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CONFERENCE ON

TOPOLOGICAL AND GEOMETRICAL PROBLEMS

RELATED TO QUANTUM FIELD THEORY

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Monopoles and 4-Manifolds
(after Seiberg-Witten, Kronheimer, Mrowka ...)

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These are preliminary lecture notes, intended only for distribution to participants

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Monopoles & 4 Manifolds.

(after Seiberg-Witten, Kronheimer-Mrowka ....)

Peter J. Braam

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1 Euler classes

1.1 Classical definition

X oriented closed k-dimensional smooth manifold

$$\begin{array}{ccc} \mathbb{R}^n & \to & E \\ & & \downarrow \pi \\ & & X \end{array}$$ rank n, oriented vector bundle.

Let $s \in \Gamma(X, E)$ be a section transverse to the 0-section. Then

$$s^{-1}(0) \equiv Z < X$$

is oriented closed submfd of X. (mfd = manifold)

<u>Definition:</u> The Euler class of E is the class

$$e(E) = [Z] \in H_{k-n}(X; \mathbb{Z}).$$

<u>Lemma:</u> This class is independent of s.

Proof: ~~...~~ Let $s_0, s_1 \in \Gamma(X, E)$ be two generic sections. Connect them in a path of sections

$$\mathfrak{s}_t \in \Gamma(X \times [0,1], p^*E) \quad . \quad p: X \times [0,1] \to X$$

s.t. $\mathfrak{s}_0 = s_0$, $\mathfrak{s}_1 = s_1$

For generic $\mathfrak{s}$, $\mathfrak{s}^{-1}(0)$ is a cobordism between $s_0^{-1}(0)$ & $s_1^{-1}(0)$. This cobordism makes $s_0^{-1}(0)$ & $s_1^{-1}(0)$ homologous in X. ⊠

1.2 Polynomial invariants

Let $\pi : E \to X$, s, Z as in 1.1

<u>Definition:</u> Let $\omega_i \in H^{d_i}(X;\mathbb{Z})$, $i = 1 \ldots m$. The polynomial invariant of Z is defined by

$$\Phi_Z(\omega_1, \ldots, \omega_m) = \begin{cases} \displaystyle\int_Z \omega_1|_Z \wedge \cdots \wedge \omega_m|_Z & \text{if } \Sigma_1^n d_i = k \\ 0 & \text{if } \Sigma_1^n d_i \neq k \end{cases}$$

Notice that Φ_Z is multilinear, symmetric in classes ω_i of even degree, antisymmetric for those of odd degree.

Also observe that in order to evaluate Φ_Z we need to solve the equation $s(x) = 0$, which defines Z.

<u>Definition:</u> The cohomological Euler class of $E \to X$ is the poincaré dual $\gamma(E) \in H^n(X;\mathbb{Z})$ of $e(E)$.

<u>Corollary</u> :

$$\Phi_Z(\omega_1, \ldots, \omega_m) = \int_X \gamma(E) \wedge \omega_1 \wedge \cdots \wedge \omega_m$$

1.3 Physics & Gauß-Bonnet theorem.

We need the concept of a <u>connection</u> ∇ on $E \to X$. This is defined as a map:

$$\nabla : \Gamma(X, E) \longrightarrow \Gamma(X, \Lambda^1 \otimes E)$$

s.t. 1) ∇ is $\mathbb{R}$-linear

2) For $s \in \Gamma(X, E)$, $f \in C^\infty(X)$

$$\nabla(fs) = df \cdot s + f \, \nabla s$$

One can also require compatibility with a metric $\langle \, , \rangle$ on E:

3) $$d \langle s_1, s_2 \rangle = \langle \nabla s_1, s_2 \rangle + \langle s_1, \nabla s_2 \rangle$$

for $s_1, s_2 \in \Gamma(X, E)$.

In local coordinates & trivialisation of E,

$$\nabla s = ds + \Sigma_i dx_i \; \alpha_i(x) \cdot s$$

so $\nabla = d + \alpha$, $\alpha = \alpha_i dx_i$,

with

$$\alpha_i \in \Gamma(U, so(E))$$

$U \subset X$ coordinate patch.

One can extend ∇ to a map:

$$\nabla : \Gamma(X, \Lambda^j \otimes E) \longrightarrow \Gamma(X, \Lambda^{j+1} \otimes E)$$

Finally ∇^2 is <u>not</u> a 2nd-order operator but

$$\nabla^2 \in \Gamma(X, so(E) \otimes \Lambda^2)$$

a two-form, with values in endomorphisms.

Physicists approach Euler class as follows:

Define a Lagrangean:

$$L_t(x, s, \alpha) = -t^2|s(x)|^2 + t\langle \alpha, \nabla s\rangle + \langle \alpha, F\alpha\rangle$$

α is constant 1-form on $\mathbb{R}^k$ with values in E (define this formally yourself, or see Matthai-Quillen, Atiyah-Jeffrey or Witten).

Now define

$$Q_t = \int_{\mathbb{R}^k} e^{L(x,s,\alpha)} \in \Omega^*(X). \quad \dagger$$

Asymptotics: $k=n$, Z set of points.

$t \to \infty$: only x s.t. $s(x)$ can contribute. Local analysis shows contribution equals $\displaystyle\int_X Q_t = \sum_{\substack{x\\ s(x)=0}} \frac{\det \nabla s_x}{|\det \nabla s_x|}$.

$t \to 0$: Expression is independent of s

$$\int_X Q_t = \int_X Pf(\nabla^2). = \text{expression in curvature}$$

Gauss Bonnet Theorem:

$$\int_X Pf(\nabla^2) = \sum_{\substack{x\\ s(x)=0}} (\pm 1)$$

$\dagger$ $\displaystyle\int_{\mathbb{R}^k} \Theta = \Theta_0\left(\frac{\partial}{\partial x_1} \wedge \dots \wedge \frac{\partial}{\partial x_k}\right)$ is the Berezin integral.

2 Donaldson Theory.

2.1 Self dual connections

If M is 4-dimensional & oriented Riemannian manifold then $*: \Omega^i(M) \to \Omega^{4-i}(M)$.
In particular we have
$$\Omega^2(M) = \Omega^2_+(M) \oplus \Omega^2_-(M),$$
a splitting into ± 1 eigenspaces.

Definition: A connection ∇ on $E \to M$ is anti-self dual iff
$$(\nabla^2)_+ = 0.$$
(here $*: \Omega^2(X, \operatorname{End} E) \circlearrowleft$ is obtained by tensoring with $\mathrm{Id}_{\operatorname{End} E}$

The automorphisms $\mathcal{G}(E)$ act on curvature by conjugation: anti self duality is unaffected.

Definition: The moduli space $\mathcal{M}(E)$ of asd connections on E
$$\left\{ \begin{array}{l} \nabla \text{ connection on } E \\ \nabla^2 \text{ is asd} \end{array} \right\} \Big/ \mathcal{G}(E).$$

This is usually studied for Hermitian $\mathbb{C}^2$ bundles, with trivialised determinant (or $SU(2)$-bundles).

Let's put this in perspective of §1

$$X = \{\text{all connections}\} / \mathcal{G}(E).$$

$$\begin{array}{l} \mathbb{E} \\ s \curvearrowleft \downarrow \\ X \ni \nabla \end{array} \qquad E_\nabla = \Omega^2_+(X, \operatorname{End} E).$$

$s(\nabla) = (\nabla^2)_+$.

Now $\mathcal{M}(\mathbb{E}) = \{ \nabla \in X ; \ s(\nabla) = 0 \}$. We can try to proceed as in 1.2 but

1) dim X & rk $\mathbb{E}$ are infinite
2) X is not smooth, not compact,
3) $\mathcal{M}(E)$ is not (always) smooth & compact.

2.2. Donaldson Polynomials.

~~[illegible]~~ & $su(2)$ bundles.

Consider only $\pi_1(M) = \{e\}$, i.e. simply connected M.

Theorems guard against disasters.

a) If $b_2^+ > 1$ no singularities in 1-parm families of moduli spaces

b) If $c_2(E) \gg 0$ $\mathcal{M}(E)$ is compact.

Polynomial invariants can be defined as in 1.2, or more specifically through:

$$\mu: H_j(M) \to H^{4-j}(\mathcal{M}(E); \mathbb{Z})$$
$$\Sigma \to p_1(E_u; \mathbb{Z}) / \Sigma$$

$E_u \to \mathcal{A} \times M / \mathcal{G}(E)$. $\quad \mathcal{A}$ = all irr. connection

So can evaluate $\mu(\Sigma)$'s on $\mathcal{M}(E) \Rightarrow$ polynomials on $H^*(M)$.

$$\Phi_M(\Sigma_1, \ldots \Sigma_k, p, \ldots, p_\ell) = \int \mu(\Sigma_1) \cup \ldots \cup \mu(\Sigma_k) \cup \mu(p_1) \ldots \mu(p_\ell)$$

$\Sigma_i \in H_2(M; \mathbb{Z})$

$p_i \in H_0(M; \mathbb{Z})$

Great variety since

$$\dim \mathcal{M}(E) = 8\, c_2(E) - 3(1 + b_2^+(M) - b^1(M))$$

3. Monopoles.

3.1 Description of the equations.

Recall that Spin is double cover of SO.
Spin - structure is a lift of structure group.
Spin (4) has 2 2-dim'l $\mathbb{C}$ - representations S^+, S^-,
these give spinor bundles $S^+, S^- \to M^4$.

Clifford algebra gives two important maps

(i) $\Lambda^1 \otimes S^\pm \to S^\mp$

(ii) $\Lambda^2_+ \otimes \mathbb{C} \xrightarrow{\sim} sl_0(S^+)$.

Fiddling a little there is also a map

(iii) $S^+ \to \Lambda_+ : \Phi \to (\bar{\Phi}\,\bar{\Phi}^t)_0$

Spin - structures exist if $w_2(TM) = 0$.
A spin^c - structure is locally like a
complex line bundle $\mathcal{L}^{1/2} \otimes S^\pm = W^\pm$

A connection on $W^\pm$ can be defined by
Levi - Civita connection <u>plus</u> a connection on $\mathcal{L}$.
The maps (i), (ii), (iii) remain in existence.

The Dirac operator

$$\not{D} : \Gamma(M, W^+) \longrightarrow \Gamma(M, W^-)$$

is composition of (i) with ∇.

Monopole equations are equations for pair

$$(a, \Phi) \in \mathcal{A}(\mathcal{L}) \times \Gamma(M, W^+).$$

$$\text{(i)} \quad D_a \Phi = 0$$

$$\text{(ii)} \quad F_a^+ = i\, \Phi \bar{\Phi}^t$$

These equations are equivariant and invariant under $\mathcal{G}(\mathcal{L}) = C^\infty(M, S^1)$.

3.2 Basic Facts about Monopole Equations.

The action of $\mathcal{G}(\mathcal{L})$ is <u>free</u> unless $\Phi \equiv 0$. Thus reducible solutions (stabilizer S^1) consist of self dual connections a on $\mathcal{L}$. We know from Donaldson theory that these are not generically present if $b^+(X) > 0$, and disappear in generic 1-param families when $b^+(X) > 1$.

For a self dual 2-form $\varepsilon \in \Gamma(M, \Lambda^+)$ can perturb the equations with:

$$\left\{ \begin{array}{l} D_a \Phi = 0 \\ (F_a)_+ = i \bar{\Phi} \bar{\Phi} + \varepsilon \end{array} \right. \qquad (*)$$

Easy computation shows that $(*)$ is submersive map

$$\Gamma(M, W^+) \times \mathcal{A}(L) \times \Gamma(M, W^+) \to \Gamma(M, W^-) \times \Gamma(M,$$

hence for generic ε the irreducible moduli space is <u>smooth</u>.

Define moduli space of monopoles as

$$\mathcal{M}(\mathcal{L}) = \{\text{solns } *\} / \mathcal{G}(\mathcal{L}).$$

and let $\mathcal{M}^*(\mathcal{L})$ be the irreducible part.
Index calculation shows if $\mathcal{M}^*(\mathcal{L}) \neq \emptyset$ it is ~~4~~

$$\dim \mathcal{M}^*(\mathcal{L}) = \tfrac{1}{4}\left(c_1(L)^2 - (2\chi + 3\sigma)\right).$$

Here $\chi = 2b^0 + b^2 - 2b^1$ & $\sigma = b^+ - b^-$ are Euler characteristic & signature of M.

The most important facts about equations are derived from the Weizenbock formula:

$$D_a^* D_a \Phi = \nabla_a^* \nabla_a \Phi + \frac{s}{4} \Phi - \frac{1}{2} (F_a)_+ \Phi .$$

This formula is widely known and easily established in local coordinates.

<u>Lemma</u>: Any solution (a, Φ) satisfies the C^0 bound $|\Phi|^2 \leqslant \max(0, -s)$ at points when $|\Phi|$ is maximal. Here s is scalar curvature on M.

Proof: When $|\Phi|$ is maximum:

$$\begin{aligned} 0 &\leqslant \nabla^* \nabla |\Phi|^2 \\ &= 2 \langle \nabla_a^* \nabla_a \Phi, \Phi \rangle - 2 \langle \nabla_a \Phi, \nabla_a \Phi \rangle \\ &\leqq -\frac{s}{2} |\Phi|^2 + \langle i(\Phi \bar{\Phi}) \Phi, \Phi \rangle \\ &= -\frac{s}{2} |\Phi|^2 - \frac{1}{2} |\Phi|^4 \end{aligned}$$

If $|\Phi|^2 \neq 0$ at maximum $\Rightarrow$ $|\Phi|^2 \leqslant -s$.

<u>Lemma</u>: $\mathcal{M}(\mathcal{L})$ is compact in the C^∞ topology.

<u>Lemma</u>: Given $d \in \mathbb{Z}_{\geq 0}$. There are only finitely many isom classes $\mathcal{L} \to M$ s.t. $\dim \mathcal{M}(\mathcal{L}) = d$.

<u>Proof</u>: $\frac{1}{4} c_1(\mathcal{L})^2 = d + \frac{1}{4}(2\chi + 3\sigma)$

But $c_1(\mathcal{L})^2 = \frac{1}{4\pi^2} \int F_a^{+\,2} - F_a^{-\,2}$. Now

$F_a^+ = i\Phi\bar{\Phi}$ can only move in compact set, therefore F_a^- can too. A compact set can only contain finitely many integral $c_1(\mathcal{L})$.

An orientation of $\mathcal{M}(\mathcal{L})$ can be obtained from a choice of orientation of

$$\left(\Lambda^{\max} H^1(M;\mathbb{R})\right) \otimes \left(\Lambda^{\max}\left(H^0(M;\mathbb{R}) \oplus H^2_+(M;\mathbb{R})\right)\right)^*$$

3.3 Invariants from Monopoles.

Let $$d = \frac{1}{4}\left(c_1(L)^2 - (2\chi + 3\sigma)\right)$$
If $b^+ > 1$, $d = 0$ then

$$n_L = \#_{\text{sign}} \mathcal{M}(L)$$

is an invariant of M. The proof has been outlined in previous sections.

We expect that $\mathcal{M}(L)$ of dimension 0 will be the most important ones. However positive dimensional $\mathcal{M}(L)$ can exist (exercise: prove this!) A μ-map can be constructed

$$\mu : H_i(M) \to H^{2-i}(\mathcal{M}(L))$$

This has prove useful to reprove Donaldson's first results (Taubes).

Notice that (unlike Donaldson theory) the geometry of M exercises strong control over the existences of solutions:

<u>Proposition</u>: On manifolds with scalar curvature s satisfying

$$\frac{1}{32\pi^2} s^2 \geq (2\chi + 3\sigma)$$

only reducible solutions exist. In case of equality a solution on a Kähler-Einstein is unique (up to $\pm$) and given by the metric.

4. Seiberg - Witten Conjecture

Recall μ - map

$$\mu : H_i(M) \to H^{4-i}(\mathcal{M}(E)).$$

Organise all donaldson polynomials in a generating function: Let $u \in H^4(\mathcal{M}(E); \mathbb{Z})$ equal $\mu(1)$ and defin

$$F(\Sigma, \lambda) = \sum_{k=c_2(E)} \langle e^{\Sigma + \lambda . u}, \mathcal{M}_k \rangle$$

$\Sigma \in H_2(M)$.

<u>Conjecture:</u> (see Witten: Monopoles & 4-manifolds).

$$F(\Sigma,\lambda) = \underline{cst} \left[e^{\left(\frac{\Sigma.\Sigma}{2} + 2\lambda\right)} \sum_x n_x e^{x.\Sigma} + i^{\Delta} e^{\left(-\frac{\Sigma.\Sigma}{2} - 2\lambda\right)} \sum_x n_x e^{-i x . \Sigma} \right]$$

$x = c_1(L)$ s.t. $d = 0$. <u>basic</u> <u>classes</u>

$\Delta = (\chi + \sigma)/4$.

<u>Notes:</u> 1) There is a physics argument using path integrals. to prove this.

2) Th general form of the equality was known to Kronheimer & Mrowka

3) There is a generalication to $SO(3)$ - bundles.

4) For Kähler manifolds the rhs is computable.

5 Kähler Manifolds.

5.1 Splitting Equations

M Kähler $\rightarrow$ Further splitting of differential forms. Let σ be Kähler form then

$$\mathbb{C} \otimes_{\mathbb{R}} \Lambda^2_+ = \Lambda^{2,0} + \Lambda^{0,2} + \mathbb{R}.\sigma$$

Also $K = \Lambda^2 T^*_{\mathbb{C}} M$ we have an instance of $\mathrm{Spin}_{\mathbb{C}}$ structure with

$$W^+ = (K^{1/2} \oplus K^{-1/2}) \otimes \mathcal{L}^{1/2}$$

Write $\Phi \in \Gamma(M, W^{1/2})$ as

$$\Phi = \alpha + \bar{\beta} \qquad \alpha \in \Gamma(K^{1/2} \otimes \mathcal{L}^{1/2}), \ \beta \in \Gamma(K^{+1/2} \otimes \mathcal{L}$$

The second equation

$$F_a^+ = i \bar{\Phi} \Phi$$

becomes:

(i) $F_a^{2,0} = \alpha \beta$

(ii) $(F_a) \lrcorner \sigma = |\alpha|^2 - |\beta|^2$

and the Dirac equation reads:

(iii) $\bar{\partial}_a \alpha + \partial_a^* \bar{\beta} = 0$

Analysis of these equations reveals:

(i) using Weitzen-bock formula we obtain
$(F^a)^{2,0} = 0$ & ($\alpha = 0$ <u>OR</u> $\beta = 0$).

(ii) has unique solution

(iii) $\bar{\partial}_a \alpha = 0$ <u>OR</u> $\bar{\partial}_a \beta = 0$.

The first point states that $\mathcal{L}^{1/2}$ has a holomorphic structure with $\bar{\partial}_{\mathcal{L}} = (\nabla_a)^{0,1}$
This equation has symmetry group $\mathcal{G}(\mathcal{L})^{\mathbb{C}} = C^\infty(M, \mathbb{C}^*)$

In fact the 2nd equation is a moment-map equation and in every stable $\mathcal{G}(L)^{\mathbb{C}}$ orbit of solutions to (i) & (iii) there is a unique $\mathcal{G}(L)$ orbit of solutions to (ii).

The alternative $\alpha = 0$ or $\beta = 0$ is decided by

$\beta = 0$ if $c_1(L^{1/2}) \cdot \sigma > 0$

$\alpha = 0$ if $c_1(L^{1/2}) \cdot \sigma < 0$.

If M simply connected then holomorphic structures on line bundles are unique and

$$\mathcal{M}(L) = \mathbb{P}\left(H^0_{hol}\left(K^{1/2} \otimes L \right) \right)$$

when $c_1(L) \cdot \sigma > 0$ (similarly in other case).

5.2 More on Kähler manifolds.

We see that in order to have non empty moduli we need $H^0_{hol}(K^{1/2} \otimes \mathcal{L}^{\pm 1/2}) \neq 0$, and $\mathcal{L}^{\pm 1/2}$ holomorphic.

Using Hodge index theorem one can deduce that this implies $\dim \mathcal{M}(\mathcal{L}) = 0$.

A further observation is that genericity of the moduli space can only be obtained after perturbation. Tracing the steps one finds that the generic solutions can be found as pairs

$$(\alpha, \beta) \in H^0_{hol}(K^{1/2} \otimes \mathcal{L}^{1/2}) \oplus H^0_{hol}(K^{+1/2} \otimes \mathcal{L}^{-1/2})$$

such that $\alpha \cdot \beta = \varepsilon$ for a small holomorphic 2-form ε. The sign is determined by dimensions of some cohomology groups.

Finally the condition that $\mathcal{L}^{1/2}$ is holomorphic is very restricting. It means that $c_1(\mathcal{L})$ must be integral and of type $(1,1)$ under the Hodge decomposition. On surfaces of general type this amounts to.

$$c_1(\mathcal{L}) = \pm K.$$

so that basic classes are very constrained.

6. Thom's Conjecture

(Kronheimer - Mrowka, Morgan - Szabo - Taubes).

6.1. Statement & Method.

Let $\Sigma \subset \mathbb{CP}^2$ be embedded C^∞-surface.

$g =$ genus (Σ)

$d = \deg(\Sigma) = \langle \omega, \Sigma \rangle$

when ω is positive generator in $H^2(\mathbb{CP}^2)$

Thom's conjecture: $(d-1)(d-2) \leq 2g$

$d \leq 4$, $d=6$: known linear index theory.

Conjecture defied 1000's pages of gauge theory!

Ideas:

(i) $\Sigma . \Sigma = d^2$ self intersection. Blow up $\mathbb{CP}^2$ at d^2

$$X = \mathbb{CP}^2 \oplus d^2\, \overline{\mathbb{CP}^2}$$

$H_2(X) = \mathbb{Z}^{d^2+1}$,

intersection form $\begin{pmatrix} 1 & & & \\ & -1 & & \\ & & \ddots & \\ & & & -1 \end{pmatrix}$

Generators: H, $H \cdot H = 1$, E_i except div $E_i^2 = -1$.

(ii) $\Sigma \to \tilde{\Sigma} \subset X$

$\tilde{\Sigma} \cdot \tilde{\Sigma} = 0$

$\tilde{\Sigma} = dH - E \qquad E = \sum_1^{d^2} E_i$

Give X metric long neck $\tilde{\Sigma} \times S^1 \times [-10^6, 10^6]$

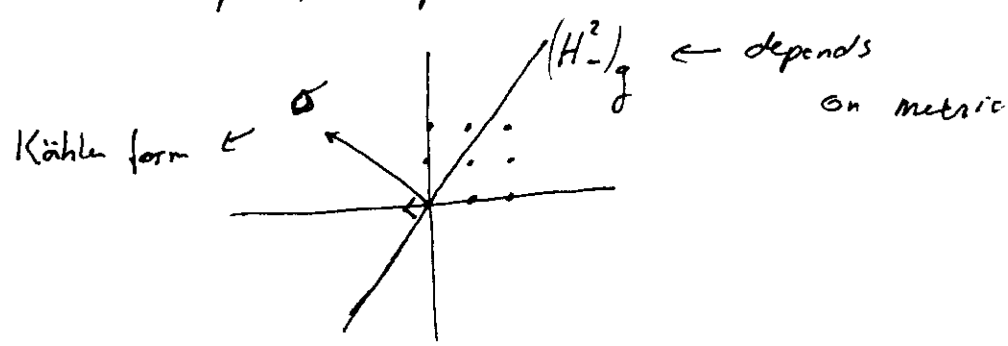
(iii) Thrm 1: $\exists$ monopoles X for $x = c_1(\mathcal{L}) = 3H - E$

Thrm 2: Nec cdn for monopoles is $|x \cdot \tilde{\Sigma}| \leq 2g-2$

$|x \cdot \tilde{\Sigma}| = d^3 - 3d \qquad (d \geq 4) \Rightarrow$ Thom C.

6.2 Existence

Notice $b^+=1$ reducibles exist when $c_1(\mathfrak{L}) \cap H^2(X;\mathbb{Z}) \cap H^2_-(X). \neq 0$.
This happens at isolated points in 1-prm family of metrics.



<u>Thrm 1</u>:

a) Reducibles exist when $\sigma \cdot (3H-E) = 0$

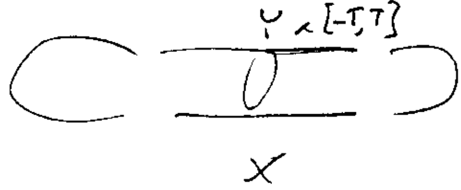
b) When $t \to \sigma_t \cdot (3H-E)$ changes sign ~~transversely~~ n_X changes by 1 (mod 2)

c) For connected sum metric $n_X = 0$ $(s \gg 0)$.

d) For long metric $n_X = 1$ (2) since ($\sigma_t \cdot c_1(\mathfrak{L})$ changes sign)

6.3 Constraints.

On neck $Y = \tilde{\Sigma} \times S^1$



As $T \to \infty$ limiting soln on $Y \times \mathbb{R}$.
Can show non trivial solution.

$$s_{Y \times \mathbb{R}} = \frac{1}{\text{Area}(\Sigma)} (1 - g).$$

Use estimate and equation:

$$\frac{1}{2} c_1(\mathcal{L}) \cdot [\Sigma] = \int_\Sigma |\Phi|^2 \leq |(1-g)|.$$

7. Food for thought.

7.1 Aspects of new theory.

a) new tool to study 4-manifolds.
b). extraordinary relation with "old" Donaldson invariants
c) new application of geometric techniques from QFT.
d) another instance non linear PDE with topological invariants.

Questions:

1). Is there an index theory for non-linear elliptic equations with symmetry groups which specialize to Donaldson & Seiberg Witten invariants?
(Recall Atiyah-Singer index theory slowly emerged from special cases).
- Palais-Smale condition for monopole equations?
- Other examples of "good" PDE
- Why is $e(\mathbb{E})$ trivial in classical sense

$$\mathbb{E} = \Gamma(W^+) \oplus \Omega_+(M) \Big/ \mathcal{G}(L)$$
$$\downarrow$$
$$\Gamma(W^+) \times A(L)$$

but $\Omega_+(M)$ has trivial $\mathcal{G}(L)$ action so
$e(\mathbb{E}) = 0$!
Same in Donaldson theory!

2) What is the algebraic topological content of TQFT's ?

a) Donaldson theory: understood Thom & Euler class
WZW model: projection on holomorphic functions

b) What is strong & weak coupling ?

c) How does Intriligator conjecture work ?

d) How does proof of conjecture work !

Expect: (i) need unearth known concepts such as Thom classes in Lagrangeans.

(ii) Give them a radically different interpretation.

It appears very significant that relation between n_x & Donaldson invariants is <u>linear</u>.

3) Find new equations which give 4-mfd invariants. Are they expressible again in terms of Donaldson or monopole invariants ?