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# A conjectured analogue of Dedekind's eta function for $K3$ surfaces

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# A conjectured analogue of Dedekind's eta function for $K3$ surfaces


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**Abstract:** A fundamental formula in the study of elliptic functions is the product formula for Dedekind's eta function or, equivalently, for the holomorphic cusp form on the upper half plane **h** which is of weight 12 with respect to the action by $PSL(2, \mathbf{Z})$. A related formula expresses the determinant of the Laplacian which acts on the space of smooth functions on an elliptic curve with a period of the elliptic curve and the Dedekind eta function. In [JT 94a], we constructed a holomorphic function on the moduli space of marked, polarized, algebraic $K3$ surfaces of fixed degree using determinants of Laplacians. The aim of this article is to state a conjecture which expresses product formula for this holomorphic form. In addition, we will present speculative relations with the representation theory of the Mathieu group $M_{24}$ as well as stating many other problems currently under investigation.


## §1. Determinants and the eta function for elliptic curves

Given $\tau = a + ib \in \mathbf{C}$ with $b > 0$, let $\Lambda_\tau$ be the two dimensional lattice in $\mathbf{C}$

$$\Lambda_\tau = \{n + m\tau \mid n, m \in \mathbf{Z}\},$$

and let $E_\tau$ be the elliptic curve $E_\tau = \mathbf{C}/\Lambda_\tau$. View $E_\tau$ as a Riemannian manifold with flat metric of area one. If $\{\lambda_n\}$ denotes the sequence of positive eigenvalues of the Laplacian which acts on smooth functions on $E_\tau$, then the associated spectral zeta function is defined for $\mathrm{Re}(s) > 1$ by

$$\zeta_\tau(s) = \sum \lambda_n^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty \sum e^{-\lambda_n t} t^s \frac{dt}{t}.$$

If the metric is represented by $\mu(z) = \frac{i}{2b} dz \wedge d\bar{z}$, then the associated Laplacian is

$$\Delta = -4b \frac{\partial^2}{\partial z \partial \bar{z}}.$$

The eigenvalues are explicitly computable, and the spectral zeta function is

$$\zeta_\tau(s) = (2\pi)^{-2s} \sum_{(n,m)\neq(0,0)} \frac{b^s}{|n\tau + m|^{2s}}.$$

The function $\zeta_\tau(s)$ is the classical non-holomorphic Eisenstein series on the hyperbolic upper half plane **h** with respect to the group $PSL(2, \mathbf{Z})$, and the study of its analytic continuation and special values are well known; see, for example, [L 87] or [W 76]. Let $q_\tau = \exp(2\pi i\tau)$, and let $\eta(\tau)$ denote the Dedekind eta function

$$\eta(\tau) = q_\tau^{1/24} \prod_{n=1}^{\infty} (1 - q_\tau^n).$$

A direct application of Kronecker's first limit formula yields the equation

$$\exp(-\zeta_\tau'(0)) = b|\eta(\tau)|^4.$$


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Following standard conventions, the special value $\exp(-\zeta_\tau'(0))$ of the spectral zeta function is known as the determinant of the Laplacian, and we shall write

$$\exp(-\zeta_\tau'(0)) = \det {}^*\Delta.$$

The asterisk reflects the fact that the zero eigenvalue has been omitted in the definition of the spectral zeta function. Let $\langle dz, dz\rangle$ denote the $L^2$ norm squared of the holomorphic one form $dz$ on $E_\tau$. With this, we can write the above formula as

$$\frac{\det^* \Delta}{\langle dz, dz\rangle} = |\eta(\tau)|^4.$$

This expression lead us to seek and obtain in [JT 94a] and [JT 94b] similar formulae in the setting of polarized, algebraic $K3$ surfaces, which we now shall briefly recall.

## §2. Basic properties of $K3$ surfaces.

Let us review some basic properties of $K3$ surfaces. For a more general and complete discussion, the reader is referred to [Ast 85].

A $K3$ surface $X$ is a compact, complex two dimensional manifold with the following properties.

a. There exists a non-vanishing holomorphic two form $\omega$.
b. $H^1(X, \mathcal{O}_X) = 0$.

For the purposes of this article, we will assume that all surfaces are projective varieties. From the defining properties, one can prove that the canonical bundle on $X$ is trivial. In [Sh 67], the following properties of $K3$ surfaces are proved. The surface $X$ is simply connected, and the homology group $H_2(X, \mathbf{Z})$ is a torsion free abelian group of rank 22. The intersection form $\langle\ ,\rangle$ on $H_2(X, \mathbf{Z})$ has the properties:

a) $\langle u, u\rangle = 0 \mod (2)$;
b) $\det(\langle e_i, e_j\rangle) = -1$, where $\{e_i\}$ is a basis of $H_2(X, \mathbf{Z})$;
c) the symmetric form $\langle\ ,\ \rangle$ has signature (3,19).

Theorem 5 from page 54 of [Ser 76] implies that as an Euclidean lattice $H_2(X, \mathbf{Z})$ is isomorphic to the $K3$ lattice $\Lambda$, where

$$H_2(X, \mathbf{Z}) \cong \Lambda = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^3 \oplus (-E_8)^2.$$

Let $X$ be a $K3$ surface, and let $\alpha = \{\alpha_i\}$ be a basis of $H_2(X, \mathbf{Z})$ with intersection matrix $\Lambda$. The pair $(X, \alpha)$ is called a marked $K3$ surface. Let $e \in H^{1,1}(X, \mathbf{R})$ be the class of a hyperplane section. The triple $(X, \alpha, e)$ is called a marked, polarized $K3$ surface.

The period map $\pi$ for a marked $K3$ surface $(X, \alpha)$ is defined by integrating the holomorphic two form $\omega$ along the basis $\alpha$ of $H_2(X, \mathbf{Z})$, meaning

$$\pi(X, \alpha) = (\ldots, \int_{\alpha_i} \omega, \ldots).$$

The Riemann bilinear relations hold for $\pi(X, \alpha)$, that is

$$\langle \pi(X, \alpha), \pi(X, \alpha)\rangle = 0 \quad \text{and} \quad \langle \pi(X, \alpha), \overline{\pi(X, \alpha)}\rangle > 0.$$

The subvariety in $\mathbf{CP}(H^2(X, \mathbf{Z}) \otimes \mathbf{C})$ described by the Riemann bilinear relations is isomorphic to $SO_0(3, 19)/SO(2) \times SO(1, 19)$. Following Piatetski-Shapiro and Shafarevich [PSS 71], the results from Burns and Rapoport [BR 75] and Todorov [To 80] combine to prove that the space of all isomorphism classes of marked, polarized $K3$ surfaces (algebraic or not) is in one to one correspondence with $SO_0(3, 19)/SO(2) \times SO(1, 19)$. If a $K3$ surface is not algebraic, one defines a polarization as a class in $H^{1,1}(X, \mathbf{R})$ lying in the Kähler cone of $X$ (see [To 80]).

If $(X, e, \alpha)$ is an algebraic, marked, polarized $K3$ surface, the degree of the polarization is the integer $d$ such that $2d - 2 = \langle e, e\rangle$. From [PSS 71] and [Ku 77] we have that the moduli space of isomorphism classes of marked, polarized, algebraic $K3$ surfaces of a fixed degree is equal to an open dense set in the symmetric space

$$\mathbf{h}_{2,19} = SO_0(2,19)/SO(2) \times SO(19).$$

Let

$$\Gamma_e = \{\phi \in \operatorname{Aut}(H^2(X, \mathbf{Z})) \mid \langle \phi(u), \phi(v)\rangle = \langle u, v\rangle \text{ and } \phi(e) = e\}.$$

The moduli space of isomorphism classes of polarized, algebraic $K3$ surfaces of a fixed degree, which we denote by $\mathcal{M}_{(X,e)}$, is isomorphic to an open, dense set in the quasi-projective variety $\Gamma_e \backslash \mathbf{h}_{2,19}$. If we allow our surfaces to have singularities which are at most double rational points, then the corresponding moduli space of isomorphism classes of marked, polarized, algebraic surfaces is equal to the entire symmetric space $\mathbf{h}_{2,19}$.

Using the period map, one can give the following description of $\mathbf{h}_{2,19}$. Let $\langle\ ,\ \rangle$ denote the bilinear form defined by the cup product on the second cohomology group $H^2(X, \mathbf{Z})$. Then $\mathbf{h}_{2,19}$ is described by

$$\mathbf{h}_{2,19} = \{\langle u, u\rangle = 0, \langle u, \bar{u}\rangle > 0 \text{ and } \langle u, e\rangle = 0\} \subset \mathbf{CP}(H^2(X, \mathbf{Z}) \otimes \mathbf{C}.$$

Let $\pi_{\text{mar},e} : M_{(X,\alpha,e)} \to M_{(X,e)}$ be the natural map which forgets the marking. From the surjectivity of the period map, it follows that $\pi_{\text{mar},e}$ coincides with the action of $\Gamma_e$ on $M_{(X,\alpha,e)}$.

## §3. A canonical family of holomorphic two forms.

In the case of elliptic curves, one has a specific family of holomorphically varying holomorphic one forms, namely the family $\{dz\}$. The construction of a canonical family of holomorphically varying holomorphic two forms can be described as follows.

1. Any marked, polarized, algebraic $K3$ surface is an element of a family of $K3$ surfaces $\mathcal{E} \to D$, where $D$ is the unit disc, such that the monodromy has a Jordan cell of dimension 3; i.e., if $T$ is the monodromy operator, on $H_2(X, \mathbf{Z})$, then $(T^N - \text{id})^3 = 0$ and $(T^N - \text{id})^2 \neq 0$ (see [To 76] and [JT 94a] for details).
2. On the generic fibre $X_t$ of this family, we have, up to sign, a unique cycle $\gamma$ such that $T\gamma = \gamma$ and any other $T$ invariant cycle is an integer multiple of $\gamma$. Further, there exists a cycle $\mu$ such that $T\mu = \gamma + \mu$.
3. Since $\mathbf{h}_{2,19}$ is contratible, there exists a globally defined, non-vanishing, holomorphically varying family of holomorphic two forms, say
$$\omega_t \in H^0(\mathbf{h}_{2,19}, \pi_* \mathcal{K}_{\mathcal{E}_{(X,\alpha)}/\mathbf{h}_{2,19}}).$$
4. In [JT 94a], it is shown that the function
$$\phi(t) = \int_\gamma \omega_t$$
is non-vanishing on $\mathbf{h}_{2,19}$.

The canonical family of holomorphic two forms is defined to be $\{\omega_t/\phi(t)\}$. We remark that when following the identical steps in the case of elliptic curves, one constructs the family of holomorphic one forms $\{dz\}$.

## §4. Definition of the analytic discriminant for $K3$ surfaces.

Let $\mathcal{T}_{(X,e)}$ be the sheaf of holomorphic vector fields on $(X, e)$. From Kodaira-Spencer deformation theory, we can identify the tangent space $T_{\mathcal{M}_{(X,e)}}$ of the moduli space of $\mathcal{M}_{(X,e)}$ at the point $(X, e)$ with $H^1(X, \mathcal{T}_{(X,e)})$. The existence of the non-vanishishing holomorphic two-form $\omega$ on $X$ implies that we can identify $H^1(X, \mathcal{T}_{(X,e)})$ with $H^1(X, \Omega)$. One can then deduce that the tangent space $T_{\mathcal{M}_{(X,\alpha,e)}}$ to the moduli space $\mathcal{M}_{(X,\alpha,e)}$ at the point $(X, \alpha, e)$ can be identified with the space

$$H^1(X, \Omega^1)_0 = \{u \in H^1(X, \Omega^1) \mid \langle u, e\rangle = 0\}.$$

We view any $\phi \in H^1(X, T_{(X,e)})$ as a linear map from $\Omega^{1,0}$ to $\Omega^{0,1}$ pointwise on $X$. Given $\phi_1$ and $\phi_2$ in $H^1(X, T_{(X,e)})$, the trace of the map

$$\phi_1\overline{\phi_2} : \Omega^{0,1} \to \Omega^{0,1}$$

at a point $x \in X$ with respect to the unit volume Calabi-Yau metric $g$ (meaning a Kähler-Einstein metric compatible with the given polarization class $e$) is simply

$$\mathrm{Tr}(\phi_1\overline{\phi_2})(x) = \sum_{k,l,m,n} (\phi_1)^k_{\bar{l}}(\overline{(\phi_2)^m_{\bar{n}}})g^{n\bar{l}}g_{k\bar{m}}.$$

We define the Weil-Petersson metric on $\mathcal{M}_{(X,\alpha,e)}$ via the inner product

$$\langle \phi_1, \phi_2 \rangle = \int_X \mathrm{Tr}(\phi_1\overline{\phi_2})\mathrm{vol}_g$$

on the tangent space $H^1(X, T_{(X,\alpha,e)})$ of $M_{(X,\alpha,e)}$ at $(X, \alpha, e)$. It is shown in [To 89] that the Weil-Petersson metric on $M_{X,\alpha,e}$ is equal to the restriction of the Bergman metric on $\mathbf{h}_{2,19}$. Therefore, the Weil-Petersson metric is a Kähler metric with Kähler form $\mu_{\mathrm{WP}}$.

For any holomorphic two form $\omega$ on $X$, let

$$\|\omega\|^2_{L^2} = -\langle \omega, \omega \rangle = \int_X \omega \wedge \bar{\omega}.$$

In [To 89] and [Ti 88] it was proved that $\log\|\omega\|^2_{L^2}$ is a potential for the Weil-Petersson metric. The following theorem from [JT 94a] proves the existence of a second potential for the Weil-Petersson metric.

**Theorem 4.1.** *Let $(X, e)$ be a polarized, algebraic $K3$ surface of degree $d$, and let $\mu$ denote the unit volume Kähler-Einstein form compatible with the given polarization. Let $\det^* \Delta^{(0,1)}_{(X,e)}$ be the determinant of the Laplacian which acts on the space of smooth $(0,1)$ forms on $X$. Let $\{\omega_{(X,e,\alpha)}\}$ be the normalized family of holomorphically varying 2-forms on the moduli space $\mathcal{M}_{(X,\alpha,e)}$. Then*

$$dd^c \log \left( \frac{\det^* \Delta^{(0,1)}_{(X,e)}}{\|\omega_{(X,\alpha,e)}\|^2_{L^2}} \right) = 0,$$

*or, equivalently*

$$-dd^c \det{}^* \Delta^{(0,1)}_{(X,e)} = -dd^c\|\omega_{(X,\alpha,e)}\|^2_{L^2} = \mu_{\mathrm{WP}}.$$

*In other words, $-\det^* \Delta^{(0,1)}_{(X,e)}$ is a potential for the Weil-Petersson metric on $\mathcal{M}_{(X,\alpha,e)}$.*

From results in [Kon 88], we have that $N_e = \#(\Gamma_e/[\Gamma_e, \Gamma_e])$ is finite. With this, we can follow the pattern observed for elliptic curves to define an analytic discriminant for polarized, algebraic $K3$ surfaces. In [JT 94a] and [JT 94b] we proved the following theorem.

**Theorem 4.2.** *Let $\mathcal{D}_e = (\Gamma_e \backslash \mathbf{h}_{2,19}) \setminus \mathcal{M}_{(X,e)}$. Then there is a holomorphic function $f_e$ on $\mathbf{h}_{2,19}$ which vanishes on $\mathcal{D}_e$ such that*

$$|f_e([X, \alpha, e])| = \left( \frac{\det^* \Delta^{(0,1)}_{(X,e)}}{\|\omega_{(X,\alpha,e)}\|^2_{L^2}} \right);$$

*whence $f_e$ does not vanish on $\mathcal{M}_{(X,\alpha,e)}$. Moreover, $f_e^{N_e}$ is an automorphic form on $\mathbf{h}_{2,19}$ with respect to the action by $\Gamma_e$.*

In section 8 below we shall discuss further properties of $f_e$ as an automorphic form.

By Theorem 4.2, we can view $f_e^{N_e}$ as a section of the line sheaf $(\pi_* \mathcal{K}_{\mathcal{X}/\mathcal{M}_{(X,e)}})^{N_e}$ which vanishes precisely on $\mathcal{D}_e$. The holomorphic function $f_e$ defined as in Theorem 4.2 will be called the analogue of Dedekind $\eta$ function for polarized, algebraic $K3$ surfaces.

To conclude, let us remark that $\mathcal{D}_e$ can be realized as the moduli space of algebraic $K3$ surfaces whose polarization $e$ is of degree $d$ and is is such that any associated projective embedding has singular double rational points. Further discussion of this point, together with interpretations in terms of singularities of the Calabi-Yau metrics, is given in [KT 87].

## §5. Realization of $\Gamma_e \backslash \mathbf{h}_{2,19}$ as a tube domain.

In order to state a product formula for the analytic discriminant defined in Theorem 4.2, we need a specific realization of the symmetric space $\mathbf{h}_{2,19}$ as a tube domain; that is, we need to define a convex cone $V^+$ in $\mathbf{R}^{19}$ and represent $\mathbf{h}_{2,19}$ as $\mathbf{R}^{19}+\sqrt{-1}V^+$. For this, we will follow the approach in [I 82], [PS 69], and [To 94]. To begin, we need two elementary lemmas in linear algebra. Throughout this section, we use the notation

$$\mathbf{h}_{2,n} = SO_0(2,n)/SO(2)\times SO(n).$$

**Lemma 5.1.** *Let $\langle\ ,\ \rangle$ be a symmetric bilinear form on $\mathbf{R}^{n+2}$ which has signature $(2,n)$. Let*

$$A = \{u \in \mathbf{P}(\mathbf{R}^{n+2}\otimes \mathbf{C}) \mid \langle u,u\rangle = 0 \text{ and } \langle u,\bar{u}\rangle > 0\}.$$

*Then:*
(a) *$A$ is isomorphic to $\mathbf{h}_{2,n}$.*
(b) *$A$ is isomorphic to $\{E \subset \mathbf{R}^{n+2} \mid \dim E = 2$ and $\langle\ ,\ \rangle\,|_E > 0\}$.*

*Proof.* For the proof see [To 80]. □

**Lemma 5.2.** *Let $Q = [\ ,\ ]$ be a bilinear form on $\mathbf{R}^n (n>1)$ of signature $(1,n-1)$. Let*

$$V = \{v \in \mathbf{R}^n \mid [v,v] > 0\},$$

*and let $V^+$ be one of the components of $V$. Then $\mathbf{R}^n + \sqrt{-1}V^+$ is isomorphic to $\mathbf{h}_{2,n}$.*

*Proof.* Let $H(V) = \mathbf{R}^n + \sqrt{-1}\mathbf{V}^+ \subset \mathbf{R}^n + \sqrt{-1}\mathbf{R}^n = \mathbf{C}^n$. Define the map

$$\psi : H(V) \to \mathbf{CP}^{n+1} \quad \text{by} \quad u \mapsto (u_1,\ldots,u_n,-1/2[u,u],1) \in \mathbf{CP}^{n+1}.$$

Consider a new symmetric bilinear form $\langle\ ,\ \rangle$ on $\mathbf{R}^{n+2}$ by

$$\langle\ ,\ \rangle = (\ ,\ ) \oplus \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

which naturally extends to a hermitian form over $\mathbf{C}^{n+2} = \mathbf{R}^{n+2}\otimes\mathbf{C}$. It is immediate that the signature of $\langle\ ,\ \rangle$ is $(2,n)$. Moreover, we have

$$\langle\psi(u),\psi(u)\rangle = 0 \text{ and } \langle\psi(u),\overline{\psi(u)}\rangle > 0.$$

By Lemma 5.1(a), $\psi$ is an embedding of $H(V)$ into $\mathbf{h}_{2,n}$. It remains to prove that $\psi$ is surjective. For this purpose, it suffices the establish the following observation. For any $g \in SO_0(2,n)$, the last coordinate of $g(\psi(u))$ is different from zero.

Assume $g(\psi(u)) = (v, v_{n+1}, 0)$. Since $\langle g\psi(u), g\psi(u)\rangle = 0$ and $\langle g\psi(u), \overline{g\psi(u)}\rangle > 0$, we then conclude

$$[v,v] = 0 \text{ and } [v,\bar{v}] > 0.$$

From Lemma 5.1(b) it follows that the two dimensional subspace in $\mathbf{R}^n$ spanned by $\mathrm{Re}(v)$ and $\mathrm{Im}(v)$ is such that $[\ ,\ ]$ restricted to this space is strictly positive. However, this is impossible since $[\ ,\ ]$ has a signature $(1, n-1)$. Therefore, the last component of $g(\psi(u))$ is never zero, and the proof of Lemma 5.2 is complete. $\square$

Recall that the lattice of primitive cohomology classes of degree $d$ is define by

$$H_d = (-E_8)^2 \oplus \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^2 \oplus e\mathbf{Z}, \text{ where } \langle e, e\rangle = 2d-2., \quad L_d = (E_8)^2 + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + e\mathbf{Z}$$

$-V^+ \cup V^+ = \{v \in L_d \otimes \mathbb{R} \mid \langle v, v\rangle > 0\}$

From [To 80], we have that the automorphisms of $L_d$ which preserve $V^+$ is isomorphic to $\Gamma_e$, where $e$ has degree $d$. With this, we have the following theorem.

**Theorem 5.3.** *Let $\mathcal{O}_+(L_d)$ be the group of automorphisms of the lattice $L_d$ which preserve the cone $V^+$, and set*

$$\mathcal{M}_d = \mathcal{O}_+(L_d)\backslash(L_d \otimes \mathbf{R} + \sqrt{-1}V^+)$$

*Then $\Gamma_e\backslash\mathbf{h}_{2,19} \cong \mathcal{M}_d$.* $L_d \otimes \mathbb{R} + \sqrt{-1}\, V^+ = K_d$

We shall let denote the isomorphism given in Theorem 5.3 by

$$\pi_e : \Gamma_e\backslash\mathbf{h}_{2,19} \to \mathcal{O}_+(L_d)\backslash(L_d \otimes \mathbf{R} + \sqrt{-1}V^+).$$

## §6. Realization of $\mathcal{D}_e$ in the tube domain.

With Theorem 5.3, we need to identity the subset in the quotient of the tube domain corresponding to $\mathcal{D}_e$. To begin, let us write a canonically defined divisor in the quotient of the tube domain, and then we will prove that this subset corresponds to the divisor $\mathcal{D}_e$.

Recall that $\mathcal{D}_e \subset \mathbf{h}_{2,19}$ can be realized as the set algebraic $K3$ surfaces whose polarization $e$ is of degree $d$ and is such that any associated projective embedding has singular double rational points. Let

$$\Delta_e = \{l \in H^2(X, \mathbf{Z}) \mid \langle l, l\rangle = -2, \langle l, e\rangle = 0\}.$$

In [PSS 71], it is proved there is a decomposition $\Delta_e = \Delta_e^+ \cup (-\Delta_e^+)$ where $\Delta_e^+ \cap (-\Delta_e^+)$ is empty. The main result of this section is the following theorem.

**Theorem 6.1.** *For each $l \in \Delta_e^+$, let*

$$H_{e,l} = \{u \in \mathbf{P}(H^2(X, \mathbf{Z}) \otimes \mathbf{C}) \mid \langle u, 1\rangle = 0\}.$$

*Then* $\pi_e(\pi_{\mathrm{mar},e}^{-1}(\mathcal{D}_e)) = \bigcup_{l \in \Delta_e^+} \left(H_{e,l} \cap \pi_e\left(\mathcal{M}_{(X,\alpha,e)}\right)\right).$

In order to prove Theorem 6.1 we need the following lemma.

**Lemma 6.2.** *Let $\mathcal{L} = \mathcal{O}_X(D)$ be a line sheaf on a $K3$ surface $X$ such that $c_1(\mathcal{L}) = 1$ and $\langle 1, 1\rangle = -2$. Then either $H^0(X, \mathcal{O}(D)) \neq 0$ or $H^0(X, \mathcal{O}(-D)) \neq 0$.*

*Proof.* From the Riemann-Roch theorem we have

$$\chi(\mathcal{O}_X(D)) = \dim_{\mathbf{C}} H^0(X, \mathcal{O}(D)) - \dim_{\mathbf{C}} H^1(X, \mathcal{O}(D)) + \dim_{\mathbf{C}} H^2(X, \mathcal{O}(D)) = -\frac{\langle 1, 1\rangle}{2} + 2 = 1,$$

hence

$$\dim_{\mathbf{C}} H^0(X, \mathcal{O}(D)) + \dim_{\mathbf{C}} H^2(X, \mathcal{O}(D)) \geq 1.$$

It follows from Serre duality and the triviality of the canonical class that $H^2(X, \mathcal{O}(D))$ and the dual of $H^0(X, \mathcal{O}(-D))$ are isomorphic. Therefore,

$$\dim_{\mathbf{C}} H^0(X, \mathcal{O}(D)) + \dim_{\mathbf{C}} H^0(X, \mathcal{O}(-D)) \geq 1$$

from which the result follows. $\square$

**Lemma 6.3.** *Let $\mathcal{L} \simeq \mathcal{O}(D)$ be a line bundle on a K3 surface X such that $H^0(X,\mathcal{L}) > 0$ and $\langle D, D\rangle = -2$. Then D is a union of non-singular rational curves. Further, D can be described by Dynkin diagrams of type $A_n, D_n, E_6, E_7, E_8$.*

*Proof.* This corollary follows directly from Lemma 6.2 and Theorem 2.7 and figure 2.8 from [A 62]. □

*Proof of Theorem 6.1.* Let

$$\mathcal{D} = \bigcup_{l\in\Delta_e^+} \left(H_{e,l} \cap \pi_e\left(\mathcal{M}_{(X,\alpha,e)}\right)\right).$$

Let $\tau \in \pi^{-1}_{\mathrm{mar},e}(\mathcal{D}_e)$. From the surjectivity of the period map, there exists a polarized $K3$ surface $(X_\tau, e_\tau)$ such that for any projective embedding associated to any power of the given polarization, the image of $X_\tau$ has at least one double rational singular point. Take a minimal resolution of these double rational points to obtain a minimal $K3$ surface $\tilde{X}_\tau$. The preimage of each double rational point from $\tilde{X}_\tau$ onto $X_\tau$ will define a divisor $D$ such that the Chern class $c_1(\mathcal{O}(D))$ will be a vector $l \in H^2(\tilde{X}_\tau, \mathbf{Z})$ for which $\langle l, e\rangle = 0$, $\langle l,\tau\rangle = 0$, and $\langle l, l\rangle = -2$. Therefore, $\pi_e(\pi^{-1}_{\mathrm{mar},e}(\mathcal{D}_e)) \subset \mathcal{D}$.

Conversely, for each $l \in H^2(X,\mathbf{Z})$ such that $\langle l,l\rangle = -2$ and $\langle l,e\rangle = 0$, there is a hyperplane

$$H_l = \{\tau \in \mathbf{h}_{2,19} \subset \mathbf{P}(H^2(L\otimes\mathbf{C})) \mid \langle \tau, l\rangle = 0\}.$$

From Lemma 6.3 and the surjectivity of the period map, we conclude that $\pi_e^{-1}(\tau) \in \pi^{-1}_{\mathrm{mar},e}(\mathcal{D}_e)$, which completes the proof of Theorem 6.1. □

## §7. The [redacted] product formula.

For each $l \in L$ such that $\langle l,l\rangle = -2$, we define the linear map

$$s_1 : V^+ \to V^+ \quad \text{by} \quad s_l(v) = v + \langle v, l\rangle l.$$

It is immediate that $s_l \in \mathcal{O}_+(L)$. Let $\Gamma_L$ be the subgroup generated by $s_l$. Then, $\Gamma_L$ is a normal subgroup in $\mathcal{O}_+(L)$, and $\Gamma_L$ acts properly and discontinuously on $V^+$. Let $\mathcal{F}(L)$ denote the fundamental domain of $\Gamma_L$ such that $\mathcal{F}(L)$ is a convex polyhedra whose walls are defined by the hyperplanes $H_l$. For a proof of these facts, see [Bour 89].

Let $\Delta = \{l \in L \mid \langle l,l\rangle = -2\}$. Each choice of $\mathcal{F}(L)$ defines a splitting of $\Delta = \Delta^+ \cup (-\Delta^+)$. Let us fix $\mathcal{F}(L)$ and let $e_1,\dots,e_{19}$ be vectors in $L$, which lie on the walls of $\mathcal{F}(L)$ and spanned $L$. Then the family $\{e_i\}$ defines a flat coordinate system $\tau_1,\dots,\tau_{19}$ in $L\otimes\mathbf{R} + \sqrt{-1}V^+$ (see [COGP 92] and [To 94]).

**Conjecture 7.1.** *Consider $f_e$ as a function on $\mathcal{M}_d$. Let $N_d = \#(\mathcal{O}_+(L_d)/[\mathcal{O}_+(L_d),\mathcal{O}_+(L_d)]$ and set $\tau = \sum_{i=1}^{19}\tau_i e_i$. For each $l \in \Delta^+$, let $q_l(\tau) = \exp(2\pi\sqrt{-1}\langle\tau,l\rangle)$ and set $q_0(\tau) = \exp(2\pi\sqrt{-1}\sum\tau_i)$. Then there is a constant $c_d$ and a set of positive integers $\{m(l)\}$ for $l \in \Delta^+$ such that*

$$f_e(\tau) = c_d q_0(\tau)^{1/N_d} \prod_{l\in\Delta^+}(1-q_l(\tau))^{m(l)} \prod_{l'\in\Delta^+}\left(1-\exp\left(2\pi i B_{l'}\right)\right)^{m(l')}$$

where for $l' \in H_d$ $\langle l', l'\rangle = 0 \rightsquigarrow l' = (a,b,c)$ $a \in L_d \rightsquigarrow B_{l'} = \{\tau\in K_d \mid a\tau - \frac{1}{2}b\langle\tau,\tau\rangle + c = 0$

For a detailed discussion concerning the structure of the integers $\{m(l)\}$, the reader is referred to the preprint [Bor 94]. A stronger version of Conjecture 7.1 would assert that, as in the case of the Dedekind eta function for elliptic curves, almost all integers $m(l)$ are equal to one.

When expanding the product in Conjecture 7.1, one obtains a Fourier series with integer coefficients. The integers are related to the number of $-2$ curves on the $K3$ surface. In this way, our discriminant can be viewed as a type of generating function. Recursive relations between the coefficients can be established and suggests that a connection with Hecke theory on $\mathbf{h}_{2,19}$ should be studied. Further investigation into these questions is under consideration (see [JT 94d]).

**Remark 7.2.** One can prove that the product defined in Conjecture 7.1 converges for all vector $\tau \in L_d\otimes\mathbf{R}_+\sqrt{-1}V^+$. Recall that $f_e$ is a section of the line sheaf $(\pi_*\mathcal{K}_{S/\mathcal{M}_{(X,e)}})$. Therefore, $f_e$ vanishes on $\mathcal{D}_e$. It is immediate from results in section 6 that the product in Conjecture 7.1 vanishes on this space as well.

In the case of elliptic curves, one can prove that the holomorphic function constructed from the determinant of the Laplacian has a product formula by proving that the product expansion has the same automorphic behavior as the holomorphic function. This fact can be verified through the Kronecker limit formula. An analogue of the Kronecker limit formula in the setting of algebraic, polarized $K3$ surfaces is currently under investigation (see [JT 94d]).

**Remark 7.3.** It was shown in [JT 94b] that $\det^* \Delta^{(0,1)}_{(X,e)} < 1$. Therefore, Conjecture 7.1 would then give an upper bound for the product in terms of the $L^2$ norm of the image of the period point in the tube domain.

## §8. An analogue of the elliptic $q$-parameter and $j$-function.

It is known that the space of weight 12 holomorphic modular forms on the hyperbolic upper half plane $\mathbf{h}$ with respect to the group $PSL(2,\mathbf{Z})$ which vanish at infinity is one dimensional (see, for example, page 88 of [Ser 73]). Analogous to this fact, we have the following conjecture.

**Conjecture 8.1.** *Let $\mathcal{M}^d_{\mathrm{p}}$ be the moduli space of polarized, algebraic $K3$ surfaces of degree d, and let $\mathcal{X}$ be the universal family of $K3$ surfaces over $\mathcal{M}^d_{\mathrm{p}}$. Then*

$$\dim H^0(\mathcal{M}^d_{\mathrm{p}}, \pi_*\mathcal{K}^{\nu d}_{\mathcal{X}/\mathcal{M}^d_{\mathrm{p}}}) = 1.$$

In [JT 94a], we constructed, following [COGP 92] an analogue of the $q$-parameter which exists in the setting of elliptic curves. Briefly, the construction is as follows.

Let $\pi : (\mathcal{X}, \mathcal{L}) \to \mathcal{D}$ be a semi-stable degenerating family of polarized $K3$ surfaces such that the monodromy operator $T \in \mathrm{Aut}(H_2(X_t,\mathbf{Z}), L_t)$ has a single Jordan cell of dimension 3 (see [To 76]). In the language of [Ku 77], such a family is of type III. Then there is a free, three-dimensional submodule $W(X_t, L_t) \subset H_2(X_t,\mathbf{Z})$ for which the action of the monodromy operator is unipotent. That is, with respect to a continously varying basis $\{A_t, B_t, C_t\}$ of $W(X_t, L_t)$ over $\mathbf{Q}$, the action of the monodromy is by the matrix

$$\begin{pmatrix} 1 & 1 & 1/2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Thus, there is a unique invariant 1-dimensional submodule, generated by $\pm A_t$ for $t \in \mathcal{D}$. Let $\omega_t$ be as in section 3, so

$$\int_{A_t} \omega_t = \pm 1.$$

The vanishing cycle is the cycle $A_t$ such that the above integral is equal to 1. An element $B_t$ in $W(X_t, L_t)$ for which $T(B_t) = B_t + A_t$ will be called a transverse cycle. Note that two transverse cycles differ by an additive factor of the form $nA_t$ where $n$ is an integer. The $K3$ modular parameter associated to the above degenerating family is defined by

$$q_\pi(t) = \exp\left(2\pi i \int_{B_t} \eta_t\right).$$

Since $\mathbf{h}_{2,19}$ is simply connected, one can use deformation arguments to show that $q_\pi$ is independent of the family and depends solely on $(X, \alpha, e)$, hence we shall write $q_{(X,\alpha,e)}$. Using results from [JT 94a], we can prove the following. If $(X, \alpha, e)$ is a Kummer surface associated to the abelian surface $\mathbf{C}^2/L(\Omega)$, where $L(\Omega)$ is the lattice associated to $I_2$, the two by two identity matrix, and $\Omega$, a matrix in the Siegel upper half space of dimension two, then $q_{(X,\alpha,e)} = \exp(\pi i \mathrm{Tr}(\Omega))$. Using the asymptotics of the periods as given in [Gr 70], we can prove

$$q_{(X,\alpha,e)}(0) = 0 \quad \text{and} \quad q'_{(X,\alpha,e)}(0) \neq 0.$$

An interesting question is to relate the above $q$ parameter and the parameter defined in Conjecture 7.1. It can be shown that when one restricts $\tau$ to a line in the tube domain generated by a rational direction (meaning a line such that some integer multiple lies in the lattice), then the two $q$ parameters coincide (see [JT 94d]).

Let us now define the $K3$ analogoue of the elliptic $j$-function, together with a conjecture concerning its Fourier expansion. As above, let us view $f_e^{N_e}$ as a holomorphic modular form on the tube domain constructed in section 5. The main result from [Ba 70], as stated on page 141, asserts that $f_e^{N_e}$ can be written as an isobaric polynomial involving Eisenstein series defined in [Ba 70]. Let $h_e$ be the weight of the polynomial. Let $S_h$ denote the set of isobaric polynomials in the Eisenstein series of weight $h$ with integer coefficients. Further work in [Ba 70] and [Ba 73] asserts that for sufficiently large $h$, a basis of $S_h$ provides a map defined over $\mathbf{Q}$ of the quotient of the tube domain into projective space such that the image is birationally equivalent to the Satake compactification of the quotient of the tube domain. Let $h = c_e h_e$ be the smallest weight divisible by $h_e$, so $c_e$ is an integer, such that $S_h$ provides such a map, and let $\{E_{h,k}\}$ with $k = 1, \ldots, N$ be a $\mathbf{Q}$ basis of $S_h$. The set of modular functions $\{E_{h,k}/f_e^{c_e}\}$ with $k = 1, \ldots, N$ is the $K3$ analogoue of the elliptic $j$-function, which is known to be expressible as the quotient of a weight twelve Eisenstein series divided by the Dedekind delta function. In the case of elliptic curves, one scales the elliptic $j$-function so that the lead coefficient in a $q$ expansion is one.

**Conjecture 8.2.** *For every $k$, there is a $c_k \in \mathbf{Q}$ such that for every type III degenerating family, the $q$ expansion $c_k E_{h,k}/f_e^{c_e}$ has positive integer coefficients.*

Going beyond Conjecture 8.2, we speculate, assuming the validity of Conjecture 8.2, that the coefficients are related to the dimensions of irreducible representations of the Mathieu group $M_{24}$, in a manner similar to that which relates dimensions of the irreducible representations of the Fischer-Griess monster simple group and the $q$ expansion of the elliptic $j$-function. We base this speculation on two facts. The first observation is a combination of the connection with $M_{24}$ and the monster, as discussed in [CN 79], together with the connection between elliptic curves and boundary components in the moduli space of polarized, algebraic $K3$ surfaces (see [Ku 77]). The second observation is a result of Mukai [Mu 88], which states that any automorphism group of a polarized, algebraic $K3$ surfaces is a certain subgroup of $M_{24}$. The reader is referred to [Mu 88] for further details of his proof. In ongoing work, we are investigating the following so far heuristic approach to Mukai's theorem.

As described in [T 85], one can embed $M_{24}$ into the automorphism group, modulo reflections, of a 26-dimensional $\mathbf{Z}$ lattice of signature $(25,1)$. More specifically, let $L_{26}$ be the set of vectors in $u \in \mathbf{R}^{26}$ with coefficients in $\frac{1}{2}\mathbf{Z}$ and such that $(u, f) \in 2\mathbf{Z}$, where $f$ is the vector in $\mathbf{R}^{26}$ with all entries equal to $1/2$, and $(,)$ is an inner product of signature $(25,1)$. Let $W$ be the subgroup of $\mathrm{Aut}(L_{26})$ generated by reflections of the form

$$s_l(u) = u + (u,l)l \quad \text{where} \quad (l,l) = -2.$$

Then one can embed $M_{24}$ into $\mathrm{Aut}(L_{26})/W$. On the other hand, it is shown in [PSS 71] that $\mathrm{Aut}(X, L)$, the automorphism group of a polarized, algebraic $K3$ surface, is equal to $\mathcal{O}^+(\mathrm{Pic(X)})/W$, where $\mathrm{Pic}(X)$ is the Picard group $H^2(X,\mathbf{Z}) \cap H^{1,1}(X,\mathbf{R})$ and $W$ is a group of reflections. Using the Hodge decomposition theorem, one can show that the signature of the inner product on $H^2(X,\mathbf{Z})$ has signature $(n,1)$ on $\mathrm{Pic}(X)$, where $n = \mathrm{rk}(\mathrm{Pic}(X)) - 1$. One can embed $\mathrm{Pic}(X)$ into $L_{26}$; namely, we need to consider subgroups in $\mathcal{O}^+(L_{26})$ which stabilize $\mathrm{Pic}(X)$. Such subgroups will define subgroups of $M_{24}$ in a natural way via intersection.

In [JT 94c], we are investigating the continuation of these ideas to the setting of Enriques surfaces. The corresponding symmetric space is $SO(2,10)$, and the corresponding simple group is $M_{12}$.

In [LY 94], the authors are using ideas from mirror symmetry to study a connection between the Fischer-Griess monster simple group and $K3$ surfaces which are complete intersections in weighted projective spaces.

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