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Diffraction and Images

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DIFFRACTION

Diffraction requires wave treatment

Light → electromagnetic waves

Optics approximation: one Cartesian component of the e.m. field

$$v(p,t)$$

is representative of the entire field
Energy is proportional to the square of this component (power flux α Poynting vector)

Complex form (coherent monocromatic)

$$v(P,t) = u(P)e^{i\omega t}$$

time dependence:

oscillation with frequency $\nu = \omega / 2\pi$

ω → source

$u(P)$ complex amplitude

complex amplitude

$$u(P) = A(P)\, e^{i\varphi}$$

A amplitude

φ phase

wavefronts: surfaces φ = constant

plane wave

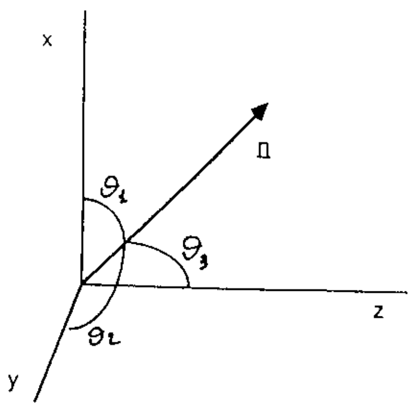
$$u(P) = Ae^{ik(\alpha x + \beta y + \gamma z)}$$

A is constant

$k = \dfrac{\omega}{c}$ is wavenumber

$k = 2\pi/\lambda$ λ wavelength

α, β, γ direction cosines of the normal to the wavefront, i. e. of the propagation direction



$$\alpha = \cos\theta_1$$
$$\beta = \cos\theta_2$$
$$\gamma = \cos\theta_3$$

$$\alpha^2 + \beta^2 + \gamma^2 = 1$$

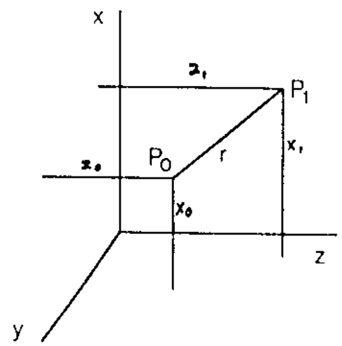
spherical wave

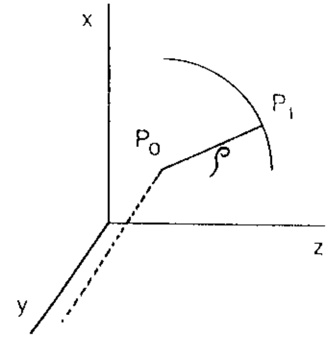
$$u(P) = \frac{A}{r} e^{ikr - i\pi/2}$$

$$r = |P_1 - P_0| = \left[(x_1 - x_0)^2 + (y_1 - y_0)^2 + (z_1 - z_0)^2\right]^{1/2}$$

$P_0(x_0, y_0, z_0)$ source

wavefronts: spheres with center in P_0





cylindrical wave

$$u(P) = \frac{A}{\sqrt{\rho}} e^{ik\rho - i\pi/4}$$

$$\rho = \left[(x_0 - x_1)^2 + (z_0 - z_1)^2\right]^{1/2}$$

wavefronts: cylinders with axis parallel to y through P_0 useful for bidimensional cases

GRIMALDI 1660

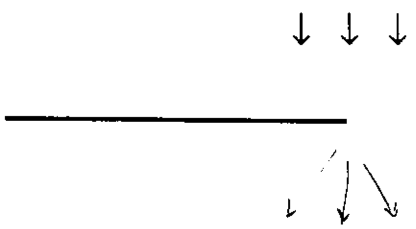
LIGHT propagates

1 - Straight

2 - Reflection

3 - Refraction

4 - Diffraction (go round obstacles)



DIFFRACTION

Occurs where there is an abrupt discontinuity in amplitude[1]

Examples

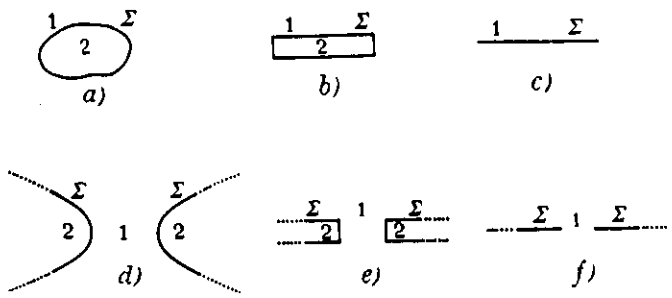


Fig. XII. 1

from: G.Toraldo di Francia and P.Bruscaglioni "Onde Elettromagnetiche" seconda edizione. Zanichelli 1988.

First observation: Grimaldi (1660)

[1]The key point in the theory is that diffraction takes place <u>where</u> the term $\nabla^2 A$ (∇^2 laplacian) is not negligible with respect to $A\,n^2 k_0^2$ (n refractive index, k_0 wave number in the free space). This implies that amplitude variations (second difference) taking place in the space of a wavelength must be negligible in order to neglect diffraction.

Due to diffraction, light reaches regions behind the screen or the obstacle, that are expected to be in the shade according to ray theory (geometrical optics).

<u>Rigorous theory</u> of diffraction requires:

solutions of Maxwell's equations with appropriate boundary conditions.

<u>Approximate solutions</u>

1) Huygens-Fresnel principle

2) Helmholtz-Kirchhoff theory and formula

3) Plane wave expansion (Toraldo-Duffieux)

Huggens-Fresnel principle { historically first; simplest

Helmholtz-Kirchhoff : general formula, classic

Plane wave expansion: introduction to Fourier optics and holography

1 - Huygens-Fresnel principle

Huygens (1678)
Each point of a wavefront can be considered a source of a spherical wave, "wavelet", propagating in the same direction with the same velocity. The wavefront at a later time is the geometrical "envelope" of the secondary waves.
Fresnel (1818): the different wavelets interfere at each point.

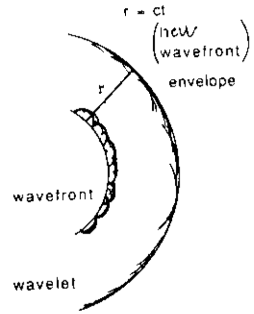


Fig 1

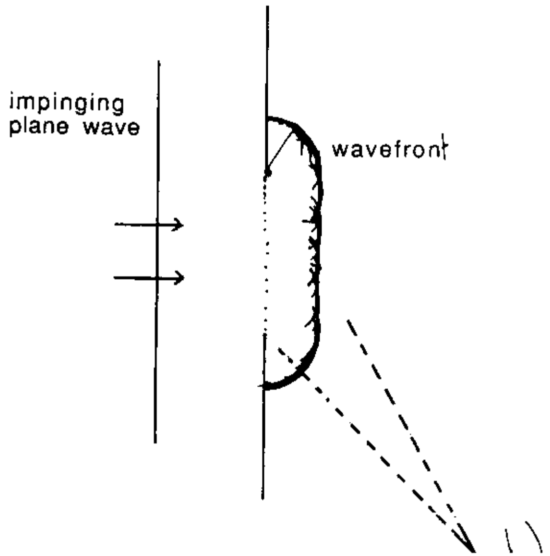
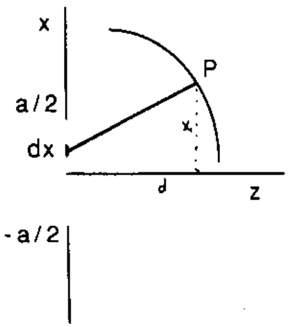


FIG. 2
diffraction: slit in absorbing screen

Use of Huygens-Fresnel principle:

pattern of a slit illuminated by a normally impinging plane wave.
From each point x a cylindrical wave. From each element dx a spherical wave of amplitude a(x)dx; a(x) is a linear density. Let us assume $a(x) = \bar{a}$ on the aperture. Its contribution to the field u(P) at point $P = (x_1, d)$ is (apart from a multiplicative factor, see Kirchhoff)

$$\frac{\bar{a}}{\sqrt{\rho}} e^{ik\rho} dx \qquad \rho = \left[(x - x_1)^2 + d^2\right]^{1/2}$$



$$u(P) = \int_{-a/2}^{a/2} \frac{\bar{a}}{\sqrt{\rho}} e^{ik\rho} dx$$

If point P is at a distance d large with respect to both x and x_1 one can develop ρ in a series and stop at the second term in the exponent:

$$k\rho = kd\left[1 + \frac{(x - x_1)^2}{d^2}\right]^{1/2} \approx kd + \frac{kx^2}{2d} - \frac{kxx_1}{d} + \frac{kx_1^2}{2d}$$

a more accurate analysis could be carried out by considering higher order terms, but it is beyond the scope of the present lessons.

The region where this approximation holds is called Fresnel region.

$$u(P) = \int_{-a/2}^{a/2} \frac{\bar{a}}{\sqrt{\rho}} e^{ik\rho} dx \approx$$

$$\approx \frac{\bar{a}\, e^{\frac{ikx_1^2}{2d}}}{d} \, e^{ikd} \int_{-a/2}^{a/2} e^{-ik\frac{xx_1}{d}} \, e^{ik\frac{x^2}{2d}} dx$$

If d is large enough for the maximum value of the exponent (at the borders) to be near zero one can write:

$$e^{i\frac{kx^2}{2d}} \approx 1 \qquad -\frac{a}{2} \le x \le \frac{a}{2}$$

this requires condition

$$\frac{ka^2}{8d} << 1 \qquad \frac{\pi a^2}{4\lambda d} << 1$$

or, what amounts to the same

$$\frac{a^2}{\lambda d} << 1 \qquad \text{Fraunhofer condition}$$

the region where

$$\frac{a^2}{\lambda d} << 1 \quad \text{Fraunhofer region}^{(2)}$$

Therefore

$$u(P) = K \int_{-a/2}^{a/2} e^{-ik\frac{xx_1}{d}} \, dx$$

where K is a complex quantity including all terms multiplying integral.
Evaluation of integral:

$$u(P) = K \left[\frac{e^{-ik\frac{xx_1}{d}}}{-\frac{ikx_1}{d}} \right]_{x=-\frac{a}{2}}^{x=a/2} = aK \left[\frac{e^{i\frac{kax_1}{2d}} - e^{-i\frac{kax_1}{2d}}}{2i\frac{kx_1a}{2d}} \right]$$

$$= aK \frac{\sin \frac{kax_1}{2d}}{\frac{kax_1}{2d}} = aK \frac{\sin\left(\frac{\pi a}{\lambda}\frac{x_1}{d}\right)}{\frac{\pi a}{\lambda}\frac{x_1}{d}}$$

$$= aK \, \text{Sinc}\left(\pi\frac{a}{\lambda}\frac{x_1}{d}\right)$$

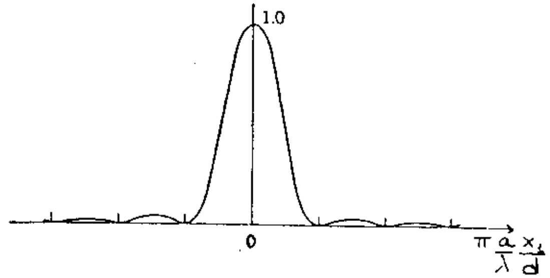
2note: Fraunhofer condition is opposite to the condition required for geometrical optics to hold $\frac{\lambda}{a} d << a \rightarrow \frac{a^2}{d l} >> 1$

The energy is proportional to $u(P)\, u^*(P)$

$$u(P)u^*(P) = a^2 |K|^2 \frac{\sin^2\left(\pi \frac{a}{\lambda}\frac{x_1}{d}\right)}{\left[\pi\frac{a}{\lambda}\frac{x_1}{d}\right]^2}$$

Note: $\frac{x_1}{d} = \sin\theta$ angular direction of point P.
This function oscillates, has maximum for $x_1 = 0$ on the axis, and subsequent zeros and maxima
First zero at

$$\pi \frac{a}{\lambda}\frac{x_1}{d} = 2\pi$$



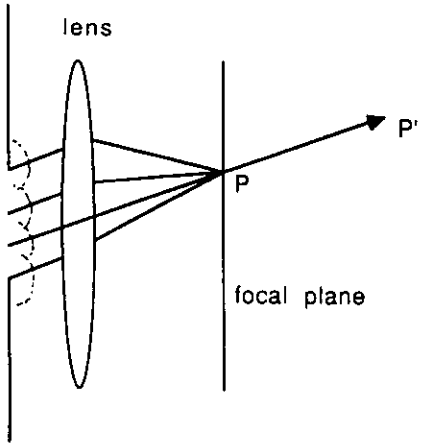
$$\theta \approx \frac{x_1}{d} = \frac{\lambda}{a/2}$$ angular semi width of diffracted beam.

For small aperture, in optics, the Fraunhofer region can be easily reached in a few meters
Example: a=1mm λ=.63 μm

$$d >> \frac{a^2}{\lambda} \qquad d >> \frac{10^{-6} m^2}{.63 \, 10^{-6} m} = 1.6 m$$

Note that dependence is with square of aperture: for a= 2 mm , one needs d >> 7 m.

The Fraunhofer diffraction pattern can be easily seen at finite distance by means of a lens (in our case a cylindric lens). The lens transfers at P, in the focal plane, the field of P' at infinity.



If there is no aperture, the border of the lens acts as an aperture, producing diffraction. All instruments present diffraction and give an image of a point source which is a "pattern", not a point. This strongly affects the resolving power of any instrument from microscopes to the larger telescopes. This effect cannot be avoided because it originates from the nature of light.

2 - KIRCHHOFF THEORY

Homogeneous, isotropic, non absorbing medium.

From Maxwell equations any Cartesian component v(P,t) satisfies:

$$\nabla^2 v(P,t) - \frac{1}{v^2}\frac{\partial^2 v(P,t)}{\partial t^2}$$ equation of d'Alembert

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$ laplacian operator

v light velocity in the medium

c in empty space

For simplicity empty space

For complex amplitude

$$\nabla^2 u(P) + k^2 u(P) = 0$$ Helmoholtz eq. or wave eq.

$$k = \frac{\omega}{c}$$

Green's Theorem

Let us consider a space V surrounded by a closed surface S.

Let u_1 and u_2 be two scalar functions regular in V and on S.

A regular function is continuous and derivable.

Green's theorem states that:

$$\int_V (u_1\nabla^2 u_2 - u_2\nabla^2 u_1)\ dV = \int_S \left(u_1\frac{\partial u_2}{\partial n} - u_2\frac{\partial u_1}{\partial n}\right) dS$$

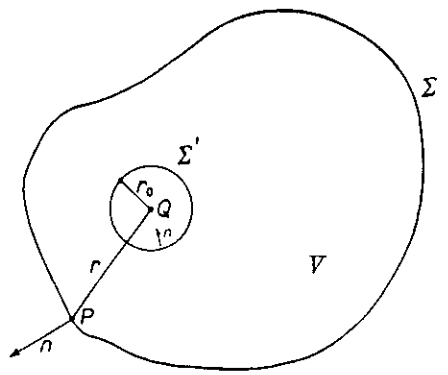
where ∇^2 is the laplacian operator,

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

and $\partial/\partial n$ denotes partial derivative in normal outward direction at a point on S.

Helmholtz-Kirchhoff formula

Derivation of field at Q from field over Σ.



Σ closed surface
Q point
let Σ' be spherical surface of radius r_0 surrounding Q

$$w = \frac{e^{ikr}}{r}$$

chose a spherical wave centered at Q (Green function)

1) $\nabla^2 w + k^2 w = 0$ wave eq. for w

2) $\nabla^2 u + k^2 u = 0$ wave eq. for complex amplitude

Multiply 1 by u and 2 by w and subtract

$$u\nabla^2 w - w\nabla^2 u = 0$$

Integrate into the space between Σ and Σ'

$$\int_V (u\nabla^2 w - w\nabla^2 u)\ dV = 0$$

from Green's formula

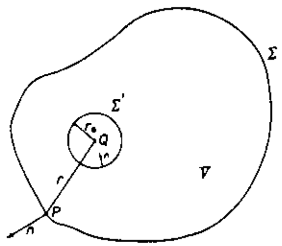
$$1)\quad \int_{\Sigma}\left(w\frac{\partial u}{\partial n}-u\frac{\partial w}{\partial n}\right)d\Sigma+\int_{\Sigma'}\left(w\frac{\partial u}{\partial n}-u\frac{\partial w}{\partial n}\right)d\Sigma'=0$$

one has (over $\Sigma' : \partial w/\partial n = -\partial w/\partial r$)

$$\int_{\Sigma'}\left(w\frac{\partial u}{\partial n}-u\frac{\partial w}{\partial n}\right)d\Sigma=\int_{\Sigma'}\left(-\frac{1}{r}\frac{\partial u}{\partial r}+ik\frac{u}{r}-\frac{u}{r^2}\right)e^{ikr}d\Sigma'=$$

$$=e^{ikr_0}\left(-\frac{1}{r_0}\overline{\frac{\partial u}{\partial r}}+ik\frac{\overline{u}}{r_0}-\frac{\overline{u}}{r_0^2}\right)4\pi r_0^2$$

$\overline{u}$ and $\overline{\frac{\partial u}{\partial r}}$ average values over Σ'



Let $r_0 \rightarrow 0$

Three terms: first and second terms $\rightarrow 0$

last term $\rightarrow -4\pi u(Q)$

From (1)

$$u(Q)=\frac{1}{4\pi}\int_{\Sigma}\left(w\frac{\partial u}{\partial n}-u\frac{\partial w}{n}\right)d\Sigma$$

and finally : Helmholtz-Kirchhoff formula

$$u(Q)=\frac{1}{4\pi}\int_{\Sigma}\frac{e^{ikr}}{r}\left\{\frac{\partial u}{\partial n}-\left(ik-\frac{1}{r}\right)u\cos(\underline{n},\underline{r})\right\}d\Sigma$$

$\cos(\underline{n},\underline{r})$= cosinus of the angle between $\underline{n}$ and $\underline{r}$

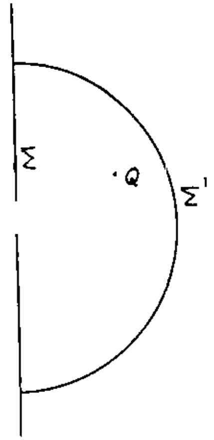
Famous equation derived by Helmholtz (1859)-Kirchhoff gave a more general case (1883).

<u>The value of the field u(Q) at point Q in the volume requires knowledge of the field and its normal derivative on all points of the surface</u> Σ.

This result is not the solution of the problem because it implies knowledge of the field and its normal derivative on Σ, that is solution of the problem on the surface region. Hypothesis for the field on the surface necessary.

Kirchhoff-Diffraction by a plane screen

Aperture in a plane screen illuminated from left



Closed surface: screen Σ+ hemisphere Σ' of large radius R

$$u(Q) = \frac{1}{4\pi}\int_{\Sigma+\Sigma'} \frac{e^{ikr}}{r}\left\{\frac{\partial u}{\partial n} - \left(ik - \frac{1}{r}\right) u \cos(\underline{n},\underline{r})\right\} d\Sigma$$

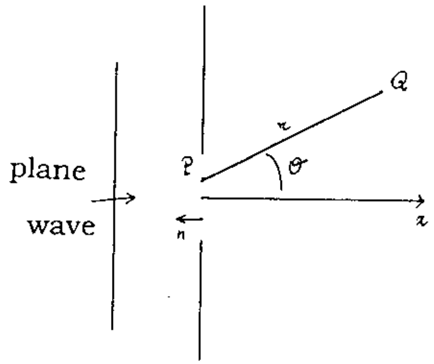
The integral over Σ' requires $(d\Sigma = r^2 d\Omega)$

$$\lim_{r\to\infty} r\left\{\left(\frac{\partial u}{\partial n} - iku\right)\right\} \to 0$$

This is known as Sommerfeld radiation condition: in practice the field vanishes at infinity at least as a diverging spherical wave.

Assumption over surface Σ:
On the opening the field and its normal derivative have the same values as in the absence of the screen and the values are zero everywhere else.

Example: a plane wave impinging ortogonally on the screen. The aperture is circular



impinging wave

$$u(P) = A\, e^{ikz}$$

$$\frac{\partial u}{\partial n} = -ik\, A\, e^{ikz}$$

$$u(Q) = \frac{1}{4\pi}\int_{\Sigma} \frac{e^{ikr}}{r}\left\{-ik\, A\, e^{ikz} - ikAe^{ikz}\cos(\underline{n},\underline{r})\right\} d\Sigma$$

(1/r<<k, term neglected) Over Σ: $z=0$

$$= \frac{-ikA}{4\pi}\int_{\Sigma}\frac{e^{ikr}}{r}[1+\cos(\underline{n},\underline{r})]d\Sigma$$

$$= \frac{-i}{\lambda}\frac{A}{2}\int_{\Sigma}\frac{e^{ikr}}{r}[1+\cos(\underline{n},\underline{r})]d\Sigma$$

$$u(Q) = \frac{-i}{\lambda}\frac{A}{2}\int_{\Sigma}\frac{e^{ikr}}{r}[1+\cos(\underline{n},\underline{r})]\ d\Sigma$$

express Huygens-Fresnel principle.

- From the aperture elementary spherical waves
- Obliquity factor $(1+\cos(\underline{n},\underline{r}))/2$
- The phase of each spherical wave is decreased by $\pi/2$ ($e^{-i\pi/2} = -i$) with respect to incident wave
- The amplitude of each elementary wave is smaller by a factor $1/\lambda$ with respect to that, A, of incident wave.

This is a more complete form of Huygens-Fresnel principle valid far from the screen. On screen even at a large distance inconsistency: u(Q) is not zero due to obliquity factor.
The inconsistency was removed in the Rayleigh-Sommerfed theory where a obliquity factor $\cos(\underline{n},\underline{r})$ was found.

Fraunhofer approximation [paraxial rays]
$P(x,y)$, $Q(x_1,y_1,d)$

$$r = \left[d^2 + (x - x_1)^2 + (y - y_1)^2\right]^{1/2}$$

$$\approx d\left\{1 + \frac{x^2 - 2xx_1 + x_1^2}{2d^2} + \frac{y^2 - 2yy_1 + y_1^2}{2d^2}\right\}$$

$$\rightarrow d - \frac{xx_1}{d} - \frac{yy_1}{d}$$

$$u(x_1, y_1, d) = \frac{-i}{\lambda}\frac{A}{2d} e^{ikd}\int_{\Sigma} e^{-ik\frac{xx_1}{d} - ik\frac{yy_1}{d}}[1 + \cos(\underline{n},\underline{r})]d\Sigma$$

source plane

$$\begin{cases} x = \rho\ \cos\varphi \\ y = \rho\ \sin\varphi \end{cases}$$

observation plane

$$\begin{cases} x_1 = \rho_1\ \cos\varphi_1 \\ y_1 = \rho_1\ \sin\varphi_1 \end{cases}$$

$$\cos(\underline{n},\underline{r}) \sim 1$$

$$d\Sigma = \rho\ d\rho\ d\varphi$$

$$\frac{xx_1}{d} + \frac{yy_1}{d} = \frac{\rho\rho_1}{d}\cos(\varphi - \varphi_1)$$

(discussion about higher order terms)

$$u(Q) = \frac{-i}{\lambda}\frac{Ae^{ikd}}{d}\int_0^a\left\{\int_0^{2\pi} e^{-ik\frac{\rho\rho_1}{d}\cos(\rho-\rho_1)}d\varphi\right\}\rho d\rho$$

Internal integral: Bessel function of zero order *[1]

$$J_0\left(\frac{k}{d}\rho\rho_1\right) = \frac{1}{2\pi}\int_0^{2\pi} e^{-i\left[k\frac{\rho\rho_1}{d}\right]\cos(\varphi-\varphi_1)}d\varphi$$

$$u(Q) = \frac{-i}{\lambda}\frac{A\,e^{ikd}}{d}2\pi\int_0^a J_0\left(\frac{k}{d}\rho_1\rho\right)\rho d\rho =$$

Recalling that

$$\int_0^z J_{\nu-1}(t)t^\nu dt = z^\nu J_\nu(z)$$

$$t = \frac{k}{d}\rho_1\rho \qquad dt = \frac{k\rho_1}{d}d\rho$$

---

*[1]Bessel function of order n $J_n(z)$ can be defined as

$$J_n(z) = \frac{1}{2\pi}\int_\alpha^{2\pi+\alpha} e^{i(n\theta - z\sin\theta)}d\theta$$

In our case n = 0, change of variable $\varphi = \theta - \alpha$ gives

$$J_0(z) = \frac{1}{2\pi}\int_0^{2\pi} e^{-iz\sin(\varphi+\alpha)}d\varphi$$

choice $\alpha = \varphi_1 + \frac{\pi}{2}$ gives $\sin(\varphi+\alpha)\cos(\varphi-\varphi_1)$ and

therefore $J_0(z) = \frac{1}{2\pi}\int_0^{2\pi} e^{-iz\cos(\varphi+\varphi_1)}d\varphi$

one finally has

$$u(Q) = \frac{-iA2\pi a}{\lambda}\frac{e^{ikd}}{d}\frac{J_1(ka\sin\theta)}{(k\sin\theta)}$$

field in Fraunhofer region, distant in the direction θ

$\frac{e^{ikd}}{d}$ spherical wave

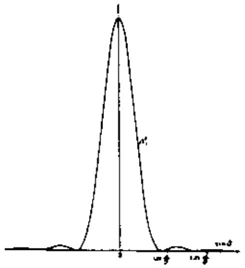
$-i = e^{-i\pi/2}$ dephasing factor

$ka = 2\pi\frac{a}{\lambda}$ parameter for angular dependence energy (intensity) in direction θ apart from unessential constant

$$I = u\,u^{*} \propto \frac{J_1^2(ka \sin\theta)}{(ka \sin\theta)^2}$$

first four zeros of $J_1(x)$

$x = 3.83\,;\ 7.02\,;\ 10.17\,;\ 13.23$



when $ka \sin\theta = 0$ maximum

when $ka \sin\theta = 3.83$ first zero

$$\frac{2a}{\lambda}\sin\theta = \frac{3.83}{3.14} = 1.22$$

Values of subsequent maxima, with respect to the central one

central	1
first	0.0175
second	0.0042
third	0.0016

It can be shown (Rayleigh 1899) that the energy flux the i-th ring is

$$\Phi_i = J_0^2(x_i) - J_0^2(x_{i+1})$$

Through the central disc and subsequent rings
Energy flux (total flux = 1)

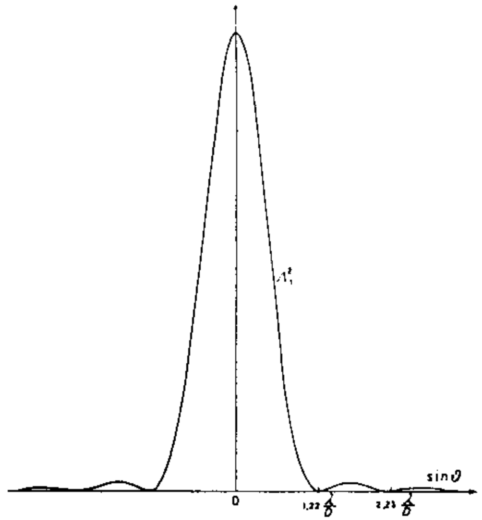
central disc	0.8378
first ring	0.0722
second "	0.0276
third "	0.0147
and so on	

The energy in the central disc of the pattern is ~ 84% of the total.
Energy is mostly concentrated in the central ring, whose total angular with is

$k\,a \sin\theta = 3.83$

$$\frac{2a}{\lambda}\sin\theta = \frac{3.83}{3.14} = 1.22$$

$2a = D$ diameter



$$\sin\theta = 1.22\frac{\lambda}{D} \rightarrow \theta = 1.22\frac{\lambda}{D}$$

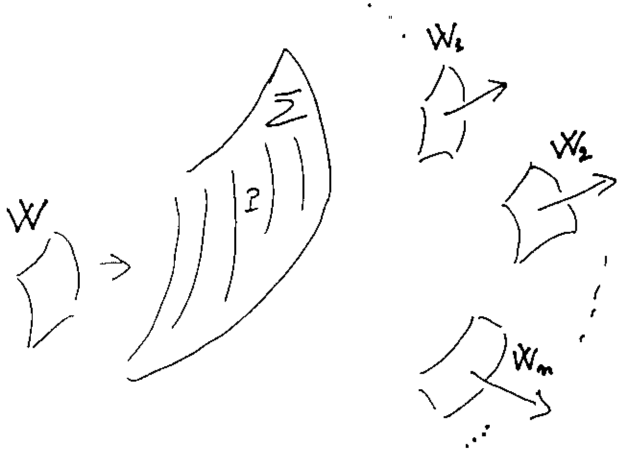
effect on Resolving power of instruments.

3 - Decomposition in plane waves

Diffraction as decomposition in plane waves is the basis of Fourier optics (Duffieux)

The decomposition by the so called "inverse interference principle", Toraldo (1941), and valid for surfaces planar or not.
Inverse interference principle:

A screen is illuminated from the left by a field W, that produces phase $\varphi(P)$ and amplitude $A(P)$ distribution at points P over the output side surface Σ.



If a system of waves outgoing from Σ is found whose interference produces the field $v(P) = A \exp(i\varphi)$ over Σ, these waves are the true diffracted waves

- uniqueness of the solution

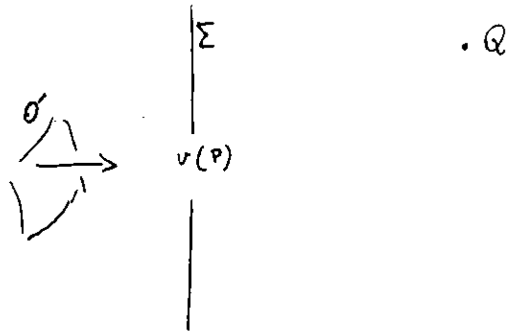
Screen:
-partially transparent: transmitted diffracted waves
-partially refecting: reflected diffracted waves
-both
(eigenfunctions)

Generally: v(P) unknown on the screen
Hypotheses about v(P) necessary.

Examples: amplitude or phase or both

1) A
2) $\varphi_0(P) = \varphi_i(P) + k\Delta(P)$
3) both

Plane screen: amplitude

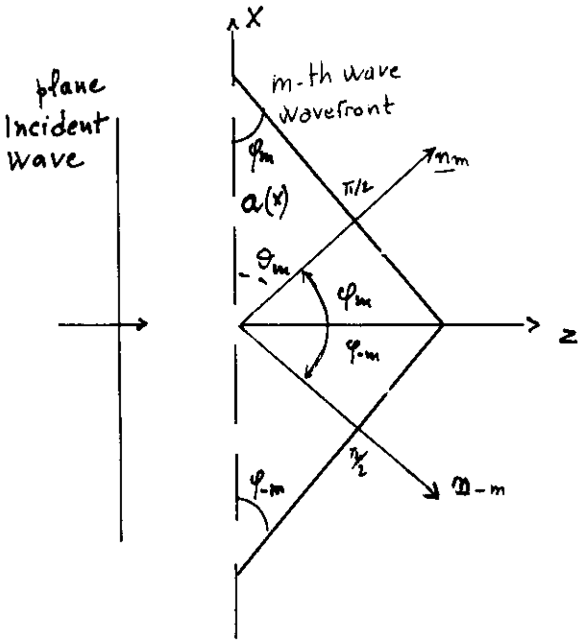


assume: v(P) on the aperture has the same value as in the absence of the screen
no need to know the normal derivative

Periodic aperture: grating

example: unidimensional periodic grating

a(x) periodic, period p



System of plane waves propagating in direction z

Fourier series for a(x)

$$a(x) = \sum_{m=-\infty}^{\infty} A_m \, e^{i2m\pi\frac{x}{p}}$$

$$A_m = \frac{1}{p}\int_{-p/2}^{p/2} a(x)\, e^{-i2m\pi\frac{x}{p}}\, dx$$

A system of plane waves $\alpha_m = \cos\theta_m = \sin\varphi_m$

$$v(x,z) = \sum_{m=-\infty}^{\infty} B_m \quad e^{ik(\alpha_m x + \gamma_m z)}$$

Condition $v(x,0) = a(x)$ gives

$$v(x,0) = \sum_{m=-\infty}^{\infty} B_m \quad e^{ik\alpha_m x}$$

Comparison with a(x) gives

$$2m\pi\frac{x}{p} = k\alpha_m x \;\rightarrow\; \alpha_m = m\frac{\lambda}{p}$$

$B_m = A_m$ $\qquad\qquad$ $\gamma_m \geq 0$

$$v(x,z) = \sum_{m=-\infty}^{\infty} A_m \quad e^{ikm\frac{\lambda}{p}x}\; e^{ik\sqrt{1-\frac{m^2\lambda^2}{p^2}}}\, z$$

for $\alpha_m = m\frac{\lambda}{p} \leq 1 \;\rightarrow\; m \leq \frac{p}{\lambda}$ $\qquad$ real waves

$\alpha_m = m\frac{m\lambda}{p} > 1 \;\rightarrow\; m > \frac{p}{\lambda}$ $\qquad$ evanescent waves

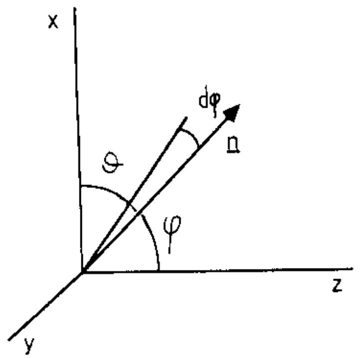
N=2max integer $\left(\frac{p}{\lambda}+1\right)$=number of real waves

$\varphi_m \approx m\frac{\lambda}{p}$ $\qquad$ for smal φ_m

PLANE SCREEN : UNIDIMENSIONAL CASE

Plane screen xy with transparency or opening; symmetric with respect to y. Complex amplitude on the screen a(x)

Plane wave of unit amplitude



$$e^{ik(\alpha x+\beta y+\gamma z)}$$

$$\alpha^2+\beta^2+\gamma=1$$

$$\alpha=\cos\theta$$
$$\beta=\cos\gamma$$
$$\gamma=\cos\varphi$$

Let us construct a continuous system of plane waves. In our case no dependence on y $\rightarrow \beta = 0$ ||
Let us consider a $d\varphi$. Let

$A\,d\varphi$

the amplitude of all waves having propagation direction between φ and $\varphi + d\varphi$

$\alpha = \sin\varphi \qquad d\alpha = \cos\varphi \quad d\varphi = \sqrt{1-\alpha^2}\,d\varphi$

$$d\varphi = \frac{d\alpha}{\sqrt{1-\alpha^2}}$$

$$v(x,z) = \int \frac{A\,e^{ik(\alpha x+\gamma z)}}{\sqrt{1-\alpha^2}}\,d\alpha$$

continuous system of diffracting waves

on the aperture $v(x,0) = a(x)$

Let us write a(x) by its Fourier transform $\overline{A}(f)$

$$\overline{A}(f) = \int_{-\infty}^{\infty} a(s)\,e^{-2\pi i f s}\,ds$$

$$a(x) = \int_{-\infty}^{\infty} e^{2\pi i f x}\,\overline{A}(f)\,df = \int_{-\infty}^{\infty} e^{2\pi i f x}\left(\int_{-\infty}^{\infty} a(s)\,e^{-2\pi i f s}\,ds\right)df$$

Let $\boxed{f = \alpha/\lambda}$

$$\rightarrow \quad a(x) = \frac{1}{\lambda}\int_{-\infty}^{\infty} e^{2\pi i \frac{\alpha}{\lambda} x}\left(\int_{-\infty}^{\infty} a(s)\,e^{-\frac{2\pi i}{\lambda}\alpha s}\,ds\right)d\alpha$$

On the aperture the integrale of plane waves

$$\rightarrow \quad v(x,0) = \int_{-\infty}^{\infty} \frac{A}{\sqrt{1-\alpha^2}}\,e^{ik\alpha}\,d\alpha$$

must have infinite limits (real and evanescent waves) and $A = A(\alpha)$

$$\frac{A}{\sqrt{1-\alpha^2}} = \frac{1}{\lambda}\int_{-\infty}^{\infty} a(s)\,e^{-\frac{2\pi i}{\lambda}\alpha s}\,ds$$

or

$$A(\alpha) = \frac{\sqrt{1-\alpha^2}}{\lambda}\int_{-\infty}^{\infty} a(x)\,e^{-ik\alpha x}\,dx$$

In terms of φ $A(\alpha) \rightarrow A(\varphi)$

$$A(\varphi) = \frac{\cos\varphi}{\lambda} \int_{-\infty}^{\infty} a(x)\, e^{-ik\sin\varphi x} dx$$

by denoting

$$A(f) = \frac{\lambda}{\sqrt{1-\alpha^2}} A(\alpha) = \int_{-\infty}^{\infty} a(x)\, e^{-i 2\pi f x} dx$$

and recall that $f = \frac{\alpha}{\lambda} = \frac{\sin\varphi}{\lambda}$

then $A(f) \rightarrow$ Fourier transform of a(x)

The amplitude A(f) of the wave diffracted in the direction φ is the component at frequency $f = \frac{\sin\varphi}{\lambda}$ of the Fourier transform of a(x)

In other words the system of diffracted waves is the Fourier Transform of the field a(x) on the screen. Diffraction → Fourier transform

---

Transform relationship

$$a(x) = \int_{-\infty}^{\infty} A(f)\, e^{2\pi i f x} df \qquad A(f) = \int_{-\infty}^{\infty} a(x)\, e^{-2\pi i f x} dx$$

ENERGY FLUX - POYNTING VECTOR

Let us recall Parceval theorem for transforms

$$\int_{-\infty}^{\infty} a(\xi)\, b^*(\xi) d\xi = \int_{-\infty}^{\infty} A(f)\, B^*(f) df$$

where A(f) Fourier transform of $a(\xi)$ and B(f) of $b(\xi)$

if $a(\xi) = b(\xi)$

$$\int_{-\infty}^{\infty} a(x) a^*(x) dx = \int_{-\infty}^{\infty} A(f) A^*(f) df$$

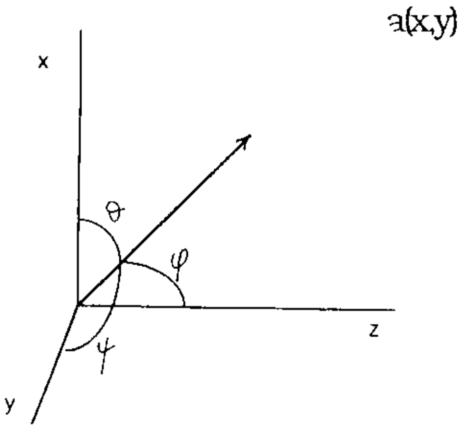
$$A(f) = \frac{\lambda A(\alpha)}{\sqrt{1-\alpha^2}} = \frac{\lambda A(\varphi)}{\cos\varphi} \qquad \alpha = \sin\varphi$$

$$\int_{-\infty}^{\infty} a(x) a^*(x) dx = \lambda \int_{\sin\varphi=-\infty}^{\sin\varphi=\infty} \frac{A(\varphi) A^*(\varphi)}{\cos\varphi} d\varphi$$

<u>left side term:</u> energy per unit time transmitted per unit y through the screen
<u>right side term</u>: the energy per unit y carried by each wave is that which a wave of intensity $A(\varphi)A^*(\varphi)d\varphi$ carries through a slit of width $\lambda / \cos\varphi$

PLANE SCREEN   Bidimensional case

Analogous to unidimensional case



$\alpha = \cos\vartheta$

$\beta = \cos\psi$

$\gamma = \cos\varphi$

One choses a system of plane waves of any direction. Amplitude in small solid angle $Ad\Omega$, $d\Omega$ solid angle, A amplitude density

$$A(\alpha,\beta)\frac{d\alpha d\beta}{\sqrt{1-\alpha^2-\beta^2}}$$

$$v(x,y,z) = \iint \frac{A(\alpha,\beta)}{\sqrt{1-\alpha^2-\beta^2}} e^{ik(\alpha x+\beta y+\gamma z)} d\alpha d\beta$$

on the aperture

$$v(x,y,0) = a(x,y)$$

a(x,y) is Fourier transformed

Procedure analogous to previous case gives the same result.

Diffracted waves are a continuous system of plane waves and evanescent waves. The diffracted field is the Fourier transform of the field on the aperture, with frequencies

$$f_x = \frac{\alpha}{\lambda} \quad \text{and} \quad f_y = \frac{\beta}{\lambda}$$

respectively. Therefore

in the space direction θ, ψ

specified by α and β ($\alpha = \cos\theta$, $\beta = \cos\psi$), one has:

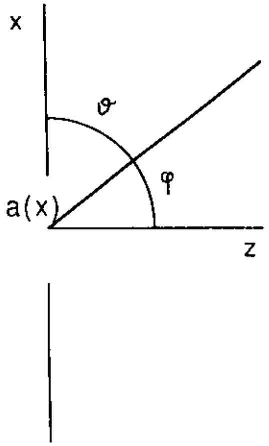
frequency components $f_x = \frac{\alpha}{\lambda}$ and $f_y = \frac{\beta}{\lambda}$ respectively.

Limited aperture: no upper limit to diffraction angles and always presence of evanescent waves. A well known property of Fourier transform: if the support of the function is finite the support of the transform is infinite.

In systems: loss of information

HOLOGRAPHY easy to explain with expansion in plane waves

EVANESCENT WAVES (surface waves or inhomogeneous waves)



bidimensional case (unidimensional aperture)

a(x) field on aperture

System of plane (real and evanescent) waves

$$v(x,z) = \int_{-\infty}^{\infty} \frac{A(\alpha)}{\sqrt{1-\alpha^2}} e^{ik(\alpha x + \gamma z)} d\alpha$$

$$\alpha = \cos\theta$$
$$\gamma = \cos\varphi$$

$$\alpha^2 + \gamma^2 = 1$$

if $\underline{\alpha > 1}$

$$\gamma = \sqrt{1-\alpha^2} = i\sqrt{\alpha^2 - 1}$$

Let us consider a wave of unit amplitude

$$e^{ik(\alpha x + \gamma z)} = e^{ik\alpha x}\, e^{-kz\sqrt{\alpha^2-1}}$$

first factor: Phase factor ⇒ propagation in x direction.

second factor: Amplitude factor ⇒ decaying of amplitude in z direction

Equiamplitude planes ≠ Equiphase planes (inhomogeneous waves)

Propagation wavelength

$$k\alpha(x_1 - x_2) = k\alpha\lambda_e = 2\pi \qquad \lambda_e = \frac{\lambda}{\alpha}$$

wavelenth shorter than real waves ($\alpha \geq 1$)

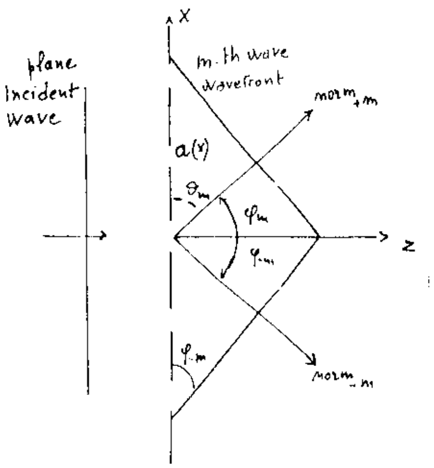
Velocity

$$v_e = \frac{\omega}{k\alpha} = \frac{v_r}{\alpha}$$

smaller than v_r of real waves

Propagation in direction x, amplitude decay in direction z. Analogously for $\alpha < -1$

Necessary support (surface) in -z direction to avoid infinite increasing (Surface waves-Leaky waves)



Periodic grating

p period

$$\alpha_m = \cos\ \theta_m$$
$$\gamma_m = \cos\varphi_m = \sqrt{1-\alpha_m^2}$$

Set of waves

$$\alpha_m = m\frac{\lambda}{p}$$

Real waves

$$m\frac{\lambda}{p} \le 1 \quad \text{or} \quad m \le \frac{p}{\lambda}$$

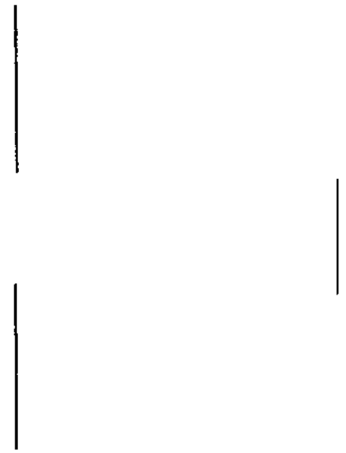
Evanescent waves

$$m > \frac{p}{\lambda}$$

If $p = \lambda + \varepsilon$	$p = \lambda$	$p = \lambda - \varepsilon$
$m = 1 \rightarrow$ real	$m = 1$ limit. case	$m = 1$ evanescent
central w $m = 0$ +2 lateral w $m = \pm 1$ +evanescent w	$\alpha_1 = 1$ $\theta_m = \pi/2$ $\gamma_m = 0$	central w $m = 0$ + evanescent w

BABINET PRINCIPLE: DIFFRACTION

by COMPLEMENTARY APERTURES



screen with circular aperture — opaque disc

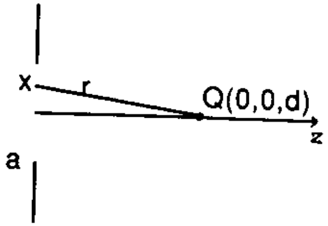
Solution of one problem allows solution of the other one

From the field diffracted from an aperture one derives the field diffracted by the complementary aperture by: adding a phase = π to all diffracted waves and adding to them a wave equal to the incident wave.

Therefore : apart from the phases, the ensemble of the waves diffracted by two complementary screens differ only for the central wave

FRESNEL ZONES

Bidimensional screen
Circular aperture
Uniformly illuminated: A



Q(0,0,d) on axis

$$r = \left[x^2 + y^2 + d^2\right]^{1/2}$$

$$1 + \cos(\underline{n}, \underline{r}) \approx 2$$

$$\cos 1^\circ = 0.9998$$
$$\cos 5^\circ = 0.996$$
$$\cos 10^\circ = 0.985$$

$$u(Q) = -\frac{iA}{\lambda}\int_\Sigma \frac{e^{ikr}}{r} d\Sigma$$

$$r \approx d\left[1 + \frac{1}{2}\frac{x^2 + y^2}{d^2}\right]$$ Fresnel region

$$u(Q) = -\frac{iA}{\lambda}\frac{e^{ikd}}{d}\int_\Sigma e^{ik\frac{x^2+y^2}{2d}} d\Sigma$$

$$= \frac{-iAe^{ikd}}{\lambda d}\int_{\rho=0}^{a}\int_{\theta=0}^{2\pi} e^{i\frac{k\rho^2}{2d}} \rho d\rho d\theta$$

$$x = \rho \cos\theta$$
$$y = \rho \sin\theta$$
$$d\Sigma = \rho\, d\rho d\theta$$

$$= -\frac{iA}{\lambda}\frac{e^{ikd}}{d}\, 2\pi\left[\frac{e^{i\frac{k\rho^2}{2d}}}{2\frac{ik}{2d}}\right]_0^a$$

$$u(Q) = A\, e^{ikd}\left[1 - e^{i\frac{ka^2}{2d}}\right]$$

⇑ impinging wave ⇑ multiplicative factor T

$$T = 1 - e^{ika^2/2d} = 1 - \cos\frac{ka^2}{2d} - i\sin\frac{ka^2}{2d}$$

For a given d, phase proportional to a^2

$ka^2/2d$	a^2	T
0	0	0 (real)
π	λd	2 (real)
2π	$2\lambda d$	0 (real)
3π	$3\lambda d$	2 (real)

and so on.

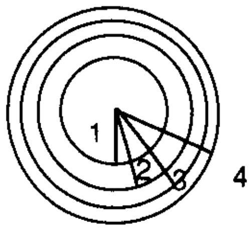
$$a^2 = n\lambda d$$
$$a = \sqrt{n\lambda d}$$

Fresnel zone of order n

$a^2 = n\lambda d$ a radius of n-th Fresnel zone

central circle
then rings

contribution from each zone cancels contribution from the preceding one



IF THE AREA CONTAINS

1) an odd number of Fresnel zones, at point Q field is maximum = 2 times the incident wave.
2) an even number: zero field at Q.

Moving along axis (d) field → maxima and zeros.

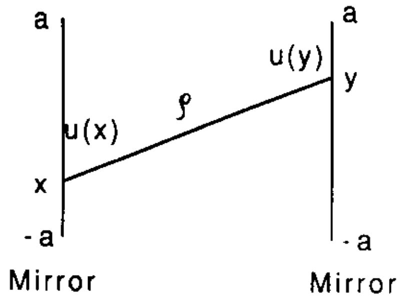
Soret grating - zone grating - is based on removing even zones and has focussing properties.

The Fraunhofer approximation $(a^2/\lambda d \ll 1)$ requires that a small portion of the first Fresnel zone is seen from point Q, at distance d.

For the case of a slit: see Goodman. Fresnel integrals.

OPEN CAVITIES FOR LASERS

Devices based on diffraction: Large Fresnel Numbers



example: bidimensional case

$$u(y) = \int_{-a}^{a} e^{-i\pi/4} u(x) \frac{e^{ik\rho}}{\sqrt{\lambda\rho}} dx$$

$u_m(y) = \sigma_m u_m(x)$ modes' eq.

σ_m complex quantity

$1 - |\sigma_m|^2$ → loss of m-th mode

$\arg \sigma_m$ → phase shift

IMAGES

Mirrors

Lenses
- traditional
- others such as aspherical or graded index

Systems, more or less sophisticated

Methods:

1 - Geometrical optics:

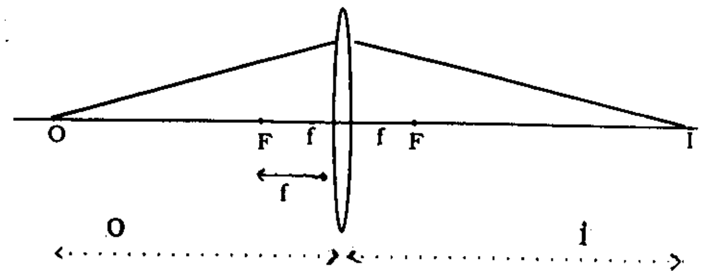
simple
rays
allows accounting for aberrations
neglects diffraction

2 - Wave optics:

allows accounting for:
diffraction
aberrations
direct by using diffraction formulas
scalar approximation
development based on linear systems theory
approximate results

1 - GEOMETRICAL OPTICS

Recall some fundamentals by thin lens



1) $$\frac{1}{o} + \frac{1}{i} = \frac{1}{f}$$

paraxial approximation, gaussian optics

2) $$\frac{1}{f} = (n-1)\left[\frac{1}{R_1} - \frac{1}{R_2}\right]$$ f focal length

Sing conventions
- $o>0$ if on left lens side
- $i>0$ if on right side
- Lens radius >0 if center on right side
  in fig $R_1>0$, $R_2<0$
- $f>0$ converging lens $f<0$ diverging lens

When $o \to \infty$, i=Focus. Perfect lens makes parallel rays converge to (or diverge as from back) focus.
Ex: $f>0$: the lens makes rays converge. It transforms plane wave into spherical converging wave, see below.

Images: Real or Imaginary

In general, for lenses or systems of elements:

From source to image optical path along each ray the same (Fermat principle). Phase along each ray the same; at image point positive interference.

In paraxial approximation, one image point corresponds to source point; the rays from source only have one cross point. In general rays do not all cross at the same point; aberrations. (Here we neglect magnification and image reversal)

In addition diffraction effect. Images by systems without aberrations are called diffraction limited.

2 - WAVE OPTICS

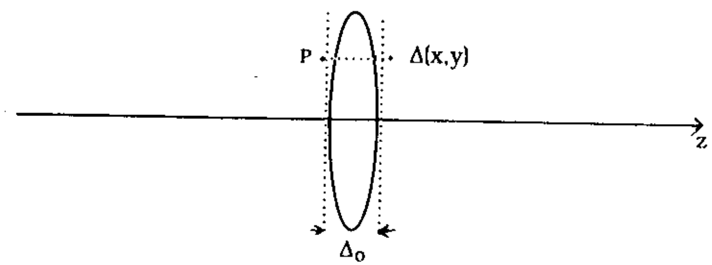
MONOCHROMATIC ILLUMINATION

THIN LENS

Lens introduces phase effect on impinging wave u(P) where P -coordinates x,y- point on entrance plane,

$$u_{out}(P) = t(P)\, u_{in}(P) \qquad t(P) = e^{i\Phi(P)}$$

t(P) thickness function



Δ_o $\Delta(P)$

$$\Phi(P) = k\, n\, \Delta(P) + k\left[\Delta_o - \Delta(P)\right]$$

Simple computation (Goodman)

$$\Delta(P)=\Delta_0 - R_1\left(1-\sqrt{1-\frac{x^2+y^2}{R_1^2}}\right)+R_2\left(1-\sqrt{1-\frac{x^2+y^2}{R_2^2}}\right)$$

Paraxial approx:

$$1-\sqrt{1-\frac{x^2+y^2}{R_1^2}} \cong 1-\frac{x^2+y^2}{2\,R_1^2}$$

$$\Phi(P) = i\,k\,n\,\Delta_0 - i\,k\,(n-1)\,\frac{x^2+y^2}{2}\left(\frac{1}{R_1}-\frac{1}{R_2}\right)$$

$$\Phi(P)=i\,k\,n\,\Delta_0 - i\,\frac{k}{2f}\left(x^2+y^2\right)$$

If $u_{in}(P) = 1$, unit amplitude plane wave, (point source at infinity) one has

$$u_{out}(P) = e^{i\,k\,n\,\Delta_0} + e^{\frac{-ik}{2f}\left(x^2+y^2\right)}$$

First term constant phase delay of no importance

Second term: quadratic approximation, at z=0, to a spherical wave converging towards the focus behind the lens, if f >0 (and then diverging) or diverging from the lens as if originating from focus before the lens if f<0.

Example f>0,

$$r=\sqrt{(f-z)^2+x^2+y^2}\cong(f-z)+\frac{1}{2(f-z)}\left(x^2+y^2\right)$$

Result:

In paraxial approximation lens adds quadratic phase term, i.e. transforms plane into spherical wave.

In first approximation this can be extended to plane waves impinging at small angles. In this case the wave is focused at a point on the focal plane.

In general case: although the lens has spherical surfaces, the wavefront departs from spherical shape. Aberrations.

PUPIL FUNCTION: To take into account the finite dimensions of apertures (and also aberrations): pupil function. Will be useful in the sequel.

For systems without aberrations (diffraction limited)

$$P(x,y) = \begin{cases} 1 & \text{inside lens aperture} \\ 0 & \text{outside} \end{cases}$$

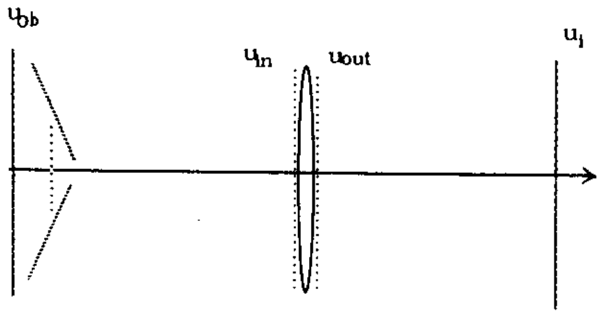
Note: here x,y point on the pupil.
Some authors include on the pupil function the field on the pupil due to a source point.

3 - WAVE OPTICS

COHERENT IMAGING: OBJECT ILLUMINATED WITH MONOCHROMATIC COHERENT FIELD

The problem of images is: given the field distribution at the object find the field distribution at the image.

3-1 LENS AND PLANE OBJECT



u_{ob} field (complex amplitude) from object

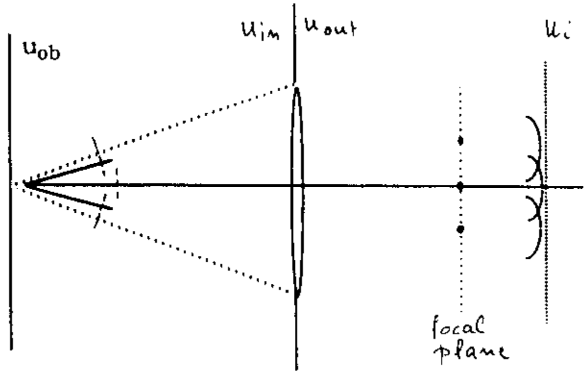
u_{in} field on input plane of the lens(entrance pupil),

u_{out} field after the lens (exit pupil),

u_{i} field on image plane (defined by geom. optics).

Problem: given u_{ob} find u_{i}

Typical diffraction problem: the field from object can be considered a diffraction field: from the object plane waves in all directions (Fourier components). u_{in} is result of interference on input plane. Field u_{out} at the output of lens can be obtained by multiplying u_{in} for the lens function t(P), and pupil function. Field u_{i} by using one of different formulas of diffraction theory, such as Kirchhoff's, which takes into account the finite dimension of the aperture.



Let us consider decomposition of diffracted field in plane waves (Fourier). Each plane wave is focused by the lens at the focus. On each plane behind the lens all waves interfere; in the image plane interference is expected to "reproduce" the object field.

However the image is never equal to the object, because not all plane waves from the object enter the lens, but only those with angle respect to axis less than a/o, a=aperture radius, o=object distance from the lens. In addition evanescent waves are lost.

Due to the limited aperture, from a plane incident wave, source at infinity, one has an Airy diffraction pattern (see diffraction); not a simple point as expected from the lens.

3-2 IMAGES BY A SYSTEM - COHERENT CASE

Let us now think of a general imaging system, of which the lens is a particular case.
First, let us consider, in the source plane, an object constituted by a simple point (source point); in general, due to diffraction and aberrations, the image is not a point but a "pattern". A point source can be represented by a Dirac delta function.

Let $h(x,y;x_0,y_0)$ denote the field at point x,y in the image produced by a source point located at point x_0,y_0 in the source plane. In a first approximation and no aberrations, $h(x,y;x_0,y_0)$ is the Airy diffraction pattern. In general it is a diffraction pattern.

Function $h(x,y;x_0,y_0)$ which represents the <u>impulse response</u> of the system is called the <u>Spread Function</u>

Let us assume to have an extended source. Let $u_{ob}(x_0,y_0)$ be the complex amplitude distribution density (surface density). Each element dx_0dy_0 gives a contribution to the field at x,y, given by

$$u_{ob}(x_0,y_0)\, h(x,y;x_0,y_0)\, dx_0dy_0$$

The field $u_i(x,y)$ at point x,y on image plane, due to the object is obtained by integrating over all the object

3) $$u_i(x,y) = \int\int_{-\infty}^{\infty} u_{ob}(x_0,y_0)\, h(x,y;x_0,y_0)\, dx_0dy_0$$

Typically the object will be of finite dimension and the integrand different from zero on a finite area.

The fact that one can easily write the total field at x,y as the integral (sum) of the contributions produced by the different points of the object is direct consequence of the linearity of Maxwell's equations. According to this linearity the total field at x,y is the superposition (interference) of the contributions from the different elements of the object.

Linearity implies use of the basic elements of linear systems. They are used here, when necessary.

If Spread Function only depends on coordinates difference

4) $$h(x,y;x_0,y_0) = h(x-x_0,\ y-y_0)$$

the system is called <u>isoplanatic</u> (or space-invariant). In practice isoplanatism means that the system "response" is independent of the object location on the source plane.
For a isoplanatic system one has

5) $$u_i(x,y) = \int\int_{-\infty}^{\infty} u_{ob}(x_0,y_0)\, h(x-x_0,\ y-y_0)\, dx_0dy_0$$

As already stated, in general $h(x-x_0,y-y_0)$ is a diffraction pattern, not a simple Dirac function as in geometrical optics approximation. Therefore the field at point x,y is affected by all source points and not only by the corresponding point of the object. This means that, due to diffraction (and aberrations), the image is a smeared version of the object. In the integral we recognize a (bidimensional) convolution operation which is the mathematical formulation of this fact.

In convolution formalism the integral can be written

6) $$u_i(x,y) = u_{ob}(x,y) \otimes h(x,y)$$

where $\otimes$ denotes the convolution operation[1].

Well known theorem, called the convolution theorem, states that the spectrum of a convolution of two functions is the product of the spectra of the two functions. In formula

7) $$U_i(u,v) = H(u,v)\, U_{ob}(u,v)$$

Capital letters denote the spectra; note that they are bidimensional Fourier transforms,

Fourier transform	of	
$U_{ob}(u,v)$	$u_{ob}(x,y)$	object
$U_i(u,v)$	$u_i(x,y)$	image
$H(u,v)$	$h(x,y)$	spread function

[1] By definition $u \otimes v = \int\int_{-\infty}^{\infty} u(x_o,y_o)v(x-x_o,y-y_o)dx_o dy_o$

H(u,v), Fourier transform of the Spread Function, is called the <u>Optical Transfer Function</u>, OTF or, as in linear systems theory, the System Transfer Function, or simply the Transfer Function. Sometimes the adjective Coherent is added to avoid confusion with the case of incoherent radiation, see below.
Eq. 6 is very important, both for theory and applications, because in the realm of spectra the convolution becomes a simple product and allows optical images to be exploited by the techniques commonly used in systems applications, such as filtering in electric systems.

3-3 SPREAD FUNCTION OF A SOURCE POINT
of unit amplitude and a thin lens without aberrations (diffraction limited)

From a source point on the axis spherical wave. In paraxial approximation, the field incident at point x,y on the lens, at distance r_o from the source, is (see diffraction lessons)

$$u_i = \bar{a}\,\frac{e^{ikr_o}}{r_o} \cong \frac{\bar{a}}{o}\, e^{iko + ik(x^2+y^2)/(2o)}$$

Here $\bar{a}$ includes the constant phase term and o denotes source-lens distance. The field u_{out} at the output plane of the lens is

$$u_{out} = \frac{\bar{a}}{o} e^{iko + i\,k\,n\,\Delta_o}\; e^{\frac{-ik}{2f}(x^2+y^2) + \frac{ik}{2o}(x^2+y^2)}$$

The quadratic phase term is an approximation to a spherical wave converging to a point at distance i from the lens

$$\frac{1}{i} = \frac{1}{o} - \frac{1}{f}$$

as from geometrical optics. The field u_i at a point x_i, y_i is obtained by using any diffraction formula, e.g. the paraxial form of Huygens-Fresnel principle (see lessons on diffraction). One obtains

$$u_i = c \iint_{aperture} e^{\frac{ik}{2}\left(\frac{1}{o}-\frac{1}{f}\right)(x^2+y^2)+\frac{ik}{2i}\left[(x-x_i)^2+(y-y_i)^2\right]} dxdy$$

where complex constant c takes into account constant amplitude and phase terms. By developing the squares the phase term can be rewritten as

$$i\Phi = \frac{ik}{2}\left(\frac{1}{o}-\frac{1}{f}+\frac{1}{i}\right)(x^2+y^2)+\frac{ik}{2i}\left[x_i^2+y_i^2-2xx_i-2yy_i\right]$$

On image plane first term in parenthesis is zero. Let us assume that also Fraunhofer condition is satisfied and neglect first and second term in square brackets. The integral reduces to

$$u_i = c \iint_{aperture} e^{\frac{-ik}{2i}\left[2xx_i+2yy_i\right]} dxdy$$

This integral was evaluated in the diffraction section, when the field diffracted from an aperture uniformly illuminated was calculated in the Fraunhofer

approximation (pg 2.9-2.13). In terms of energy the result is the well known Airy pattern. Therefore, in the considered limits, and apart from a complex constant factor, the spread function of a source point is the diffraction pattern, in the Fraunhofer region, of a uniformly illuminated aperture. In other words, the spread function of a source point at finite distance is proportional to that of the source at infinity. Note also that when the source distance is infinite the pattern location is on the focal plane. This corresponds to the well known fact that in general, a lens transfers on its focal plane the Fraunhofer patter of the field on its aperture.

SPREAD FUNCTION = FOURIER TRANSFORM OF THE PUPIL FUNCTION.

VALID IN GENERAL. Easy to understand by plane wave development.

CUT OFF FREQUENCY

3 - 4 LINE SPREAD FUNCTION

Let the object be a line, infinitely thin and concident with the y axis:

$$u_{ob}(x_o, y_o) = \delta(x_o)$$

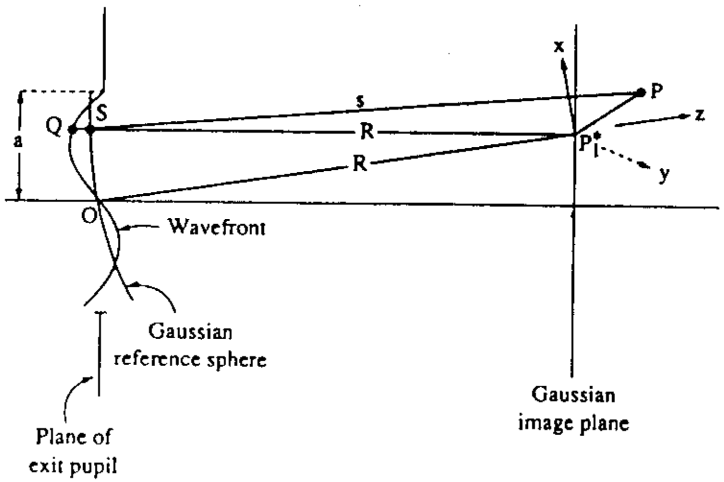
From 5) one obtains

$$u_i(x) = \int_{-\infty}^{\infty} h(x, y_o)\, dy_o$$

3 - 5 ABERRATIONS

According to Wolf function $W(x,y)$ is the departure from spherical shape in the exit pupil. Phase distortion due to aberrations $kW(x,y)$.

Effect of aberration on diffraction patter: lowering the maxima, filling the zeros and rising the minima.



W(x,y) from Born and Wolf

Quality of a system is described by Strehl ratio for source point:

$$S = \frac{\text{Intensity in the (nominal) central maximum}}{\text{Theoretical intensity with no aberrations}}$$

General definition suitable also for partial coherence and for aberrations introduced by propagation in random media, such as turbulent atmosphere.

An image is well corrected if S not less than 0.8.

It can be shown, for small enough aberrations, that intensity at point P is

$$i(P) \cong 1 - k^2 (\Delta\Phi_P)^2$$

where $(\Delta\Phi_P)^2$ mean square deformation of the wavefront and k wavenumber. It follows that condition $S \geq 0.8$, requires $|\Delta\Phi| \leq \lambda / 14$.
(Criterion by Rayleigh $\lambda / 4$ for spherical aberration)

Best focus.

Point on axis. Aberrations function W_a, with respect to a sphere centered on best focus is expanded in terms of r, position on pupil, or ($\Omega \approx r/d_i$ field angle)

$$W_a = a_4 r^4 + a_6 r^6 + \ldots = W_4 + W_6 + \ldots$$

W_4 primary aberrations, W_6 secondary aberrations and so on.
For points off axis in general there are also angles, total power of primary aberrations is always 4, including angles. Primary Seidel aberrations.
(Extended theory: Born and Wolf, Goodman, see also R.W. Ditchburn for instrument applications)

EFFECT OF ABERRATIONS ON OTF

Generalized Pupil Function including W

$$P(x,y) = \exp(ikW) \quad \text{inside pupil}$$
$$= 0 \quad \text{outside}$$

W, phase factor

W does not affect total intensity, but adds phase factor to the different (spatial) frequencies

Blurring of the image.

Example in terms of rays (recall source point): the normal to the wavefront, (ray) changes direction and the rays no longer have a common point.

In terms of plane wave development of the field, each wave has a change in phase, and they are no longer focussed at the same point.

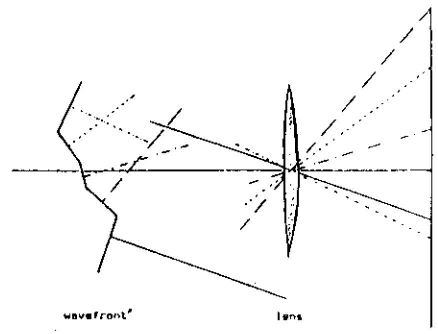
In general lowering of maxima, disappearing of zeros, increase of minima

ABERRATIONS DUE TO MEDIUM (Turbulence)

Aberrations due to medium before the system: typically due to propagation in turbulent medium:

First approximation (small fluctuation)

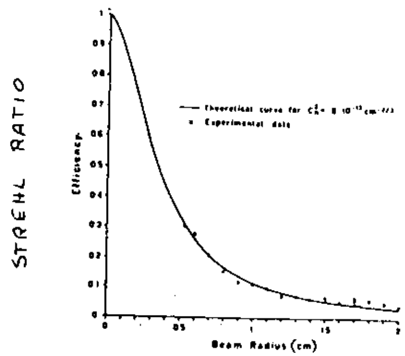
-phase effect



Spots produced in the image plane explained in terms of plane elements of the wavefront.

In general

strong phase and amplitude effects



example: Strehl ratio, atmospheric measurement example: diffraction pattern by 3 apertures of different increasing radius

R = .5 cm R = 1 cm R = 2 cm

Correction requires systems based on adaptive optics.
In some cases use of Zernike polynomials

3 - 6 IMAGING - INCOHERENT CASE

The most common light found in nature, emitted by bodies much larger than the wavelength, is incoherent radiation. The emitting atoms of a body, emit randomly, in time and space, wave trains which are completely uncorrelated, unless the atoms are very near each other, with respect to wavelength. Only the laser emits coherent radiation. At a point outside an emitting source (not a laser) the field is constituted by many wave trains with random phase, which interfere with each other but continuously and rapidly change. One cannot think of a “wave”, as in the case coherence, but rather of energy. For incoherent radiation one has to deal with the modulus square of the field.

Although the general case is partial coherence, both in time and space, we will consider here only the limiting case of incoherent quasi-monochromatic light, as the case corresponding to the coherent monochromatic one already considered.

Quasi-monochromatic light has a bandwidth $\Delta\nu$ which is very small with respect to the central frequency ν, that is $\Delta\nu/\nu \ll 1$.

Quantity of interest here is the average value of the intensity in a long time with respect to the period of oscillation (infinite time). In practice the response time of the eye or of typical instruments. In this case approximating time and space incoherent radiation with monochromatic (time coherent) radiation ν, is a good approximation. Frequency =central frequency of incohrerent radiation, Therefore the radiation is only spatially incoherent.

The instantaneous intensity $I_{inst}(P)$ at a point P is the field square (see diffraction). The space coherence of a field is described by the field correlation function $B_u(P,P')$ defined as

$$B_u(P,P') = \langle u(P)\, u^*(P') \rangle \qquad (8)$$

Asterisk as usual denotes complex conjugate and brackets infinite time average. The average intensity I(P) is given by (assume homogeneity)

$$I(P) = \lim_{P' \to P} \langle u(P)\, u^*(P') \rangle$$

For spatially incoherent radiation:

$$B_u(P,P') = I(P)\, \delta(P-P') \qquad (9)$$

The intensity in the image of incoherent radiation is

$$I(x,y) = \langle u_i(x,y) u_i^*(x,y) \rangle = \int\int_{-\infty}^{\infty}\int\int_{-\infty}^{\infty} \langle u_{ob}(x_o,y_o) u_{ob}(x'_o,y'_o) \rangle$$

$$h(x-x_o,y-y_o) h^*(x-x'_o,y'-y_o)\, dx_o dy_o dx'_o d'y_o$$

where average and integral operations have been interchanged and the fact that the impulse response does not depend on time has been taken into account. This relationship holds for partially coherent light and could be further developed.

In the case of complete incoherence, introduction of Eq. 9 gives[2] the important final result:

$$I(x,y) = \int\int_{-\infty}^{\infty} I(x_o,y_o)\, |h(x-x_o,y-y_o)|^2 dx_o dy_o \qquad (10)$$

Conclusions for INCOHERENT CASE:

- Intensity

- Convolution relationship between source intensity and (incoherent) point spread function

- The incoherent point spread function is the modulus square of the coherent spread function.

Example. The incoherent spread function of a source on the axis of a thin lens free from aberrations (already considered for the coherent case): is the Airy function, (Airy function is defined as the modulus square of the Fraunhofer diffraction pattern, see lessons on diffraction) centered on the geometrical image point.

[2] Recall that $\int_{-\infty}^{\infty} f(x)\delta(x-a)\,dx = f(a)$

Let $\mathcal{I}_o$ and $\mathcal{I}_i$ and $\mathcal{H}$ Fourier Transforms of the intensities of object, image and spread function respectively. By convolution theorem:

11) $$\mathcal{I}_i = \mathcal{H}\,\mathcal{I}_o$$

$\mathcal{H}$ is called Incoherent Optical Transfer Function; its modulus:

MODULATION TRANSFER FUNCTION, MTF.

Generally normalization to 1 at zero frequencies, where there is the maximum (see e.g. Goodman).

General relationship between incoherent, $\mathcal{H}$ (normalized), and coherent, H, transfer functions:

12) $$\mathcal{H}(f_x,f_y) = \frac{\int_{-\infty}^{\infty}\int H(u,v)H^*(u+f_x,v+f_y)\,du\,dv}{\int_{-\infty}^{\infty}\int |H(u,v)|^2\,du\,dv}$$

valid for systems both with and without aberrations.

For coherent systems one has (see diffraction)

$$H(u,v) = P(\lambda i u, \lambda i v)$$

λ wavelength, i image distance from the lens.

For incoherent system, introduction of H(u,v) into Eq. 12 shows that $\mathcal{H}$ (normalized) is the spatial autocorrelation function of the pupil function:

$$\mathcal{H}(f_x,f_y) = \frac{\int_{-\infty}^{\infty}\int P(\lambda i u,\lambda i v)P(\lambda i u+f_x,\lambda i v+f_y)\,du\,dv}{\int_{-\infty}^{\infty}\int P(u,v)\,du\,dv}$$

Recall P(x,y) real function of modulus one. Denominator=pupil area. Numerator the common area of pupil and displaced pupil.

FOR INCOHERENT SYSTEMS THE SPECTRUM IS DIFFERENT THAN FOR COHERENT

In particular it has a larger widht (due to convolution of the pupil function)

Consequence the same system gives different images with coherent or incoherent radiation.

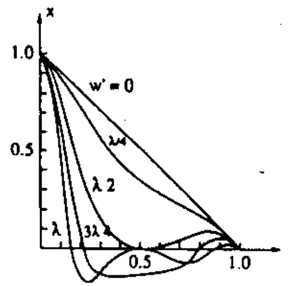
Advantages and disadvantages depend also on the object.

Effect of aberrations on incoherent systems

ALWAYS DECREASE MTF

In general lower the contrast of each spatial frequency component, leaving the cut off unchanged. However the higher frequencies can be severely reduced, so that, in practice cut off can be much lower than in the diffraction limited case.

Aberrations can also give rise to negative values of OTF in some ranges of frequencies. Consequence: constrast reversal in image, that is intensity maxima can become zeros and viceversa.
Typical example of this case is defocusing error



Defocus OTF for a square pupil.

4 - RESOLVING POWER

1- Rayleigh criterion (v diffraction)

2- OTF or MTF half width

3 Degrees of freedom of images

Superresolution