into Eq. (3.7), for example, leads to the following equation:

$$\begin{aligned} &C_{ki}^{\circ}(x^a)u_i(x^a, t) \\ &\quad + \sum_{m=1}^{M}\sum_{\alpha=1}^{N} \int_{\partial D^m} \Big\{ \sum_{k=1}^{K}\sum_{\beta=1}^{L} \Big(\int_{t_{k-1}}^{t_k} W_{ki}(x^a, \tau; y(\xi), 0)\phi_\beta(t-\tau)\,d\tau \Big) \\ &\quad \times u_i(x^{(m,\alpha)}, t_{(k,\beta)}) \Big\} N^\alpha(\xi)J_S(\xi)\,dS(\xi) \\ &= \sum_{m=1}^{M}\sum_{\alpha=1}^{N} \int_{\partial D^m} \Big\{ \sum_{k=1}^{K}\sum_{\beta=1}^{L} \Big(\int_{t_{k-1}}^{t_k} U_{ki}(x^a, \tau; y(\xi), 0)\phi_\beta(t-\tau)\,d\tau \Big) \\ &\quad \times t_i(x^{(m,\alpha)}, t_{(k,\beta)}) N^\alpha(\xi)J_S(\xi)\,dS(\xi) , \end{aligned} \tag{4.7}$$

where the body force is disregarded, the quiescent past is assumed for simplicity, and $x^{(m,\alpha)}$ and $t_{(k,\beta)}$ refer to the node α of the space element m and the node β of the time element k.

As an example, if we assume a linear variation in time within each time-step according to

$$\phi_1 = \frac{t_k - t}{\Delta t_k} , \qquad \phi_2 = \frac{t - t_{k-1}}{\Delta t_k} , \qquad \Delta t_k = t_k - t_{k-1} ,$$

the time convolution integral in Eq. (4.7) in each time-step reduces to

$$\begin{aligned} &\Big\{ \int_{t_{k-1}}^{t_k} W_{ki}(x^a, \tau; y(\xi), 0) \frac{t_k - \tau}{\Delta t_k}\,d\tau \Big\} u_i(x^{(m,\alpha)}, t_{(k,1)}) \\ &\quad + \Big\{ \int_{t_{k-1}}^{t_k} W_{ki}(x^a, \tau; y(\xi), 0) \frac{\tau - t_{k-1}}{\Delta t_k} \Big\} u_i(x^{(m,\alpha)}, t_{(k,2)}) \end{aligned}$$

with a similar result for terms involving U.

Since $\boldsymbol{U}$ and $\boldsymbol{W}$ involve time functions as shown in Eqs. (1.46) and (1.48), the convolution integral may be evaluated by the use of the following results:

$$\int_{t_{k-1}}^{t_k} \tau \frac{t_k - \tau}{\Delta t_k} H\left(\tau - \frac{r}{c_\alpha}\right) \mathrm{d}\tau = \begin{cases} \dfrac{1}{6\Delta t_k}\left(t_k^3 - 3t_k t_{k-1}^2 + 2t_{k-1}^3\right), & t_{k-1} \geq \dfrac{r}{c_\alpha}, \\ \dfrac{1}{6\Delta t_k}\left\{t_k^3 - 3t_k\left(\dfrac{r}{c_\alpha}\right)^2 + 2\left(\dfrac{r}{c_\alpha}\right)^3\right\}, & t_k \geq \dfrac{r}{c_\alpha} \geq t_{k-1}, \\ 0, & \dfrac{r}{c_\alpha} \geq t_k, \end{cases}$$

$$\int_{t_{k-1}}^{t_k} \tau \frac{\tau - t_{k-1}}{\Delta t_k} H\left(t - \frac{r}{c_\alpha}\right) \mathrm{d}\tau$$
$$= \begin{cases} \dfrac{1}{6\Delta t_k}\left(2t_k^3 - 3t_k^2 t_{k-1} + t_{k-1}^3\right), & t_{k-1} \geq \dfrac{r}{c_\alpha}, \\ \dfrac{1}{6\Delta t_k}\left\{2t_k^3 - 3t_k^2 t_{k-1} + 3t_{k-1}\left(\dfrac{r}{c_\alpha}\right)^2 - 2\left(\dfrac{r}{c_\alpha}\right)^3\right\}, & t_k \geq \dfrac{r}{c_\alpha} \geq t_{k-1}, \\ 0, & \dfrac{r}{c_\alpha} \geq t_k, \end{cases}$$

$$\int_{t_{k-1}}^{t_k} \frac{t_k - \tau}{\Delta t_k} \delta\left(\tau - \frac{r}{c_\alpha}\right) \mathrm{d}\tau = \begin{cases} \dfrac{1}{\Delta t_k}\left(t_k - \dfrac{r}{c_\alpha}\right), & t_{k-1} \leq \dfrac{r}{c_\alpha} \leq t_k, \\ 0, & \dfrac{r}{c_\alpha} \geq t_k, \dfrac{r}{c_\alpha} \leq t_{k-1}, \end{cases}$$

$$\int_{t_{k-1}}^{t_k} \frac{\tau - t_{k-1}}{\Delta t_k} \delta\left(\tau - \frac{r}{c_\alpha}\right) \mathrm{d}\tau = \begin{cases} \dfrac{1}{\Delta t_k}\left(\dfrac{r}{c_\alpha} - t_{k-1}\right), & t_{k-1} \leq \dfrac{r}{c_\alpha} \leq t_k, \\ 0, & \dfrac{r}{c_\alpha} \geq t_k, \dfrac{r}{c_\alpha} \leq t_{k-1}, \end{cases}$$

$$\int_{t_{k-1}}^{t_k} \frac{t_k - \tau}{\Delta t_k} \dot{\delta}\left(\tau - \frac{r}{c_\alpha}\right) \mathrm{d}\tau = \begin{cases} 1, & t_{k-1} \leq \dfrac{r}{c_\alpha} \leq t_k, \\ 0, & \dfrac{r}{c_\alpha} \geq t_k, \dfrac{r}{c_\alpha} \leq t_{k-1}, \end{cases}$$

where the relation

$$\int_a^b \delta^{(k)}(\tau - \alpha) f(\tau)\, \mathrm{d}\tau = (-1)^{(k)} f^{(k)}(\alpha), \qquad a \leq \alpha \leq b$$

has been used.

It should be noted that the convolution integral has meaning only in the interval of the past time characteristic cone. It may also be said that the stability condition for time stepping is less stringent than that of conventional explicit difference marching process, since the time is implicitly contained in the integral equation.

A more comprehensive discussion on the numerical treatment of elastodynamic boundary integrals can be found in the next chapter of this book.

4.3. *Combination of integral equation method and finite element method*

In order to solve non-homogeneous or nonlinear problems for infinite or semi-infinite domains that are frequently encountered in practical applications, the combination of the boundary integral equation method (BIEM) and the finite element method (FEM) is most useful [56, 83]. Of course, the finite element method is used for the interior domain where non-homogeneity or nonlinearity is predominant and the boundary integral equation method is utilized for the rest of the infinite or semi-infinite domain which may well be regarded as homogeneous and linear.

A weak form (virtual work) of the elastodynamic problem in the Fourier-transformed domain can be expressed for the interior domain with an arbitrary vector $\hat{\boldsymbol{v}}(\boldsymbol{x}; \omega)$ as [136]

$$\int_D \hat{\boldsymbol{\epsilon}} : \hat{\boldsymbol{\sigma}}\, \mathrm{d}V = \int_D \rho\omega^2 \hat{\boldsymbol{v}} \cdot \hat{\boldsymbol{u}}\, \mathrm{d}V + \int_{\partial D_2} \hat{\boldsymbol{v}} \cdot \hat{\bar{\boldsymbol{t}}}\, \mathrm{d}S \tag{4.8}$$

where $\hat{\boldsymbol{\epsilon}}$ means strain given in matrix form as

$$\hat{\boldsymbol{\epsilon}} = \mathbf{C}\hat{\boldsymbol{v}} \tag{4.9}$$

$\mathbf{C}$ is an operator whose components are given by

$$\tfrac{1}{2}\left(\delta_{im} \frac{\partial}{\partial x_j} + \delta_{jm} \frac{\partial}{\partial x_i}\right) \tag{4.10}$$

and $\hat{\bar{\boldsymbol{t}}}$ is the prescribed traction on ∂D_2.

By expressing now the displacement vectors and the surface traction vectors for each element in terms of the shape function N^k and the respective nodal values $\tilde{\boldsymbol{u}}$, $\tilde{\boldsymbol{v}}$ and $\tilde{\boldsymbol{t}}$ as in Eq. (4.2),

$$\hat{\boldsymbol{u}} = \mathbf{N}\tilde{\boldsymbol{u}}, \qquad \hat{\boldsymbol{v}} = \mathbf{N}\tilde{\boldsymbol{v}}, \qquad \hat{\boldsymbol{t}} = \mathbf{N}\tilde{\boldsymbol{t}} \tag{4.11}$$

with

$$\mathbf{N} = [\mathbf{N}^1 \quad \mathbf{N}^2 \quad \mathbf{N}^3 \quad \cdots], \qquad \mathbf{N}^k = \begin{bmatrix} N_1^k & 0 & 0 \\ 0 & N_2^k & 0 \\ 0 & 0 & N_3^k \end{bmatrix},$$

$$\tilde{\boldsymbol{u}}^{\mathrm{T}} = [\boldsymbol{u}^1 \quad \boldsymbol{u}^2 \quad \boldsymbol{u}^3 \quad \cdots], \qquad \hat{\boldsymbol{u}}^{\mathrm{T}} = [u_1 \quad u_2 \quad u_3]$$

and substituting them into Eq. (4.8) together with the constitutive relation

$$\dot{\boldsymbol{\sigma}} = \mathbf{D}\dot{\boldsymbol{\varepsilon}} = \mathbf{DC}\tilde{\boldsymbol{u}} \tag{4.12}$$

(where $\mathbf{D}$ is the elastic matrix), we have the following finite element equation

$$\sum_e \Big(\int_{D^e} \mathbf{B}^t \mathbf{D} \mathbf{B} \,\mathrm{d}V - \rho\omega^2 \int_{D^e} \mathbf{N}^t \mathbf{N} \,\mathrm{d}V \Big) \tilde{\boldsymbol{u}}^e = \sum_e \Big(\int_{\partial D^e} \mathbf{N}^t \mathbf{N} \,\mathrm{d}S \Big) \tilde{\boldsymbol{t}}^e \tag{4.13}$$

where $\mathbf{B}$ is the strain matrix given by

$$\mathbf{B} = \mathbf{CN} \tag{4.14}$$

and $\sum_e$ means to take sum over the relevant elements.

We simply write the system of algebraic equations (4.13) as

$$\mathbf{K}\tilde{\boldsymbol{u}} = \mathbf{H}\tilde{\boldsymbol{t}}\,. \tag{4.15}$$

Combining Eqs. (4.15) and (4.5) with the aid of compatibility and equilibrium conditions at the interface between the interior and the exterior domains, we have a system of equations. In order to fix ideas, consider an example as shown in Fig. 2, in which a part of the interior domain is embedded in a half-space exterior domain.

We denote the boundaries of the interior and the exterior domains by S_i and S_e, respectively, their common interface by S, and also use these symbols as subscripts for their related quantities. Here the boundary S_e is assumed temporarily finite.

We have then the finite element equation for the interior domain and the boundary integral equation for the exterior domain as

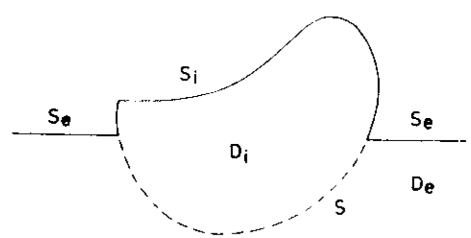


Fig. 2. A model for the BIE–FEM combined method.

$$\begin{bmatrix} \mathbf{K}_{II} & \mathbf{K}_{IS_i} & \mathbf{K}_{IS} \\ \mathbf{K}_{IS_i} & \mathbf{K}_{S_iS_i} & \mathbf{K}_{S_iS} \\ \mathbf{K}_{IS} & \mathbf{K}_{S_iS} & \mathbf{K}_{SS} \end{bmatrix} \begin{bmatrix} \tilde{\boldsymbol{u}}^i_I \\ \tilde{\boldsymbol{u}}^i_{S_i} \\ \tilde{\boldsymbol{u}}^i_S \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{H}_{S_i}\tilde{\boldsymbol{t}}^i_{S_i} \\ \mathbf{H}_S\tilde{\boldsymbol{t}}^i_S \end{bmatrix} \tag{4.16}$$

$$[\mathbf{W}_{S_e} \quad \mathbf{W}_S] \begin{bmatrix} \tilde{\boldsymbol{u}}^e_{S_e} \\ \tilde{\boldsymbol{u}}^e_S \end{bmatrix} - [\mathbf{U}_{S_e} \quad \mathbf{U}_S] \begin{bmatrix} \tilde{\boldsymbol{t}}^e_{S_e} \\ \tilde{\boldsymbol{t}}^e_S \end{bmatrix} = [\boldsymbol{F}_I] \tag{4.17}$$

where $\boldsymbol{F}_I$ is a field due to the incident wave (cf. Eq. (3.4)) and the subscript F implies quantities related to the interior domain.

Using the continuity conditions on S (note that the positive unit normal is taken as pointing from the interior to the exterior domain)

$$\tilde{\boldsymbol{u}}^i_S = \tilde{\boldsymbol{u}}^e_S - \hat{\boldsymbol{u}}_S\,, \qquad \tilde{\boldsymbol{t}}^i_S + \tilde{\boldsymbol{t}}^e_S = \mathbf{0} \tag{4.18}$$

and the boundary conditions, say,

$$\tilde{\boldsymbol{t}}^i_{S_i} = \bar{\boldsymbol{t}}_{S_i}\,, \qquad \tilde{\boldsymbol{t}}^e_{S_e} = \bar{\boldsymbol{t}}_{S_e}\,, \tag{4.19}$$

we have the combined system of equations as follows:

$$\begin{bmatrix} \mathbf{K}_{II} & \mathbf{K}_{IS_i} & \mathbf{K}_{IS} & 0 & 0 \\ \mathbf{K}_{IS_i} & \mathbf{K}_{S_iS_i} & \mathbf{K}_{S_iS} & 0 & 0 \\ \mathbf{K}_{IS} & \mathbf{K}_{S_iS} & \mathbf{K}_{SS} & -\mathbf{H}_S & 0 \\ \mathbf{0} & \mathbf{0} & \mathbf{W}_S & -\mathbf{U}_S & \mathbf{W}_{S_e} \end{bmatrix} \begin{bmatrix} \tilde{\boldsymbol{u}}^i_I \\ \tilde{\boldsymbol{u}}^i_{S_i} \\ \hat{\boldsymbol{u}}_S \\ \tilde{\boldsymbol{t}}^e_S \\ \tilde{\boldsymbol{u}}^e_{S_e} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{H}_{S_i}\bar{\boldsymbol{t}}_{S_i} \\ \mathbf{0} \\ \boldsymbol{F}_I + \mathbf{U}_{S_e}\bar{\boldsymbol{t}}_{S_e} \end{bmatrix} \tag{4.20}$$

As the right-hand side of the above equation is known from the incident wave field and prescribed boundary conditions, we can solve this equation for the nodal displacements $\tilde{\boldsymbol{u}}$ in D_i, on S_i and S_e, and for the displacement $\hat{\boldsymbol{u}}$ and the traction $\tilde{\boldsymbol{t}}$ on S.

4.4. *Miscellaneous topics in numerical treatment*

4.4.1. *Numerical integration*

In numerical procedures, evaluation of integrals over each element is of basic importance. Element size must be chosen in general sufficiently small so as to express the shortest wavelength appropriately, that is, at least four or five nodal points must be included in the shortest wavelength.

The boundary integrals, whenever non-singular, are usually evaluated with the aid of Gaussian quadrature formula, or double exponential formula [119].

The double exponential formula is such as to compute the integral

$$I = \int_{-1}^{1} f(x)\,\mathrm{d}x \tag{4.21}$$

approximately by using the trapezoidal rule with step size h according to the formula

$$I_h = \frac{\pi}{2} h \sum_{n=-\infty}^{\infty} f(\tanh(\tfrac{1}{2}\pi \sinh nh)) \frac{\cosh nh}{\cosh^2(\frac{1}{2}\pi \sinh nh)} . \tag{4.22}$$

where the integration variable x has been changed by

$$x = \tanh(\tfrac{1}{2}\pi \sinh t)$$

and the integral has been transformed to an infinite one. This formula can easily be extended to the two-dimensional case.

Since the evaluation of boundary integrals occupies a significant amount of computing time, it is advisable to optimize, if possible, by specifying certain maximum error bounds [72].

When the field point coincides with the node of the same element of integration, special attention must be paid. In the numerical integration of the simple layer kernel $\boldsymbol{U}$ for that case, the singular element is divided into a number of triangular subelements in such a manner as the common apex of all subelements to be located on the field point. Then each triangular subelement is mapped on a flat triangle and numerical integration is performed by introducing polar coordinates centered at the singular apex. In the integration of the double layer kernel $\boldsymbol{W}$, since the fundamental solutions consist of the sum of the static counterparts and the regular parts as will be seen in section 4.4.2, the integral can be obtained as a sum of the integral of the regular part and the static contribution, which is actually obtainable as the sum, with minus sign, of the off-diagonal submatrices of the elastostatics. This is based on the fact that the sum of the free-term C^c and the singular integral of elastostatics can be evaluated altoghether by applying arbitrary rigid-body displacements [72, 78, 105–107].

4.4.2. *Expansion of the time-harmonic fundamental solutions*

In the evaluation of the integrals containing time-harmonic fundamental solutions, it is advised to expand them and delete unfavourable terms beforehand, particularly for small argument. Some manipulations can yield the following results: From Eq.(1.40),

$$\hat{U}_{jk}(\boldsymbol{x}, \boldsymbol{y}; \omega) = A(\hat{U}_1 \delta_{jk} - \hat{U}_2 r_{,j} r_{,k}) , \quad \hat{U}_1 = \hat{U}_1 + \hat{U}_1^{\mathrm{D}} , \quad \hat{U}_2 = \hat{U}_2 + \hat{U}_2^{\mathrm{D}} , \tag{4.23a}$$

where

(i) for the three-dimensional case:

$$\left.\begin{aligned} A &= \frac{1}{4\pi\mu} , \quad \hat{U}_1 = \frac{\lambda + 3\mu}{2(\lambda + 2\mu)} \frac{1}{r} , \quad \hat{U}_2 = -\frac{\lambda + \mu}{2(\lambda + 2\mu)} \frac{1}{r} , \\ \hat{U}_1^{\mathrm{D}} &= \sum_{n=1}^{\infty} \frac{i^n k_T^n r^{n-1}}{n!(n+2)} \left\{ (n+1) + \left(\frac{k_L}{k_T}\right)^{n+2} \right\} , \\ \hat{U}_2^{\mathrm{D}} &= \sum_{n=1}^{\infty} \frac{i^n k_T^n r^{n-1}}{n!} \frac{n-1}{n+2} \left\{ 1 - \left(\frac{k_L}{k_T}\right)^{n+2} \right\} , \end{aligned}\right\} \tag{4.23b}$$

$$\hat{U}_{jk} = \frac{1}{8\pi\mu(\lambda + 2\mu) r} \{ (\lambda + 3\mu)\delta_{jk} + (\lambda + \mu) r_{,j} r_{,k} \} , \tag{4.24}$$

(ii) for the two-dimensional case:

$$A = \frac{i}{4\mu} , \quad \hat{U}_1 = \frac{i(\lambda + 3\mu)}{\pi(\lambda + 2\mu)} \ln r , \quad \hat{U}_2 = \frac{i(\lambda + \mu)}{\pi(\lambda + 2\mu)} ,$$

$$\begin{aligned} \hat{U}_1^{\mathrm{D}} = \frac{1}{2(\lambda + 2\mu)} \Bigg| &(\lambda + 3\mu) \\ &+ \frac{i}{\pi} \left\{ (\lambda + \mu) + 2(\lambda + 3\mu)\gamma + 2(\lambda + 2\mu) \ln \frac{k_T}{2} + 2\mu \ln \frac{k_L}{2} \right\} \Bigg| \\ &+ \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2} \left(\frac{k_T r}{2}\right)^{2n} \Bigg| \frac{1}{2(n+1)} \left\{ 2n + 1 + \left(\frac{k_L}{k_T}\right)^{2n+2} \right\} A + B \\ &- \frac{1}{2(n+1)} D + \frac{1}{2(n+1)} \left\{ 1 - \left(\frac{k_L}{k_T}\right)^{2n+2} \right\} E \Bigg| , \end{aligned} \tag{4.23c}$$

$$\begin{aligned} \hat{U}_2^{\mathrm{D}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{(n!)^2} \left(\frac{k_T r}{2}\right)^{2n} \Bigg| &-\frac{n}{n+1} \left\{ 1 - \left(\frac{k_L}{k_T}\right)^{2n+2} \right\} A + B - \left(\frac{k_L}{k_T}\right)^{2n+2} C \\ &- \frac{n}{n+1} D + \frac{1}{n+1} \left\{ 1 - \left(\frac{k_L}{k_T}\right)^{2n+2} \right\} E \Bigg| , \end{aligned}$$

$$A = 1 + \frac{2i}{\pi} \{ \ln r + \gamma \} , \quad B = \frac{2i}{\pi} \left\{ \ln \frac{k_T}{2} - \phi(n) \right\} ,$$

$$C = \frac{2i}{\pi} \left\{ \ln \frac{k_L}{2} - \phi(n) \right\} , \quad D = \frac{2i}{\pi} \left\{ \ln \frac{k_T}{2} - \left(\frac{k_L}{k_T}\right)^{2n+2} \ln \frac{k_L}{2} \right\} ,$$

$$E = \frac{i}{\pi} \{ \phi(n) + \phi(n+1) \} , \quad \gamma = 0.57721 \text{ (Euler constant)} ,$$

$$\phi(n) = \sum_{m=1}^{n} \frac{1}{m} ,$$

$$\hat{U}_{jk} = \frac{1}{4\pi\mu(\lambda + 2\mu)} \left\{ (\lambda + 3\mu) \ln \frac{1}{r} \delta_{jk} + (\lambda + \mu) r_{,j} r_{,k} \right\} , \tag{4.25}$$

(iii) for the anti-plane case:

$$\hat{U}_{33}(x, y; \omega) = \mathring{U}_{33} + \hat{U}^{D}_{33},$$

$$\mathring{U}_{33} = \frac{1}{2\pi\mu}\ln(1/r), \tag{4.23d}$$

$$\hat{U}^{D}_{33} = \frac{i}{4\mu} - \frac{1}{2\pi\mu}\left(\ln\frac{k_1}{2} + \gamma\right) + \frac{i}{4\mu}\sum_{n=1}^{\infty}\frac{(-1)^n}{(n!)^2}\left(\frac{k_1 r}{2}\right)^{2n}(A + B).$$

In these expressions, $\mathring{U}_{jk}$ are fundamental solutions for elastostatics. They show that the order of singularity of elastodynamic fundamental solutions is the same as that of elastostatics.

The second fundamental solutions defined by Eq. (1.44) can also be expanded with simultaneous deletion of unfavourable terms by the use of $\hat{U}_1$, $\hat{U}_2$ given above.

For the anti-plane case, we have

$$\hat{W}_{33}(x, y; \omega) = \mathring{W}_{33} + \hat{W}^{D}_{33}, \tag{4.26a}$$

$$\mathring{W}_{33} = -\frac{1}{2\pi r}\frac{\partial r}{\partial n},$$

$$\hat{W}^{D}_{33} = \frac{ik_1}{4}\left[\sum_{n=0}^{\infty}\frac{(-1)^n}{n!(n+1)!}\left(\frac{k_1 r}{2}\right)^{2n+1}\left\{A - E + \frac{2i}{\pi}\ln\frac{k_1}{2}\right\}\right]\frac{\partial r}{\partial n}. \tag{4.26b}$$

4.4.3. *Circumvention of singularities*

The existence of a geometric singularity such as a corner, an edge, etc. and of an abrupt change in imposed boundary conditions may pollute the solution of integral equations. The standard method to improve the solution is to take properties of singularities into account. Eigenfunction expansion solutions may be used at the singularity which are incorporated into the boundary integral equation [48, 129]. Singular boundary elements are also devised to evaluate boundary integrals [5, 74].

At a corner where several subdomains meet or two subdomains meet the exterior domain, the "double point" technique is also available [14, 108, 127].

Singularity problems are sometimes circumvented by introducing "discontinuous elements" or "partially discontinuous elements" [103], in which boundary functions (quantities) are expressed by using free nodes (arbitrarily chosen points) other than geometric nodes.

4.4.4. *Volume integrals*

Volume integrals can sometimes be converted into surface integrals by the use of Gauss's divergence theorem or the formula given by Eq. (3.24) for the constant element [25, 118].

4.4.5. *Numerical inversion of the Fourier transform*

The numerical Fourier transform and its inversion are conveniently carried out with the aid of the Fast Fourier Transform (FFT) algorithm [9, 21]. The numerical Laplace transform inversion has also been studied in some detail in [87] and the conclusion has been drawn that among the many proposed methods, Durbin's method [34] seems the most suitable one for dynamic problems. Ref. [87] also discusses the numerical direct Laplace transform based on Durbin's method.

5. Dynamic problems in viscoelastic media

5.1. *Viscoelastic media*

With the assumption of infinitesimal deformations, the constitutive relation for a linear viscoelastic medium is expressed by [15]

$$\sigma_{ij}(x, t) = \int_{-\infty}^{t} G_{ijkl}(t - \tau)\frac{d\varepsilon_{kl}(x, \tau)}{d\tau}\,d\tau, \tag{5.1a}$$

where $G_{ijkl}(t)$ characterizes the mechanical behavior of the viscoelastic medium and is termed the relaxation tensor. Using the convolution integral defined over $(-\infty, t]$, we may write Eq. (5.1a) as

$$\sigma_{ij}(x, t) = G_{ijkl}(t) * d\varepsilon_{kl}(x, t). \tag{5.1b}$$

For the homogeneous isotropic medium,

$$G_{ijkl}(t) = \lambda(t)\delta_{ij}\delta_{kl} + \mu(t)(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \tag{5.2}$$

where $\lambda(t)$, $\mu(t)$ are two independent relaxation functions. Hence,

$$\sigma_{ij}(x, t) = \lambda(t) * d\varepsilon_{kk}(x, t)\delta_{ij} + 2\mu(t) * d\varepsilon_{ij}(x, t). \tag{5.3}$$

Here we assume that

$$G_{ijkl}(t) = \mathring{G}_{ijkl} + \bar{G}_{ijkl}(t) \tag{5.4}$$

and $\bar{G}_{ijkl}(t) \to 0$ as $t \to \infty$.

Specifying the strain history as being a time-harmonic function

$$\varepsilon(x, t) = \varepsilon_0(x)\,e^{-i\omega t} \tag{5.5}$$

and introducing it into Eq. (5.1), we have, with the change of variables $t - \tau$ to η

$$\boldsymbol{\sigma}(\boldsymbol{x}, t) = \mathrm{i}\omega\left[\int_0^{\infty} G(\eta)\,\mathrm{e}^{\mathrm{i}\omega\eta}\,\mathrm{d}\eta\right]\boldsymbol{\varepsilon}_0(\boldsymbol{x})\,\mathrm{e}^{-\mathrm{i}\omega t}\,. \tag{5.6}$$

Since $G(t) = \mathbf{0}$ for $t < 0$, we can replace the lower limit of the integtration by $-\infty$. Hence the integral in (5.6) can be expressed as the Fourier integral

$$\hat{G}(\omega) = \int_{-\infty}^{\infty} G(\eta)\,\mathrm{e}^{\mathrm{i}\omega\eta}\,\mathrm{d}\eta\,. \tag{5.7}$$

Rewriting Eq. (5.6) as

$$\boldsymbol{\sigma}(\boldsymbol{x}, t) = \boldsymbol{G}^*(\omega)\boldsymbol{\varepsilon}_0(\boldsymbol{x})\,\mathrm{e}^{-\mathrm{i}\omega t} \tag{5.8}$$

we have

$$G^*(\omega) = \mathrm{i}\omega\hat{G}(\omega)\,, \tag{5.9}$$

which is called the complex relaxation tensor. This tensor may be also written as

$$G^*(\omega) = G^0(\omega) + \mathrm{i}G^1(\omega)\,. \tag{5.10}$$

At zero frequency, it follows that for exponentially decaying type relaxation functions

$$G^0(0) = \mathring{G} = \bar{G}(t)|_{t\to\infty}\,, \qquad G^1(0) = \mathbf{0}\,. \tag{5.11}$$

At infinite frequency, it can be shown with the change of variable $\omega\eta = \tau$, that

$$G^0(\infty) = \mathring{G} + \bar{G}(0) = G(t)|_{t=0}\,, \qquad G^1(0) = \mathbf{0}\,. \tag{5.12}$$

These expressions imply that viscoelastic solids behave as elastic solids for very high and low frequencies.

From the more general point of view, the Fourier transform of Eq. (5.1) reduces to

$$\hat{\boldsymbol{\sigma}}(\boldsymbol{x}, \omega) = \mathrm{i}\omega\hat{G}(\omega)\hat{\boldsymbol{\xi}}(\boldsymbol{x}; \omega) \tag{5.13}$$

and hence, using Eq. (5.9), we have

$$\hat{\boldsymbol{\sigma}}(\boldsymbol{x}, \omega) = \boldsymbol{G}^*(\omega)\hat{\boldsymbol{\varepsilon}}(\boldsymbol{x}; \omega)\,. \tag{5.14}$$

5.2. *Dynamic problems for viscoelastic media*

When the dynamic problems for viscoelastic media are considered in the Fourier-transformed domain, the same formulations as in elastodynamics can be immediately derived by simply replacing elastic tensors by the complex relaxation tensors.

The complex relaxation tensor for an isotropic homogeneous viscoelastic medium is given by

$$G^*_{ijkl}(\omega) = \lambda^*(\omega)\delta_{ij}\delta_{kl} + \mu^*(\omega)(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})\,, \tag{5.15}$$

the Fourier-transformed equation of motion becomes, ignoring the body force,

$$\mu^*(\omega)\nabla^2\hat{\boldsymbol{u}}(\boldsymbol{x}; \omega) + [\lambda^*(\omega) + \mu^*(\omega)]\boldsymbol{\nabla\nabla}\cdot\hat{\boldsymbol{u}}(\boldsymbol{x}; \omega) + \rho\omega^2\hat{\boldsymbol{u}}(\boldsymbol{x}; \omega) = \mathbf{0}\,, \tag{5.16}$$

and the boundary conditions are the same as those in elastodynamics. Therefore, we can utilize the same techniques as those developed for elastodynamics.

Consider the time-harmonic displacement wave

$$\boldsymbol{u} = A\boldsymbol{d}\exp[\mathrm{i}(k\boldsymbol{x}\cdot\boldsymbol{p} - \omega t)]\,, \tag{5.17}$$

where ω is the circular frequency, k is the wave-number and $\boldsymbol{p}$ and $\boldsymbol{d}$ are unit vectors in the directions of propagation and motion, respectively. Substituting Eq. (5.17) into Eq. (5.16), we have

$$(\lambda^* + \mu^*)k^2(\boldsymbol{p}\cdot\boldsymbol{d})\boldsymbol{p} + (\mu^*k^2 - \rho\omega^2)\boldsymbol{d} = \mathbf{0}\,. \tag{5.18}$$

This equation can be satisfied by the alternatives

(i) $\boldsymbol{d} = \boldsymbol{p}$, and hence $(\lambda^* + 2\mu^*)k^2 = \rho\omega^2$,

(ii) $\boldsymbol{p}\cdot\boldsymbol{d} = 0$, and hence $\mu^*k^2 = \rho\omega^2$.

In Eq. (5.18) either k or ω can be considered as the real-valued independent variable. In the following we take ω as the independent real variable. Hence k must be a complex number of the form

$$k^* = k^0 + \mathrm{i}k^1\,, \qquad k^0, k^1 : \text{real}\,. \tag{5.19}$$

Case (i) implies a P-wave with a displacement $\boldsymbol{u}$ given as

$$\boldsymbol{u} = A\boldsymbol{d}\exp(-k_1^1\boldsymbol{x}\cdot\boldsymbol{d})\exp[\mathrm{i}(k_1^0\boldsymbol{x}\cdot\boldsymbol{d} - \omega t)] \tag{5.20}$$

and a wavenumber k_1^* determined from

$$k_1^* = \left(\frac{\rho}{\lambda^* + 2\mu^*}\right)^{1/2} \omega = k_1^{\mathrm{II}} + \mathrm{i} k_1^{\mathrm{I}} . \tag{5.21}$$

Now putting

$$\frac{\rho}{\lambda^* + 2\mu^*} = \frac{1}{c_1^2} e^{2i\alpha} , \qquad c_1, \alpha : \text{real} , \tag{5.22}$$

we have

$$k_1^{\mathrm{II}} = \frac{\omega}{c_1} \cos \alpha , \qquad k_1^{\mathrm{I}} = \frac{\omega}{c_1} \sin \alpha . \tag{5.23}$$

Case (ii) expresses an S-wave with a displacement $\boldsymbol{u}$ given as

$$\boldsymbol{u} = A\boldsymbol{d} \exp(-k_1^{\mathrm{I}} \boldsymbol{x} \cdot \boldsymbol{p}) \exp[\mathrm{i}(k_1^{\mathrm{II}} \boldsymbol{x} \cdot \boldsymbol{p} - \omega t)] \tag{5.24}$$

and a wave-number k_1^* determined from

$$k_1^* = (\rho/\mu^*)^{1/2} \omega = k_1^{\mathrm{II}} + \mathrm{i} k_1^{\mathrm{I}} \tag{5.25}$$

with

$$k_1^{\mathrm{II}} = \frac{\omega}{c_1} \cos \beta , \qquad k_1^{\mathrm{I}} = \frac{\omega}{c_1} \sin \beta , \tag{5.26}$$

where β is given by $\rho/\mu^* = e^{2i\beta}/c_1^2$.

Furthermore, since $\boldsymbol{d}$ can be decomposed into the horizontal and vertical directions, $\boldsymbol{p} \cdot \boldsymbol{d} = 0$ implies the two cases such as motion in the vertical plane (SV-wave) and in the horizontal plane (SH-wave).

Fundamental solutions can be derived from the ones for elastodynamics by simply replacing the wavenumbers k_1 and k_1 by k_1^* and k_1^*, respectively. Finally we have the same integral representations and integral equations as the ones for time-harmonic elastodynamics, i.e. Eqs. (2.10), (3.5) and (3.2), (3.6) and (3.26), (3.28), using the fundamental solutions for viscoelastodynamics.

For the time–space domain, we have integral representations similar to those of Eqs. (2.12)–(2.14) and integral equations similar to those of Eqs. (3.7)–(3.9) [38], in which we must replace the elastodynamic fundamental solutions by the corresponding viscoelastic ones, which may be derived from the time-harmonic viscoelastodynamic fundamental solutions by the aid of inverse Fourier transform.

6. Examples

In this section, we present some numerical examples, which may help to understand the usefulness and applicability of the integral equation method.

6.1. *Time-harmonic solutions*

In order to have a general idea for the applicability of the integral equation method and also to confirm the accuracy of the numerical results some simple time-harmonic example problems are solved first.

(1) A circular hole subject to a time-harmonic SH-wave

As the simplest case, we computed displacements and stresses around a circular hole of radius a in an infinite domain subject to a time-harmonic plane displacement SH wave. The results were obtained with quadratic isoparametric boundary elements and an 8-point Gaussian quadrature formula. Figure 3 shows some results for different wave-numbers and for different numbers of elements, in which analytical curves are calculated from the formula given in the series expansion form [101] by taking 21 terms into account.

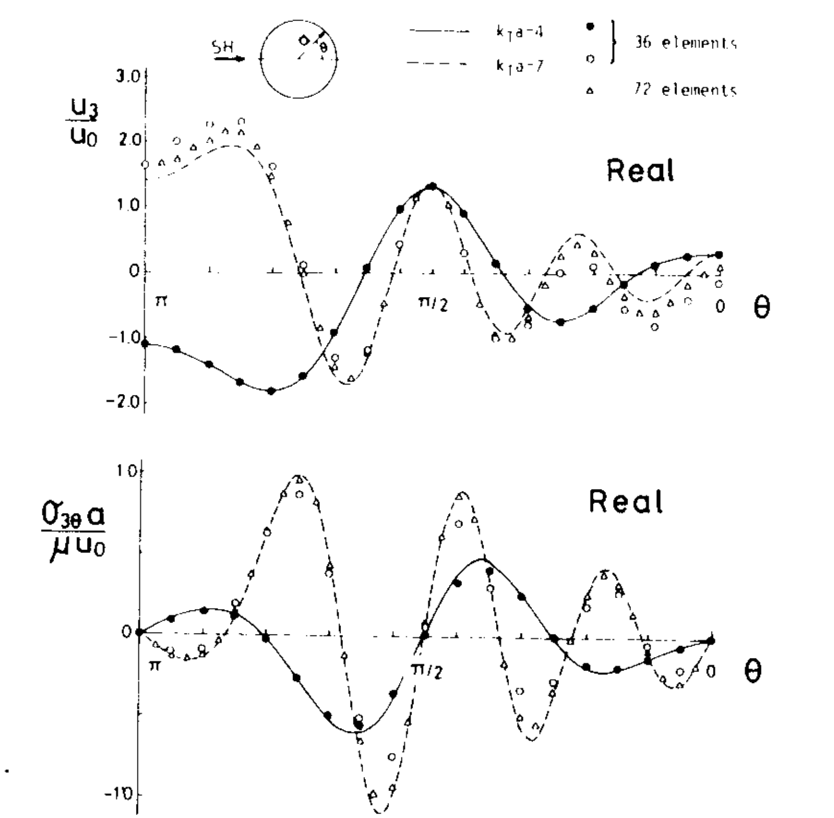


Fig. 3. Boundary displacements and hoop stresses around a hole due to a plane harmonic displacement SH-wave. Symbols: BIEM. Curves: analytical. u_0: amplitude of the incident wave.

The real part implies the solution at the moment when the peak of the incident wave arrives at the center of the hole. The results show reasonable agreement with each other. The wave-number $k_1 a = 7$ corresponds to a wavelength of about 0.86 times the radius of the hole.

(2) A circular hole subject to a time-harmonic P-wave

In Fig. 4, hoop stresses obtained by the use of BIEM are compared with the analytical ones [100] when the hole is subject to a time-harmonic plane stress P-wave. The analysis was carried out with 24 quadratic isoparametric boundary elements and an 8-point Gaussian quadrature formula. The results show a good agreement with each other.

(3) A circular lined tunnel in a half-space subject to a time-harmonic SH wave

Figure 5 shows the anti-plane displacement amplitudes on the outer surface of the lining and on the ground surface due to a time-harmonic plane displacement SH wave with different incident angles. The analysis was done with 120 constant boundary elements on the basis of the half-space Green's function for the integral kernels. These results by the BIEM agree very well with the analytical ones.

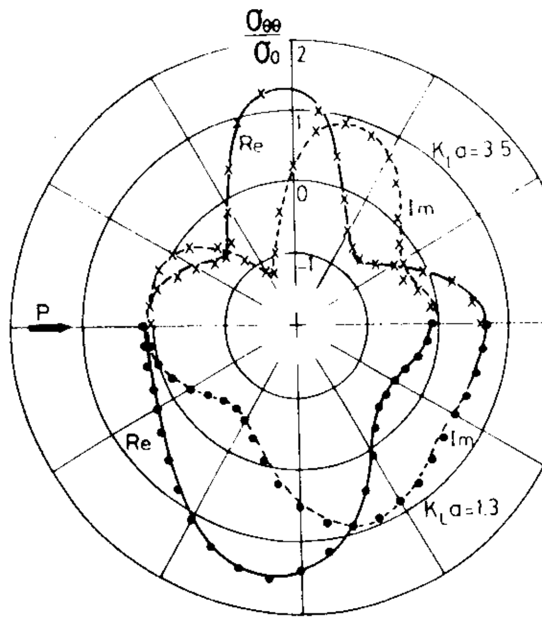


Fig. 4. Hoop stresses around a hole due to the plane harmonic stress P-wave. Symbols: BIEM. Curves: analytical. σ_0: magnitude of the incident wave.

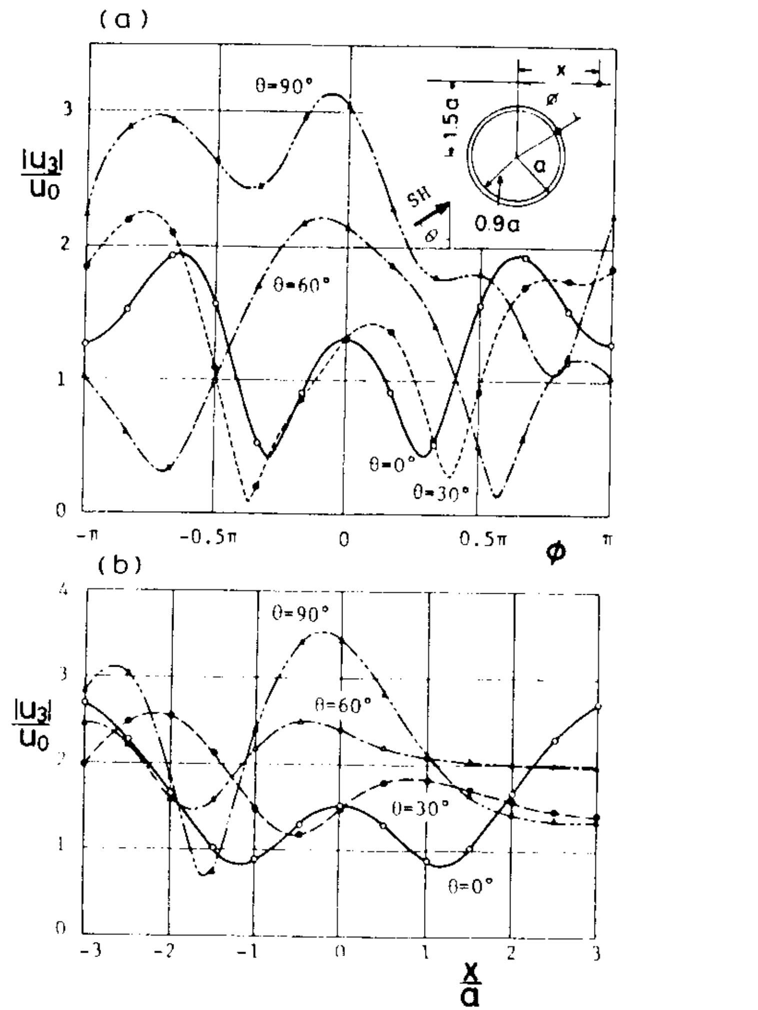


Fig. 5. Displacements on the outer surface of the linning and on the ground surface due to a plane harmonic displacement SH-wave. Symbols: BIEM. Curves: analytical. ($c_1^l/c_1^e = 9$, $k_{1e}a = 1.57$; l: lining, e: soil). u_0: amplitude of the incident wave.

(4) A spherical cavity subject to a plane time-harmonic P-wave [59]

As an attempt to provide results from a three-dimensional analysis, displacements around a spherical cavity due to a plane time-harmonic P-wave were analysed. Figure 6 shows the results in comparison with the analytical ones [101]. The results show a good agreement even for the rather coarse discretization of the boundary (30 quadratic isoparametric elements and 16-point Gaussian quadratic formula) that was used.

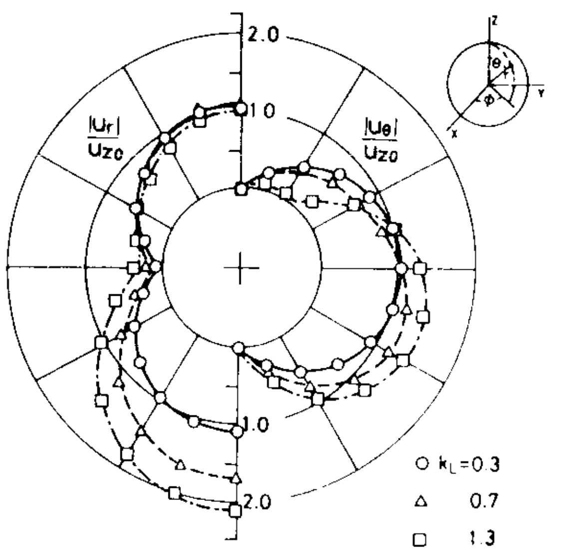


Fig. 6. Boundary displacements around a spherical cavity due to a plane harmonic displacement P-wave. Poisson's ratio $\nu = 0.25$. Symbols: BIEM. Curves: analytical. u_{z0}: amplitude of the incident wave.

6.2. *Transient responses*

6.2.1. *BIEM in the Fourier-transformed domain*

(1) A circular hole in an infinite space subject to a step SH-wave

Figure 7 shows displacement distributions on a circular hole subject to a step plane SH-wave as compared with anaytical ones calculated from a formula [101]

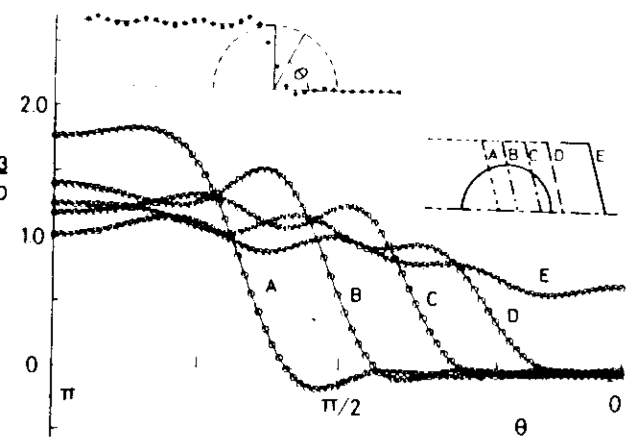


Fig. 7. Transient displacement and hoop stress distributions around a hole subject to a plane step displacement SH-wave. Symbols: BIEM. Curves: analytical. u_0: amplitude of the incident wave.

with 21 terms taken into account. In the analysis, 160 quadratic isoparametric boundary elements with an 8-point Gaussian quadrature formula were used. The shape of the incident wave is shown in the small figure, which accommodates wave-numbers up to $k_1 a = 7$. The BIEM solutions show an excellent agreement with the analytical ones.

(2) A circular hole in an infinite plane subject to a step P-wave

Figure 8 shows the displacement and hoop stress histories calculated at several boundary points of a hole subject to a step P-wave in comparison with analytical

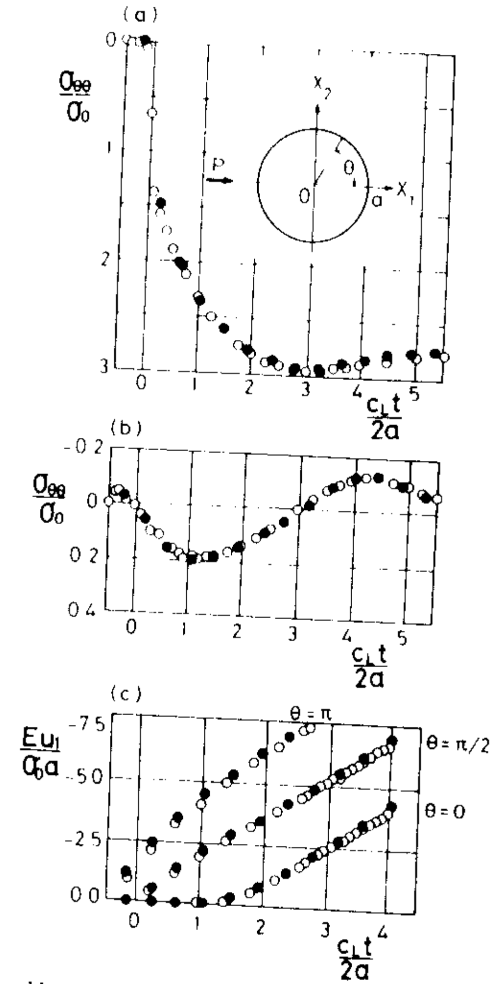


Fig. 8. Displacement and hoop stress distributions at several boundary points on the hole subject to a plane step stress P-wave. Poisson's ratio $\nu = 0.25$. (a) Hoop stress at $\theta = \pi/2$, (b) Hoop stress at $\theta = \pi$, (c) Displacement u_1. Solid circles: BIEM. Open circles: Garnet and Crouzet-Pascal [42]. σ_0: magnitude of the incident wave.

ones obtained by Garnet and Crouzet-Pascal [42]. The numerical results with 24 isoparametric quadratic elements agree well with the analytical ones for up to approximately $k_1 a = 7$.

6.2.2. *BIEM in time–space domain*

(1) A circular hole subject to a step SH-wave [41]

A time-space domain formulation is suitable for transient analysis, specifically for the analysis of the early stage of responses to travelling waves.

Figure 9 shows histories of displacements and stresses at several points on the

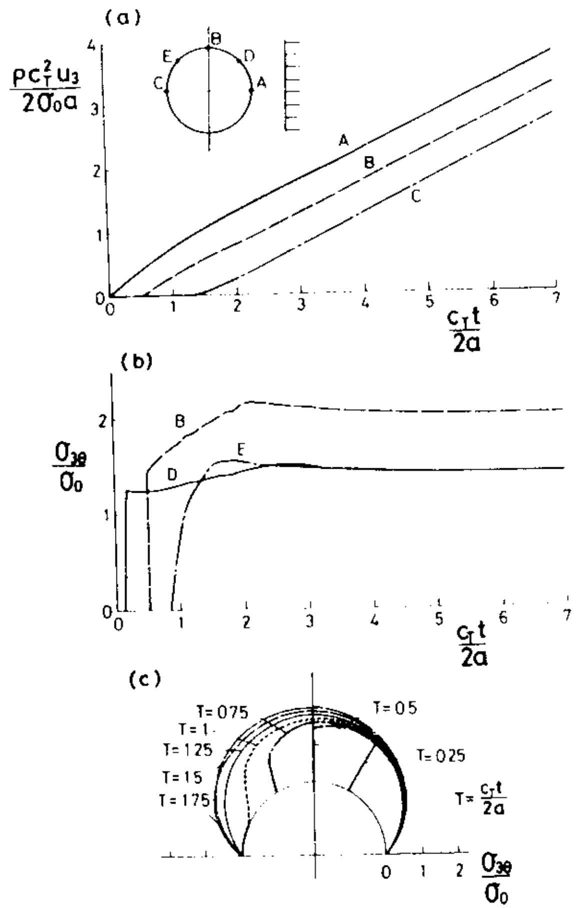


Fig. 9. Displacement and hoop stress histories at several points on the hole subject to a step stress SH-wave analysed by the time–space domain BIEM. (a) Displacements. (b) Hoop stresses. (c) Hoop stresses. σ_0: magnitude of the incident wave.

hole in an infinite space subject to a step SH-wave. The hoop stress distributions are also shown in Fig. 9. In the analysis, 16 boundary elements were used with a linear time shape function and the time-step was selected such that the wave-front crosses the hole in 8 steps. The results show good stability.

(2) A circular hole subject to a step SV-wave

Figure 10 shows stress histories at several points on a circular hole due to a step SV-wave calculated by the BIEM in time–space domain [66] as compared with those by the BIEM in the Fourier-transformed domain. These results show again quite good agreement with each other.

6.3. *Eigenfrequency problems* [57, 98]

Eigenfrequencies are roots of the determinant of the discretized homogeneous boundary integral equation. The roots are usually searched for by the direct substitution of presumed values.

Table 1 shows the eigenfrequencies of the Helmholtz equation in a rectangular domain with $2a \times a$, which is fixed along the long edge and traction-free on the short edge. In Table 1, m and n mean the nodal numbers on the long and short edges, respectively. In the calculation, the boundary was discretized into 54 equal constant elements and an 8-point Gaussian quadrature formula was used with a increment of wavenumber $k_1 a = 0.1$.

Figure 11 shows eigenmodes of a dam and reactions from the rigid base. The numerical figures denote the eigenfrequencies. The figures in parentheses imply

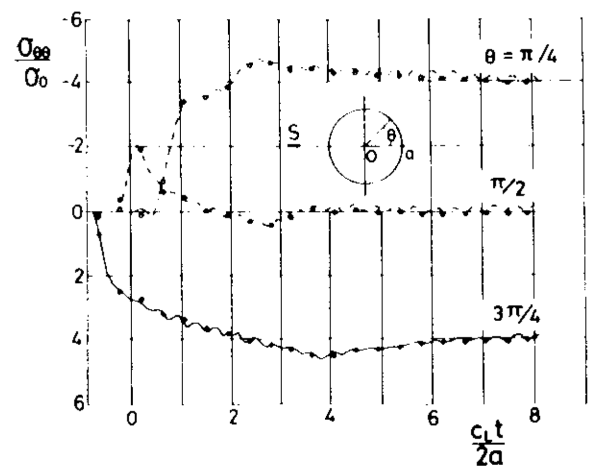
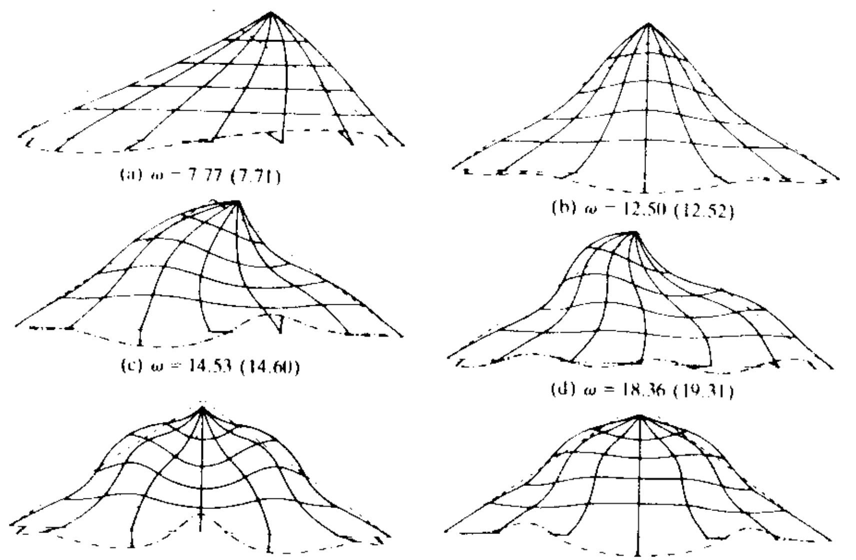


Fig. 10. Hoop stress histories at several points on a circular hole due to a step stress SV-wave. Comparison was made for the results by the time-space type BIEM (curves) and those by the Fourier-transformed type BIEM (symbols). σ_0: magnitude of the incident wave.

Table 1
Eigenfrequencies of the Helmholtz equation in a $2a \times a$ rectangular domain, which is fixed along the long edge and traction-free on the short edge. $\Delta k_1 a = 0.1$.

	$n = 1$		$n = 2$		$n = 3$		$n = 4$	
m	BIEM	Anal.	BIEM	Anal.	BIEM	Anal.	BIEM	Anal.
0	6.3	6.283	12.6	12.566	18.9	18.850	25.1	25.133
1	7.0	7.025	13.0	12.953	19.1	19.110	25.3	25.328
2	8.9	8.886	14.0	14.050	19.9	19.869	25.9	25.906
3	11.3	11.327	15.7	15.708	21.1	21.074	26.8	26.842
4	14.0	14.050	17.8	17.772	22.6	22.654	28.1	28.099
5	16.9	16.918	20.1	20.116	24.5	24.537	29.6	29.638
6	19.9	19.869	22.6	22.654	26.6	26.657	31.4	31.416



(a) $\omega = 7.77$ (7.71)
(b) $\omega = 12.50$ (12.52)
(c) $\omega = 14.53$ (14.60)
(d) $\omega = 18.36$ (19.31)
(e) $\omega = 20.23$ (20.12)
(f) $\omega = 22.46$ (23.10)

Fig. 11. Eigenmodes, reactions and eigenfrequencies of a dam structure on a rigid base. Poisson's ratio $\nu = 0.45$. Eigenfrequencies: BIEM (FEM [19]).

the values obtained by the use of the FEM [19]. The results agree well with each other.

6.4. *Miscellaneous examples*

(1) Displacement response curves of a column

Figure 12 shows response curves for the horizontal displacement at the top of a column subject to SV-waves. The critical angle of incidence is $\theta_{cr} = 35.26°$ in this

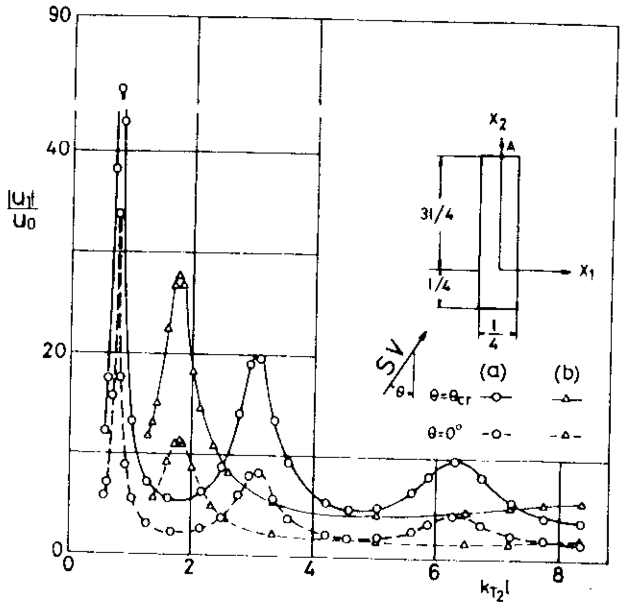


Fig. 12. Response curves of the horizontal displacement at the top of a column A due to SV-waves. (a) Young's modulus $E_1/E_2 = 1.0$, density $\rho_1/\rho_2 = 1.0$. (b) $E_1/E_2 = 10$, $\rho_1/\rho_2 = 1.0$ (1: column, 2: soil, Poisson's ratio $\nu_1 = \nu_2 = 0.25$). u_0: amplitude of the incident wave.

example. The magnification factor in the displacement becomes very large when the wave comes in at the critical angle. This means that dominant frequencies can be obtained more accurately by means of waves with the critical incident angle.

(2) Displacement distributions in a deposit-filled valley due to a time-harmonic SV-wave [61]

As a rather practical application, an analysis was made for the determination of the dynamic response of a filled valley with two layers of deposits subject to a time-harmonic plane SV-wave.

Figure 13 shows the displacement distributions. The real (imaginary) part implies that the peak (node) of the incident wave arrives at the origin of the coordinates (see the small figure). In the analysis the combined method of BIEM with FEM is applied using quadratic isoparametric elements as shown in the small figure. The boundaries on both sides of the valley are extended to more than twice the incident wavelength.

(3) Viscoelastic analsis of a deposit-filled valley

A similar problem as in (2) but with the two layers of the deposits being viscoelastic was analysed using the BIE–FE combined method. In this case, the

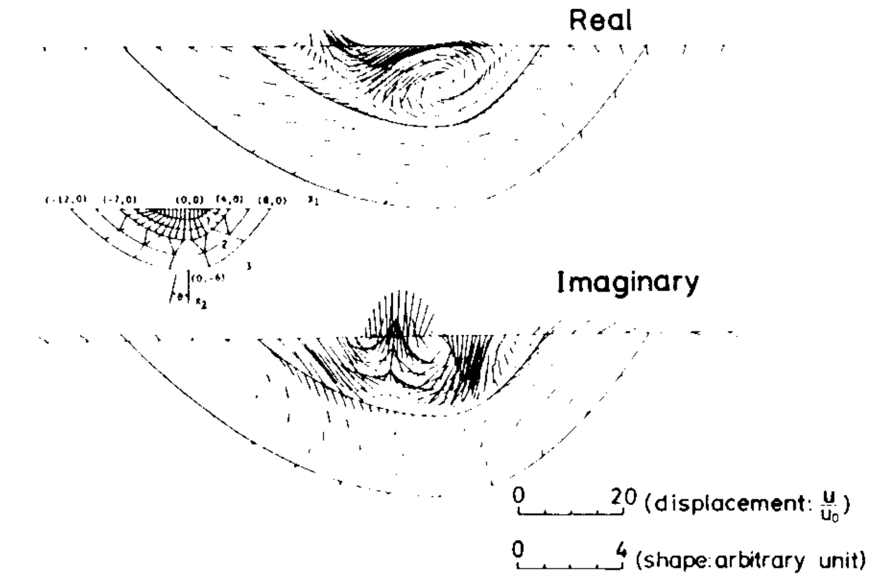


Fig. 13. Displacements in an alluvial valley subject to a plane harmonic SV-wave with wavelength $\lambda_0 = 24$ (arbitrary unit) and amplitude u_0. Young's modulus: $E_1 = E_3/25$, $E_2 = E_3/5$, $E_3 = 10$ GPa. Density: $\rho_1 = 2$ t/m^3, $\rho_2 = \rho_3 = 2.4$ t/m^3. Poisson's ratio: $\nu_1 = 0.4$, $\nu_2 = 1/3$, $\nu_3 = 1/4$. Incident angle $\theta = 20°$.

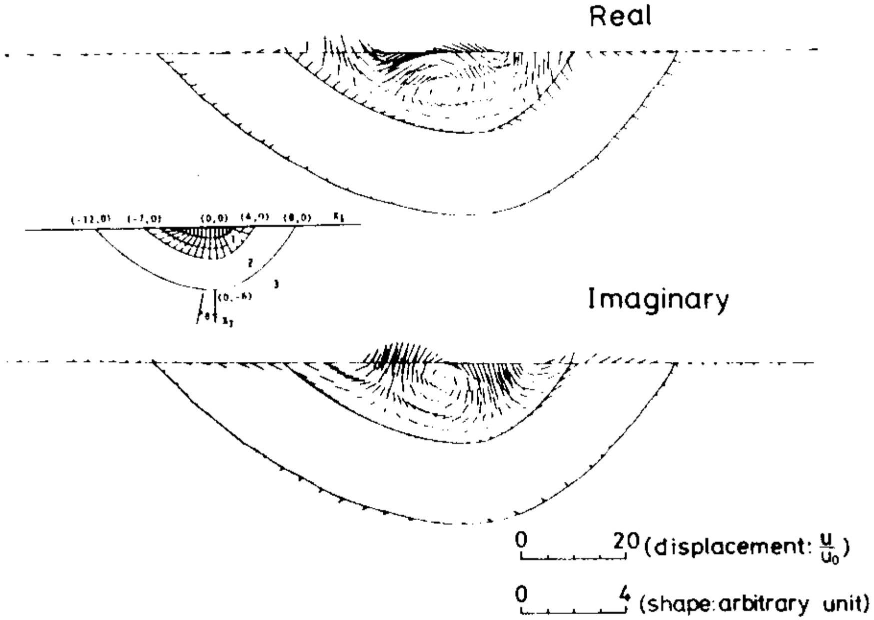


Fig. 14. Displacements in a viscoelastic alluvial valley subject to a plane harmonic SV-wave with wavelength $\lambda_0 = 24$ (arbitrary units) and amplitude u_0. Material constants are the same as those in Fig. 13 with $k_1^I/k_1^{II} = 0.07$, $k_2^I/k_2^{II} = 0.05$, $k_3^I/k_3^{II} = 0$ in the complex wave-number $k_\alpha^* = k_\alpha^{II} + ik_\alpha^I$. Incident angle $\theta = 20°$.

model was modified slightly so as to use the FEM only in the central deposit domain denoted by 1 and the BIEM for the rest.

The imaginary wave-numbers used in this calculation were $k_2^I = 0.07k_2^{II}$ and $k_1^I = 0.05k_1^{II}$ for deposit domains 2 and 1, respectively. Figure 14 shows the displacement distributions due to the same incident wave as the one mentioned in Fig. 13.

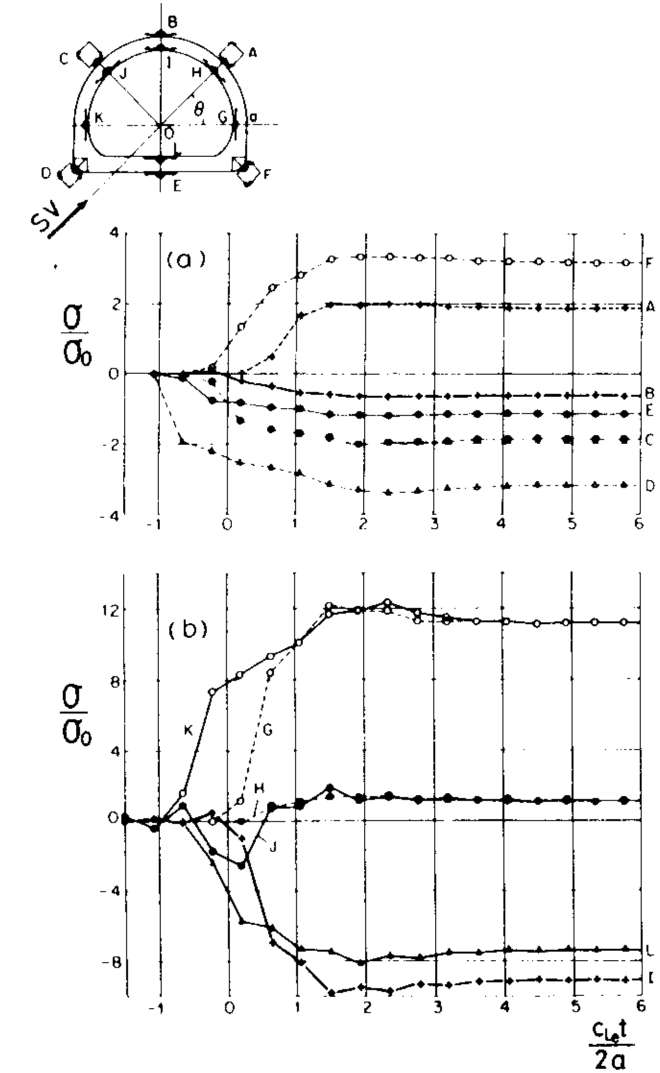


Fig. 15. Transient stresses around a lined tunnel subject to a step stress SV-wave. (a) Stresses on the surrounding medium. (b) Stresses on the inner surface of the lining. Poisson's ratio: $\nu_e = 1/4$, $\nu_i = 1/6$. Density: $\rho_i/\rho_e = 1.0$. Young's modulus: $E_i/E_e = 5$. (i: lining, e: soil).

(4) Transient stresses around a lined tunnel [65, 66]

In Fig. 15, we show transient stresses around a horseshoe-shaped lined tunnel during the passage of an incident step SV-wave. The model is depicted in the small figure. Use is made of 48 quadratic isoparametric boundary elements and an 8-point Gaussian quadrature formula.

(5) Transient responses of an earth dam [94]

Transient responses of an earth dam with a center-core due to the passage of the plane shear Ricker wavelet were analysed. The model is shown in Fig. 16(a). The shear modulus μ and Poisson's ratio ν vary linearly as shown in Fig. 16(a) with μ^0 = arbitrary constant and $\nu^0 = 0.28$. The plane Ricker wavelet

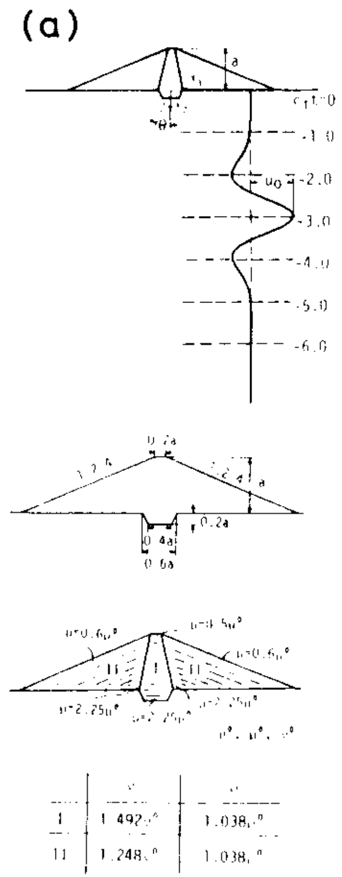


Fig. 16 Transient displacement of an earth dam subject to a shear Ricker wavelet of oblique incidence with angle $\theta = 30°$. (a) Model and the incident wave. (b) Movements of the dam. u_0: the maximum amplitude of the incident wave.

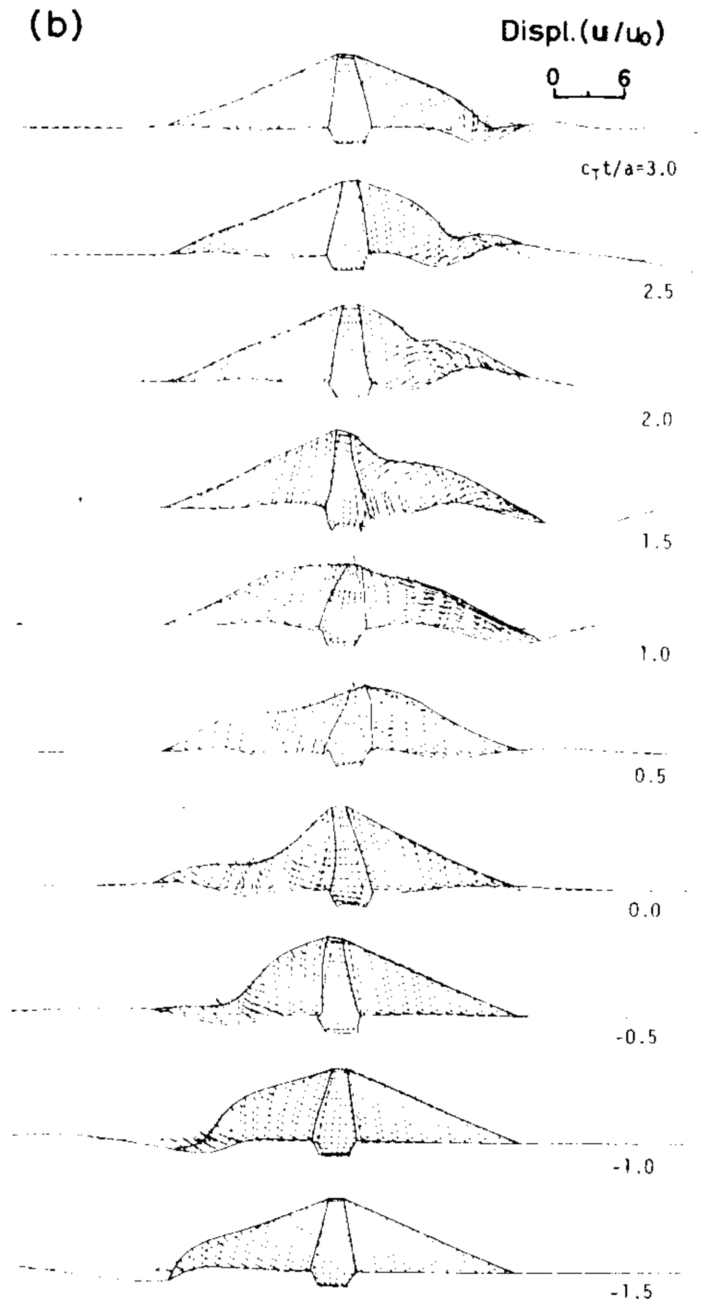


Fig. 16 (continued).

propagating in the $\boldsymbol{p}$ direction is defined as

$$\boldsymbol{u}^{\mathrm{I}}(\boldsymbol{x}, t) = u_0 \boldsymbol{d}(\zeta^2 - \tfrac{1}{2})\, e^{-\zeta^2}$$

with

$$\zeta = \pi\left(t - \frac{\boldsymbol{x} \cdot \boldsymbol{p}}{c_{\mathrm{T}}}\right)\Big/ t_{\mathrm{p}}, \qquad \frac{t_{\mathrm{p}} c_{\mathrm{T}}^{0}}{a} = \frac{5\pi}{a k^{0}_{\mathrm{T\,max}}}$$

where $\boldsymbol{d}$ is the unit displacement vector, c_{T} is the incident wave velocity, c_{T}^{0} is the wave velocity in the half-plane, t_{p} stands for the characteristic time and $k^{0}_{\mathrm{T\,max}}$ denotes the maximum value of the transverse wave-number used in the inverse Fourier transform which was $2\pi/a$ in this analysis.

Figure 16(b) shows the transient displacements of the center-cored earth dam subject to the transverse Ricker wavelet of oblique incidence with angle $\theta = 30°$.

For some other applications, the reader is advised to see Refs. [1, 29–33, 36, 43, 68, 84, 86, 99, 109–111, 121–124, 132–135].

7. Concluding remarks

In this chapter we discussed some fundamentals of boundary integral equation methods for elasto- and viscoelastodynamics. Integral equations for elastodynamic problems both in the time–space and the Fourier-transformed domains were formulated. The latter case was also extended to viscoelastic problems with a slight modification in the fundamental solutions. The combination technique of integral equation method with the finite element method was also briefly presented.

Numerical implementations for these formulations were considered and the efficiency and applicability of these formulations for engineering problems were demonstrated by some examples.

In conclusion, boundary integral equation methods as well as their combination with the finite element method can be very effectively applied to linear elasto- and viscoelastodynamic problems. Even for nonlinear problems, these methods may also be applicable with the aid of the equivalent linearization technique.

Recent progress in the area of integral equation methods is remarkable. Further research will find out more efficient and versatile ways to apply them to elasto- and viscoelastodynamic problems.

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