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INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS
I.C.T.P., P.O. BOX 586, 34100 TRIESTE, ITALY, CABLE: CENTRATOM TRIESTE

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Conference on
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Analyses for Soliton Control in Optical Fibres

Y. Kodama

Ohio State University
Dept. of Mathematics
Columbus, Ohio
U.S.A.

Analyses for Soliton Control in Optical Fibers.

Purpose is to present several useful and necessary tools for analysis of soliton propagation in a ultra-fast transmission line.

Basic idea is to give a "dynamical system approach" to soliton equations.

Recall: a finite dimensional system (ode)

$$\frac{d\vec{x}}{dt} = \vec{f}(\vec{x}), \qquad \vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n$$

$$\vec{f}(\vec{x}) = \begin{pmatrix} f_1(x_1, \cdots, x_n) \\ \vdots \\ f_n(x_1, \cdots, x_n) \end{pmatrix}$$

$f_i(x_1 \cdots x_n)$: polynomial in x (or entire function)

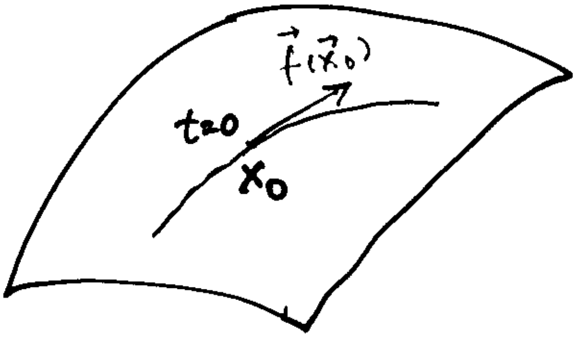
Then the solution can be written in the form with the initial condition $\vec{x}(0) = \vec{x}_0$.

$$\vec{x}(t) = \exp\left[t \, \vec{f}(\vec{x}_0) \cdot \nabla \right] \vec{x}_0$$

$$= \vec{x}_0 + t \, \vec{f}(\vec{x}_0) + \frac{1}{2} t^2 \, \vec{f}(\vec{x}_0) \cdot \nabla \vec{f}(\vec{x}_0) + \cdots$$

where
$$\vec{f}(\vec{x}_0) \cdot \nabla = \sum_{i=1}^{n} f_i(\vec{x}_{0i}) \frac{\partial}{\partial x_{0i}}$$

(directional derivative)



Change of Coordinates can be formulated as follow:
(Simplification of equation) (Normal Form)

Consider

$$\Phi_\varepsilon : \mathbb{R}^n \longrightarrow \mathbb{R}^n$$
$$\vec{y} \longmapsto \vec{z} = \vec{\Phi}_\varepsilon(\vec{y}) \quad , \text{ with } \begin{cases} \vec{\Phi}_0(\vec{y}) = \vec{y} \\ \vec{\Phi}_1(\vec{y}) = \vec{x} \end{cases}$$

Infinitesimal change of coordinates:

$$\frac{d}{d\varepsilon}\vec{\Phi}_\varepsilon(\vec{y}) = \frac{d\vec{z}}{d\varepsilon} = \vec{\phi}(\vec{z}) \quad : \text{generator of the transformation}$$

The solution can be obtained by

$$\vec{z} = \vec{\Phi}_\varepsilon(\vec{y}) = \exp\left[\varepsilon\,\vec{\phi}(\vec{y})\cdot\nabla_y\right]\vec{y} \quad , \quad \vec{\Phi}_0(\vec{y}) = \vec{y}$$

$$\Rightarrow \quad \vec{x} = \exp\left[\vec{\phi}(\vec{y})\cdot\nabla_y\right]\vec{y}$$
$$= \vec{y} + \vec{\varphi}(\vec{y}) + \frac{1}{2}\vec{\varphi}(\vec{y})\cdot\nabla\vec{\varphi}(\vec{y}) + \cdots$$

With this coordinate change,

$$\frac{d\vec{x}}{dt} = \frac{d\vec{y}}{dt}\cdot\vec{\nabla}\left(e^{\vec{\varphi}(\vec{y})\cdot\nabla}\vec{y}\right) = \vec{g}(y)\cdot\vec{\nabla}\left(e^{\vec{\varphi}(\vec{y})\cdot\nabla}\vec{y}\right)$$
$$= \vec{f}\left(e^{\vec{\varphi}\cdot\nabla}\vec{y}\right) = e^{\vec{\varphi}\cdot\nabla}\vec{f}(y) = e^{\vec{\varphi}\cdot\nabla}\left(\vec{f}(y)\cdot\nabla\right)\vec{y}$$

$$\Rightarrow \quad \vec{g}(y)\cdot\nabla := \vec{X}_g = e^{\vec{\varphi}\cdot\nabla}\,\vec{f}(y)\cdot\vec{\nabla}\,e^{-\vec{\varphi}(y)\cdot\nabla}$$
$$= e^{\vec{X}_\varphi}\circ\vec{X}_f\circ e^{-\vec{X}_\varphi}$$

$$\Rightarrow \quad \boxed{\vec{X}_g = \vec{X}_f + [\vec{X}_\varphi, \vec{X}_f] + \frac{1}{2}[\vec{X}_\varphi, [\vec{X}_\varphi, X_f]] + \cdots}$$

Lie Transformation

Then if we have

$$\frac{dx}{dt} = f_0(x) + f_1(x) + \cdots$$

$$\Big\downarrow \; x = e^{\varphi(y)\cdot\nabla}$$

$$\frac{dy}{dt} = g_0(y) + g_1(y) + \cdots$$

Set $\quad \varphi(y) = \varphi_1(y) + \varphi_2(y) + \cdots$

$\Rightarrow \quad X_{g_0} = X_{f_0} \quad \Rightarrow \quad f_0 = g_0$

$$X_{g_1} = X_{f_1} + [X_{\varphi_1}, X_{g_0}]$$

$$\text{or} \quad [X_{f_0}, X_{\varphi_1}] = X_{f_1} - X_{g_1}$$

$\vdots$

<u>Example</u>: $\quad x \in \mathbb{R}^2$

$$\frac{d}{dt}x = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} x + f_2(x) + \cdots$$

$$\Big\downarrow \quad x = \Phi_1(y) = y + \varphi_1(y) + \cdots$$

$$\frac{d}{dt}y = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} y + \begin{pmatrix} ax_1 - bx_2 \\ bx_1 + ax_2 \end{pmatrix}(x_1^2 + x_2^2) + O(x^5)$$

Now let us extend these into an ∞-dim system (pde), that is,

- choose a proper coordinate for the system
- define the vector field on the space
- define the (Lie) transformation between systems

For the case of the nonlinear Schrödinger eq.

$$\frac{\partial q}{\partial z} = \frac{i}{2}\frac{\partial^2 q}{\partial T^2} + i|q|^2 q \quad , \quad q = q(z,T)$$

Coordinates:

$$(q, q^*, q_T, q_T^*, q_{TT}, q_{TT}^*, \cdots) \equiv [q, q^*]$$

Vector field:

$$\frac{d}{dz} q = \frac{i}{2} q_{TT} + i q^2 q^* = K_0[q, q^*] \in \mathbb{C}[q, q^*]$$

Derivative of $F[q, q^*]$

$$\frac{dF}{dz} = \frac{dq}{dz}\frac{\partial F}{\partial q} + \frac{dq^*}{dz}\frac{\partial F}{\partial q^*} + \frac{dq_T}{dz}\frac{\partial F}{\partial q_T} + \cdots$$

$$= \sum_{n=0}^{\infty}\left\{\left(\frac{dq}{dz}\right)_{nT}\frac{\partial F}{\partial q_{nT}} + \left(\frac{dq^*}{dz}\right)_{nT}\frac{\partial F}{\partial q^*_{nT}}\right\}$$

$$\equiv \frac{d\vec{q}}{dz}\cdot\nabla F = \vec{K}_0\cdot\nabla F \equiv X_{K_0} F$$

$$\Rightarrow \quad \vec{X}_F \equiv \vec{F}[q, q^*] \cdot \nabla$$

Lie transformation

$$q = e^{\vec{\Phi} \cdot \nabla} v \quad , \qquad \Phi = \Phi[v, v^*]$$

$$= v + \Phi(v) + \frac{1}{2} \vec{\Phi} \cdot \nabla \Phi + \cdots$$

$$X_G = e^{X_\Phi} \circ X_F \circ e^{-X_\Phi}$$

where
$$\begin{cases} \dfrac{dq}{dz} = X_F \, q \\[2ex] \dfrac{dv}{dz} = X_G \, v \end{cases}$$

Now we apply these to an analysis of soliton propagation.

The IST method :

$$\begin{cases} L\Psi := \begin{pmatrix} i\partial_T & q \\ -q^* & -i\partial_T \end{pmatrix}\Psi = \lambda \Psi \\ \dfrac{\partial \Psi}{\partial z} = \begin{pmatrix} -i\zeta^2 + \frac{i}{2}|q|^2 & iq\zeta - \frac{1}{2}q_T \\ iq^*\zeta + \frac{1}{2}q_T^* & i\zeta^2 - \frac{i}{2}|q|^2 \end{pmatrix}\Psi \end{cases}$$

$$\Longleftrightarrow \quad \frac{\partial q}{\partial z} = \frac{i}{2}\frac{\partial^2 q}{\partial T^2} + i|q|^2 q \;, \quad \frac{\partial \lambda}{\partial z} = 0$$

Direct Scattering Problem :

$$(q, q^*) \longrightarrow S \quad : \text{ Scattering data}$$

$$\Psi(\infty) = S\,\Psi(-\infty) \qquad (\lambda, a, b)$$

Inverse Scattering Problem :

$$S \longrightarrow (q, q^*)$$

1.- Soliton solution:

$$q(T,Z) = \eta \operatorname{sech} \eta (T + \kappa Z + \theta_0) \exp\left[i\kappa Z + \tfrac{i}{2}(\eta^2 - \kappa^2) + i\delta_0\right]$$

$$S = (\kappa, \eta, \theta_0, \delta_0)$$

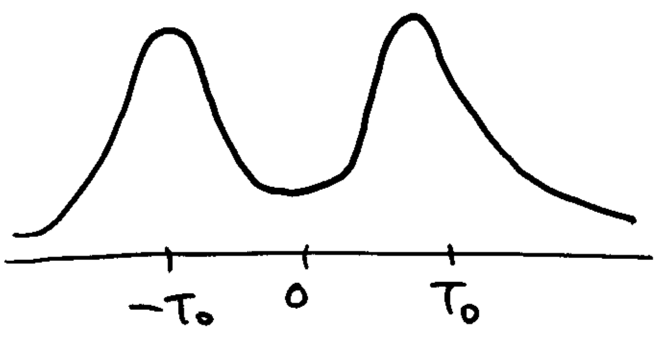
$$\begin{cases} a(\zeta) = \dfrac{\zeta - \lambda}{\zeta - \lambda^*} \\ b(\zeta) = 0 \quad \text{for } \zeta \in \mathbb{R} \\ \lambda = \tfrac{1}{2}(\kappa + i\eta) \end{cases}$$

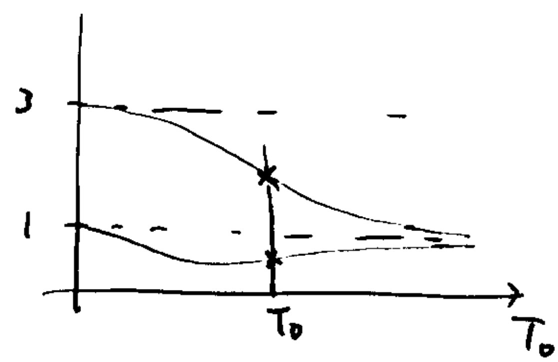
$$b(\lambda) \sim \theta_0, \delta_0$$

2 - Soliton solution: Especially $\kappa_1 = \kappa_2 = 0$, $\eta_1 \neq \eta_2$

$$q(T,Z) = \frac{2}{\Delta}\left(\frac{\eta_1 + \eta_2}{\eta_1 - \eta_2}\right)\left(\eta_1 \cosh \eta_2 T\, e^{\frac{i}{2}\eta_1^2 Z} + \eta_2 \cosh \eta_1 T\, e^{\frac{i}{2}\eta_2^2 Z}\right)$$

$$\Delta = \cosh(\eta_1 + \eta_2)T + \left(\frac{\eta_1 + \eta_2}{\eta_1 - \eta_2}\right)^2 \cosh(\eta_1 - \eta_2)T + \frac{4\eta_1\eta_2}{(\eta_1 - \eta_2)^2}\cos\tfrac{1}{2}(\eta_1^2 - \eta_2^2)Z$$





$$q(T,Z) \simeq A \operatorname{sech} A(T - T_0) + A \operatorname{sech} A(T + T_0)$$

$$\eta_{1,2} = A\left[1 + \frac{2T_0}{\sinh 2T_0} \pm \operatorname{sech} T_0\right]$$

NLS eq. has ∞-many conservation Laws:

$$\mathcal{E} = \int_{-\infty}^{\infty} |q|^2 \, dT = 2 \sum \eta_\ell$$

$$\mathcal{M} = \frac{i}{2} \int (q q_T^* - q_T q^*) \, dT = -2 \sum \kappa_\ell \eta_\ell$$

$$\mathcal{H}_0 = \frac{1}{2} \int (|q|^4 - |q_T|^2) \, dT = -2 \sum \left(\kappa_\ell^2 \eta_\ell - \frac{1}{3} \eta_\ell^3 \right)$$

$$\vdots$$

$$\mathcal{H}_n \sim \frac{1}{n} \sum_{\ell=1}^{N} \left[(\lambda_\ell^*)^n - \lambda_\ell^n \right], \quad \lambda_\ell = \frac{1}{2} (\kappa_\ell + i \eta_\ell)$$

Thus $\mathcal{H}_n = \int H_n[q, q^*] \, dT$ can be written in the scattering data $\{\lambda_\ell\}$.

Perturbation Method for the NLS eq.

$$\frac{dq}{dz} = K_0[q, q^*] + \epsilon P[q, q^*]$$

$$\begin{cases} K_0 = \frac{i}{2} q_{TT} + i |q|^2 q \\ P = \cdots\cdots \end{cases}$$

Evolution of these conserved quatities

$$\frac{dH}{dz} = \int_{-\infty}^{\infty} \frac{d}{dz} H[q, q^*] \, dT$$

$$= \int_{-\infty}^{\infty} (\vec{K}_0 \cdot \nabla H)[q, q^*] \, dT \quad = 0$$

$$+ \ \varepsilon \int_{-\infty}^{\infty} (P \cdot \nabla H) [q, q^*] \, dT$$

How to solve

$$\frac{dH}{dz} = \varepsilon \int_{-\infty}^{\infty} (P \cdot \nabla H) [q, q^*] \, dT$$

From the IST.

$$q = q_{sol} + q_{rad} = \sum_{\nu \in \Lambda} q_\nu \varphi_\nu$$

($\{\varphi_\nu\}_{\nu \in \Lambda}$ gives a basis of certain ∞-dim Complete metric space)

An analogy of Galerkin method (Finite dim. Approx)

$$q \simeq q_{sol}$$

$$\Rightarrow \begin{cases} \dfrac{d\eta}{dz} = \dfrac{1}{2} \displaystyle\int (q^* P + q P^*) \, dT & \text{for 1-soliton} \\[2ex] \dfrac{d\kappa}{dz} = \dfrac{i}{2\eta} \displaystyle\int (q_T^* P - q_T P^*) \, dT - \dfrac{\kappa}{2\eta} \displaystyle\int (q^* P + q P^*) \, dT \end{cases}$$

In general,

$$\frac{d\mathcal{S}}{dz} = \int (K_0 \cdot \nabla S)\, d\tau + \varepsilon \int (P \cdot \nabla S)\, dT$$

for the scattering data $\mathcal{S}$.

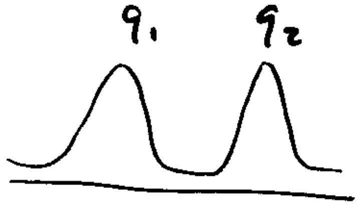
$$\left\{ \begin{aligned} \frac{d\eta}{dz} &= \int \mathrm{Re}\{P e^{-i\varphi}\} \operatorname{sech} \tau \, d\tau \\ \frac{d\kappa}{dz} &= -\int \mathrm{Im}\{P e^{-i\varphi}\} \operatorname{sech} \tau \tanh \tau \, d\tau \\ \frac{dT_0}{dz} &= -\kappa + \frac{1}{\eta^2} \int \mathrm{Re}\{P e^{-i\varphi}\} \, \tau \operatorname{sech} \tau \, d\tau \\ \frac{d\sigma_0}{dz} &= \frac{1}{2}(\eta^2 + \kappa^2) + T_0 \frac{d\kappa}{dz} \\ &\quad + \frac{1}{\eta} \int \mathrm{Im}\{P e^{-i\varphi}\} \operatorname{sech} \tau (1 - \tau \tanh \tau)\, d\tau \end{aligned} \right.$$

Examples of the perturbation term P and Results will be discussed further in S. Wabnitz's talk.

Examples:

① $$P[q, q^*] = \delta q + \beta \frac{\partial^2 q}{\partial T^2} + \gamma_1 |q|^2 q + \gamma_2 |q|^4 q$$
$$+ i\alpha_0 T q + i\sigma_R q \frac{\partial}{\partial T}|q|^2$$
$$+ \beta_1 |q|^2 q_T + \beta_2 (|q|^2 q)_T + \beta_3 \frac{\partial^3 q}{\partial T^3}$$
$$+ \cdots$$

② Soliton-Soliton Interaction:



$$q = q_1 + q_2$$

$$|q|^2 q = (q_1 + q_2)^2 (q_1 + q_2)^*$$
$$= [q_1^2 q_1^* + q_1^2 q_2^* + 2|q_1|^2 q_2]$$
$$+ [q_2^2 q_2^* + q_2^2 q_1^* + 2|q_2|^2 q_1]$$

$$\begin{cases} P_1[q, q^*] = i\left(q_1^2 q_2^* + 2|q_1|^2 q_2\right) \\ P_2[q_2, q_2^*] = i\left(q_2^2 q_1^* + 2|q_2|^2 q_1\right) \end{cases}$$

③ Lie Transform . ..

$$P = \beta_1 (|q|^2 q)_T + \beta_2 |q|^2 q_T + \beta_3 q_{TTT}$$

$$q = Q + i\left(-3\beta_3 + \tfrac{i}{2}\beta_1\right)Q_T + (6\beta_3 - 2\beta_1 - \beta_2) Q \int^T |Q(T')|^2 dT'$$

$$\frac{\partial Q}{\partial z} = \frac{i}{2}\frac{\partial^2 Q}{\partial T^2} + i|Q|^2 Q + \beta_3 \left(6|Q|^2 Q_T + Q_{TTT}\right) + \cdots$$

$$Q = \eta_1 \operatorname{sech} \eta_1 (T - T_0(z)) \exp[i K_1 T + i \sigma_0(z)]$$

$$\begin{cases} \dfrac{dT_0}{dt} = -K_1 - \beta_3(\eta_1^2 - 3K_1^2) \\ \dfrac{d\sigma_0}{dt} = \dfrac{1}{2}(\eta^2 - K^2) - \beta K(3\eta_1^2 - K_1^2) \end{cases}$$

Namely

$$q = \eta \operatorname{sech} \eta (T - T_0(z)) \exp[\cdots]$$

$$+ A_1 \operatorname{sech} \eta (T - T_0(z)) \exp[\cdots]$$

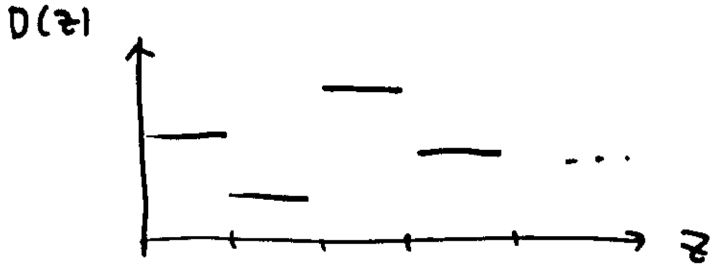
$$+ iA_2 \operatorname{sech} \eta (T - T_0(z)) \tanh \eta (T - T_0(z)) \exp[\cdots] + \cdots$$

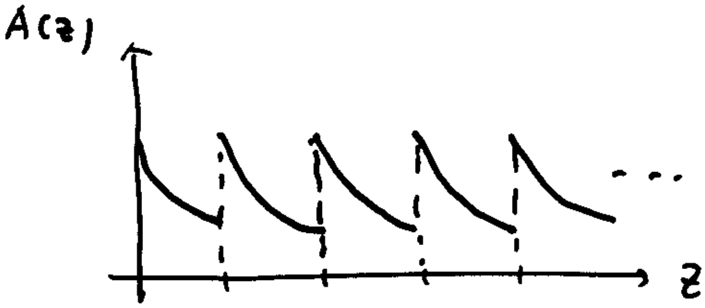
4. Varying Coefficient Problem.

$$\frac{dq}{dz} = \frac{i}{2} D(z) \frac{\partial^2 q}{\partial T^2} + i |q|^2 q - \Gamma q + G(z) q$$

$$\downarrow \quad q \to \sqrt{A(z)}\, q = a(z)\, q , \quad \frac{\partial a}{\partial z} = -\Gamma a + G(z) a .$$

$$\frac{dq}{dz} = \frac{i}{2} D(z) \frac{\partial^2 q}{\partial T^2} + A(z) |q|^2 q$$





Extend the ∞-dim space to :

$$[q, q^* ; z] \equiv (q, q^*, q_T, q_T^*, q_{TT}, q_{TT}^*, \cdots ; z)$$

$$\frac{dq}{dz} = K [q, q^* : z]$$

$$= K_0 [q, q^*] + \tilde{K} [q, q^* : z]$$

$$X_0[q, q^*] \equiv \frac{i}{2}\langle D\rangle \frac{\partial^2 q}{\partial T^2} + i\langle A\rangle |q|^2 q$$

$$\tilde{X}[q, q^*; z] \equiv \frac{i}{2}\tilde{d}(z)\frac{\partial^2 q}{\partial T^2} + i\,\tilde{A}(z)|q|^2 q$$

$D(z) = \langle D\rangle + \tilde{d}(z)$ ← random with zero average 0

$A(z) = \langle A\rangle + \tilde{\alpha}(z)$ ← periodic with zero average 0.

If $\left\{\begin{array}{lll} D(z) \text{ has a characteristic distance} & & z_d \\ A(z) & \text{"} & z_a \end{array}\right\}$

and $z_d \sim z_a \ll z_0$: characteristic distance of NLS
$(= 1)$

introduce a small parameter

$$\varepsilon \equiv \frac{z_d}{z_0} \ll 1$$

Then use the averaging method.

Note integration is a <u>smoothing operation</u>