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CONFERENCE ON

TOPOLOGICAL AND GEOMETRICAL PROBLEMS

RELATED TO QUANTUM FIELD THEORY

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Elementary charge modifications and immersed spheres in 4-manifolds

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These are preliminary lecture notes, intended only for distribution to participants

Recall Seiberg Witten Invariant for X^4:

$$\boxed{SW : \mathcal{C} \longrightarrow \mathbb{Z}}$$

$\mathcal{C}$ = set of spin$_c$-structures on $X \underset{\uparrow}{=} \left\{ L \in H^2(X;\mathbb{Z}) \;\middle|\; \begin{array}{l} L\cdot x \equiv x^2 \ (2) \\ x \in H^2(X) \end{array} \right\}$

no $\mathbb{Z}$-torsion in H_2

Given $L \in \mathcal{C}$ consider $(A, \psi) \in \mathcal{A}(L) \times \Gamma(S^+)$

- $\not{\partial}_A \psi = 0$ $\quad\cdot\quad$ $F_A^+ = (\psi \otimes \psi^*)_0$

$\mathcal{M}_L$ = solutions $/ \mathrm{Map}(X, S^1)$ $\qquad \dim \mathcal{M}_L = \dfrac{c_1^2(L) - (3\sigma + 2e)}{4}$

$$SW(L) = \begin{cases} 0 & \dim \mathcal{M}_L < 0 \\ \#\text{pts (with sign)} & \dim \mathcal{M}_L = 0 \\ \langle h^d, [\mathcal{M}_L] \rangle & \dim \mathcal{M}_L = 2d \end{cases}$$

h = gen of $H^2(\mathbb{C}P^\infty; \mathbb{Z}) = H^2(\mathcal{B}_L^*; \mathbb{Z})$

- $\mathcal{M}_L$ compact (Weitzenboch)
- $s > 0 \Rightarrow$ only solutions are $\psi = 0$; $F_A^+ = 0$ (Weitzenboch)
- $SW(L) \neq 0$ for only finitely many L (Weitzenboch)
- $SW(\pm K_X) = \pm 1$ $\quad K_X$ canonical class of complex (Kahler) X.

Other Fundamental Properties

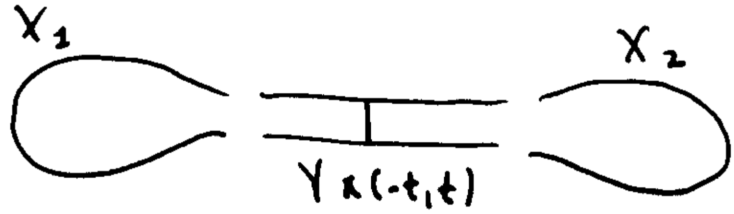
- Suppose $X = X_1 \cup_Y X_2$ where Y has positive scalar curvature and $b^+_{X_1} > 0$, $b^+_{X_2} > 0$, then $SW \equiv 0$

E.G. $Y = S^3$, Lens spaces, Poincaré sphere.

- ($\Rightarrow$ $S^2 \hookrightarrow M^4$ with positive self-intersection $\Rightarrow SW \equiv 0$)
- Poincare sphere $= \Sigma(2,3,5)$ embeds in many complex X but one side is always negative definite.

$[M, S^1]_0$

Pf:



$\partial X_1 = Y$
$\partial X_2 = \bar{Y}$
Y: + curvature

$\Rightarrow$ large t get only reducible solutions on tube

Given char. line bundle L, $\exists!$ ASD connection on $L|_{Y \times \mathbb{R}} \leftrightarrow G(L)|_Y$

and $\mathcal{M}_L(X)$ $\xrightarrow[\text{large } t]{}$ pair of solutions $(A_1,\Psi_1),(A_2,\Psi_2) \in \mathcal{M}_{L_1}(X_1) \times \mathcal{M}_{L_2}(X_2)$
(based moduli)

Get $\widetilde{\mathcal{M}}_L(X) \longrightarrow \widetilde{\mathcal{M}}_{L_1}(X_1) \times \widetilde{\mathcal{M}}_{L_2}(X_2)$ $\qquad (\mathcal{M}_L = \widetilde{\mathcal{M}}_L/S^1)$

Conversely - same old story - can glue to get inverse

$$\Rightarrow \widetilde{\mathcal{M}}_L(X) \simeq \widetilde{\mathcal{M}}_{L_1}(X_1) \times \widetilde{\mathcal{M}}_{L_2}(X_2)$$

provided neither L_1 or L_2 admits reducible solutions!

Note: $S^1 \times S^1$ acts freely on $\widetilde{\mathcal{M}}_{L_1} \times \widetilde{\mathcal{M}}_{L_2}$; $\mathcal{M}_L = [\widetilde{\mathcal{M}}_{L_1} \times \widetilde{\mathcal{M}}_{L_2}]/S^1$

so $\mathcal{M}_L$ has S^1-action! ∎

OR $\dim \mathcal{M}_L(X) = \dim \mathcal{M}_{L_1}(X_1) + \dim \mathcal{M}_{L_2}(X_2) + 1$

Observe: char. line bundles on $X \# \overline{\mathbb{C}P}^2$ are of the form

$L \pm (2k+1)E$ where L is char. line bundle on X

(written additively $L \leftrightarrow c_1(L)$).

Compute dimensions:

$$\dim(L \pm (2k+1)E) = \frac{1}{4}\left\{ c_1^2(L) - (2k+1)^2 - (3(\sigma(X)-1) + 2(e(X)+1)) \right\}$$
$$= \dim \mathcal{M}_L(X) - k(k+1) = d(L) - k(k+1).$$

As a warm up: Consider case $d(L) = 0 \Rightarrow k = 0$;

compare $n_L(X)$ with $n_{L\pm E}(X \# \overline{\mathbb{C}P}^2)$ $\quad$ ($n_L = \#$ SW solution's)

Recall: $\overline{\mathbb{C}P}^2$ has +'ve scalar curvature; only SW soln is reducible:

$(A_{\pm E}, 0)$: So

$$\mathcal{M}_{L\pm E} = [\tilde{\mathcal{M}}_L(X) \times \tilde{\mathcal{M}}_{\pm E}(\overline{\mathbb{C}P}^2)]/S^1$$

$$(A, \psi) \mapsto (A \# A_{\pm E}, \psi + 0) \qquad = [\tilde{\mathcal{M}}_L(X) \times \{(A_{\pm E}, 0)\}]/S^1$$

$$= \mathcal{M}_L(X) \times \{(A_{\pm E}, 0)\}$$

$$\Rightarrow \boxed{n_{L\pm E}(X \# \overline{\mathbb{C}P}^2) = n_L(X)} \quad (d(L) = 0)$$

More generally: $\boxed{\text{If } k(k+1) \leq 2d = d(L), \text{ then } n_{L\pm(2k+1)E}(X \# \overline{\mathbb{C}P}^2) = n_L(X)}$

So if $d(L) > 0$, get "new" 0-dim'l solutions on $X \# \overline{\mathbb{C}P}^2_s$.
So all higher index solutions on X are seen as 0-index solutions on $X \#_k \overline{\mathbb{C}P}^2$.

Blow up formula here is (apriori) dull.

In contrast: The blowup formula of Fintushel + myself in Donaldson theory is an "interesting" formula involving an elliptic curve

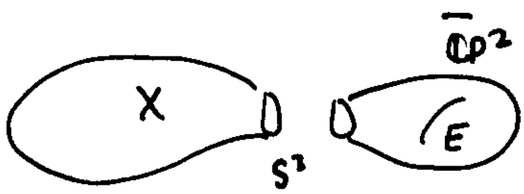
$$z^2 = 4(y-e_1)(y-e_2)(y+x/3)$$

$$e_1 = \frac{x}{6} + \frac{\sqrt{x^2-4}}{2} \; ; \; e_2 = \frac{x}{6} - \frac{\sqrt{x^2-4}}{2} \; ; \; x \text{ the point class.}$$

In SW theory, blow up formula dull - but "interesting" elliptic curve arises in transition from SW $\leftrightarrow$ Donaldson.

(SW Blow up): If $k(k+1) \le 2d = d(L)$ then $n_{L \pm (2k+1)}(X \# \overline{\mathbb{CP}}^2) = n_L(X)$

Pf. (general idea):



$$\begin{array}{ccc} L & \# & [(2k+1)E = E_k] \\ \downarrow & & \downarrow \\ X & \# & \overline{\mathbb{CP}}^2 \end{array}$$

$$\dim \mathcal{M}_{E_k}(\overline{\mathbb{CP}}^2) = -\frac{(2k+1)^2 - (-3+6)}{4} = -k(k+1) - 1.$$

$\overline{\mathbb{CP}}^2$ has +'ve scalar curvature so only reducible SW sdn: $(A_{(2k+1)E}, 0)$

If $\dim \mathcal{M}_X(L) \geq k(k+1)$ can "graft" in $(A_{(2k+1)E}, 0)$

Why: Obstruction bundle is given by "Kuranishi picture": a bundle associated to base point fibration $S^1 \to \widetilde{\mathcal{M}}_L(X)$

$$\downarrow$$
$$\mathcal{M}_L(X)$$

$$\widetilde{\mathcal{M}}_L \times \mathbb{C}^{k(k+1)} \leftarrow \text{cokernel of } \partial_{A_k} \text{ on } \overline{\mathbb{CP}}^2$$

Kuranishi map $\longrightarrow$ $\Phi \curvearrowright S^1$, $\downarrow$ $\mathcal{M}_L(X)$

$$[\mathcal{M}_{\tilde{L}}(X \# \overline{\mathbb{CP}}^2)] = [\Phi^{-1}(0)] \text{ in } \mathcal{B}_{\tilde{L}}(X \# \overline{\mathbb{CP}}^2).$$

This bundle splits as sum of $\mathcal{B}$ = base point fibration

Again: If $\dim \mathcal{M}_L = 0$

$$M_{L\pm E} \cong \left[\hat{M}_L(X) \times \tilde{M}_{\pm E}(\overline{\mathbb{C}P^2})\right] / S^1$$

Recall, $\overline{\mathbb{C}P^2}$ has + scalar curvature & only soln on $\overline{\mathbb{C}P^2}$ is reducible: $(A_{\pm E}, 0)$.

So $\quad M_{L\pm E} \cong \tilde{M}_L(X) \times \{(A_{\pm E}, 0)\} / S^1 \cong M_L(X) \times \{(A_{\pm E}, 0)\}$

$$\cong M_L(X).$$

$$(A, \psi) \longleftrightarrow (A \# A_{\pm E}, \psi \# 0).$$

$\Rightarrow \quad n_{L\pm E}(X \# \overline{\mathbb{C}P^2}) = n_L(X) \quad$ when $d(L) = 0$.

More generally – Assume $d(L) = 2d \geq 0$ and $n_L = \langle h^d, [M_L] \rangle$

Let $k \ni k \cdot (k+1) \leq 2d$. So $\dim M_{L \pm (2k+1)E} = 2d - k(k+1) \geq 0$.

Shorthand : $\bar{L} = L \pm (2k+1)E$.

If $k > 0$ it is <u>not</u> <u>true</u> that $\tilde{M}_{\bar{L}} \cong \hat{M}_L(X) \times \tilde{M}_{\pm(2k+1)E}(\overline{\mathbb{C}P^2})$

because $\tilde{M}_{\pm(2k+1)E}(\overline{\mathbb{C}P^2}) = \{(A_{\pm(2k+1)E}, 0)\}$ a pt.

But $\dim \tilde{M}_{\bar{L}} = \dim \tilde{M}_L(X) - k(k+1)$.

How can we see this?

Get local model by examining the elliptic complex

$$0 \to \Omega^0 \to \Omega^1 \oplus \Gamma(W^+) \longrightarrow \Omega^2_+ \oplus \Gamma(W^-) \to 0$$

Our soln has the form $(A,\psi) * (\underset{\text{constant}}{A_{\pm(2k+1)E}}, 0)$

~~So a local model looks like $M_L(X)$ except that~~

Kuranishi map $\quad H^1 \xrightarrow{\Phi} H^2 \quad \ni \quad M_{\tilde{L}}(X\#\overline{CP}^2) \underset{\text{locally}}{\cong} \Phi^{-1}(0)$.

~~Using the [illegible] of $\tilde{G}^4 \times \ldots S^3$ we see that~~

Compare with Donaldson's Thm. 4.53 of "Connections, cohomology, ..."

$$\begin{array}{ccc} H^1_{\text{complex for }(X,L)} \oplus 0 & \xrightarrow{\Phi} & H^2_{\cancel{\text{complex}}\text{ for }(X,L)} \oplus \operatorname{coker} \not{D}_{G_2}. \\ \text{locally } \cong & & \uparrow 0 \qquad\qquad \Big| \\ M_L(X) & & \text{must be } \mathbb{C}^{k(k+1)/2 = N} \end{array}$$

Work in the based moduli space :

$$\begin{array}{ccc} \tilde{\xi} & \longrightarrow & \mathbb{C}^N = \operatorname{coker} \not{D} \\ \downarrow \uparrow \Phi & & \downarrow \\ \tilde{M}_L(X) \times \{pt\} & \longrightarrow & \{(A_{(2k+1)E}, 0)\} \end{array}$$

Kuranishi $\Rightarrow$ $\tilde{M}_{\tilde{L}}(X\#\overline{CP}^2) = \Phi^{-1}(0)$ for section Φ - <u>must be S^1-eqv't.</u>

ξ is trivial ! - not S^1-equivariantly though :

Divide out :

$$\begin{tikzcd} \tilde{\xi} = \tilde{M}_L(X) \times \mathbb{C}^N \arrow[d] \arrow[r, "/S^1"] & \tilde{M}_L(X) \times_{S^1} \mathbb{C}^N = \xi \arrow[d] \\ \tilde{M}_L(X) & M_L(X) \arrow[u, bend left, "\Phi"] \end{tikzcd}$$

Clearly, ξ assoc. to basept. fibr. $\Rightarrow$ $\xi = \oplus \beta^N$ where

$\beta =$ line bdle $\tilde{M}_L(X) \times_{S^1} \mathbb{C} \longrightarrow M_L(X)$.

$[M_{\bar{L}}(X \# \overline{CP}^2)] = [\Phi^{-1}(0)]$ in $B_{\bar{L}}(X \# \overline{CP}^2)$.

(viewing $M_L(X) \hookrightarrow B_{\bar{L}}(X \# \overline{CP}^2)$ by $(A,\psi) \longrightarrow (A \# A_{\text{above}}, \psi + 0)$)

Recall, $c_1(\mathbb{L})|_{x \times t} = \mathcal{O}_L \times \Gamma(W^+) \simeq \tilde{B}_L(X)$ $\because$ based gauge gp is contractible.

$$\begin{array}{c} \mathcal{O}_L \times \Gamma(W^+) \\ \downarrow \\ B_L(X) \end{array}$$

Thus, 0-set of generic section of β is $PD(c_1(\beta)) = PD(h) = V_x$ ← pt. divisor

Now $d(\bar{L})/2 = d - N$. $\quad$ ($d(L) = 2d \quad k(k+1) = 2N$)

So $\langle h^{d(\bar{L})/2}, [M_{\bar{L}}(X \# \overline{CP}^2)] \rangle = \langle h^{d-N}, [\Phi^{-1}(0)] \rangle$

$= \langle h^{d-N}, M_L(X) \cap (\bigcap_N V_x) \rangle = \langle h^d, [M_L(X)] \rangle = n_L(X)$

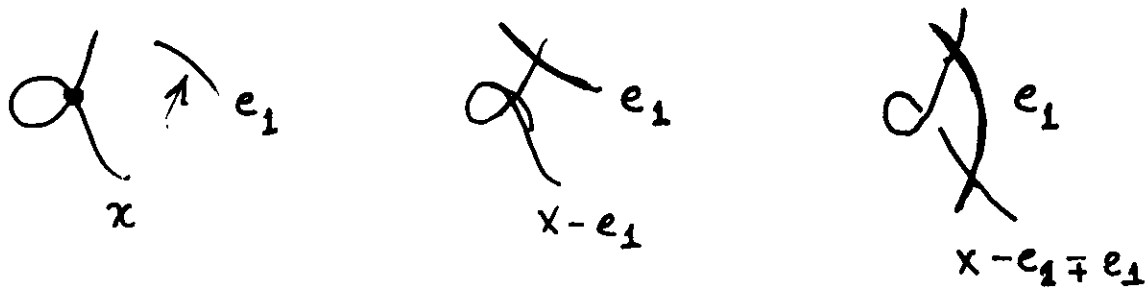
$\Rightarrow$ if $k \cdot (k+1) < 2d = d(L)$ then

$n_{L \pm (2k+1)E}(X \# \overline{CP}^2) = n_L(X)$. $\quad$ Blowup Formula

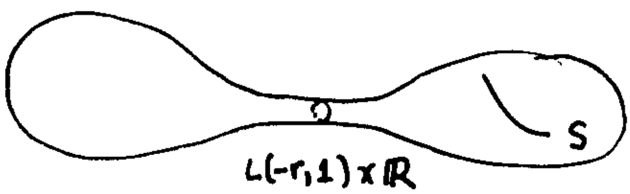
Thom conjecture for immersed S^2's + generalized adjunction formulae

<u>Observation</u>: $x \in H_2(X;\mathbb{Z})$ represented by immersed $S^2 \overset{\alpha}{\looparrowright} X$ with p positive double points and n negative " ".

Then $x - 2e_1 - 2e_2 - \cdots - 2e_p$ is represented by <u>embedded</u> S^2 in $X \#_p \overline{\mathbb{C}P}^2 \#_n \overline{\mathbb{C}P}^2$



So can "reduce" problems of immersed spheres to problems of <u>embedded</u> spheres (since we know blow up formulae



$S \cdot S = -r \leq 0$

Do as for blow-up:

Fundamental Lemma: $S^2 \hookrightarrow X$ $\quad S\cdot S = -r \leq 0$.

Suppose $\begin{matrix} L \\ \downarrow \\ X \end{matrix}$ characteristic, $n_L \neq 0$. Suppose

$|S\cdot L| = kr + R \quad k>0 \quad 0 \leq R \leq r-1$. Then

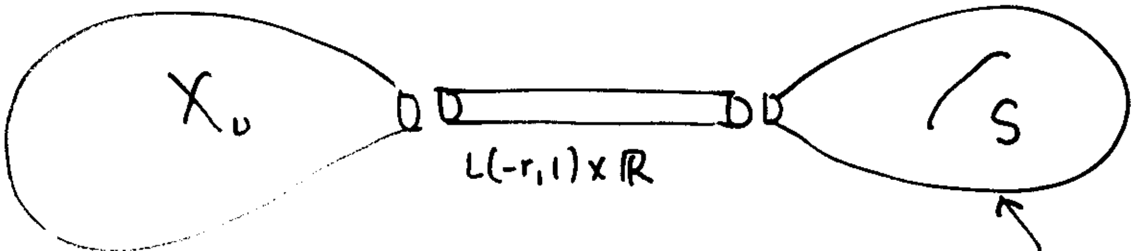
$$L' = \begin{cases} L+2S & L\cdot S>0 \\ L-2S & L\cdot S<0 \end{cases} \quad \text{has } n_{L'} = n_L.$$

and $\dim \mathcal{M}_{L'} = \dim \mathcal{M}_L + (k-1)r + R$. (increases!)

[Note: assumption $\Leftrightarrow$ $S\cdot S + |S\cdot L| \geq 0$ i.e. a violation of expected adjunction formula $2\cdot 0 - 2 \geq S^2 + |S\cdot L|$]

Pf: Say $S\cdot L > 0$ $\quad S\cdot L = kr+R$ – Stretch neck

neck has +'ve scalar curvature



$L+2S|_{X_0} = L|_{X_0}$; now use same argument as in blow up formula:

$\dim \mathcal{M}_{L|X_0} < 0$; here $\dim \mathcal{M}_{L+2S|X_0}$ is less negative

so can glue in.

(charge modification).

Get generalized adjunction formulas for immersed spheres by blowing up until embedded and using fundamental lemma.

Thm: Let $x \in H_2(X;\mathbb{Z})$ be rep by immersed S^2 with p positive double points. Let L char with $n_L \neq 0$ where $\dim \mathcal{M}_L = \sum_{i=1}^{\ell} r_i(r_i+1)$ $(\ell \leq p)$. Then either

- $2p-2 \geq x^2 + |x \cdot L| + 4\sum r_i$

or

- $n_{L \pm 2x} \neq 0$ and

$$2p \geq x^2 + |x \cdot L| + 4\sum r_i + \dim \mathcal{M}_L - \Delta_{max}$$

Were $\Delta_{max} = \max\{\dim \mathcal{M}_{L'} \mid L' \text{ char}, n_{L'} \neq 0\}$.

Note:
- no restrictions on sign of x^2
- works for $b_+ = 1$ (but must keep track of chamber).

Cor: Consider $X = \mathbb{C}P^2 \#_q \overline{\mathbb{C}P}^2$ and let $x = dH + \sum_1^q a_i E_i \in H_2(X;\mathbb{Z})$
If $d \geq 2$ and $d^2 - 3d \geq \sum_1^q (a_i^2 - a_i) + 2p$
Then x cannot be represented by an immersed S^2 with p pos. double points. (New; especially when $x^2 < 0$).

Special Case:
Cor: $(q=0)$ If dH rep by immersed S^2, then

$$p \geq \frac{(d-1)(d-2)}{2}.$$

(p = #positive double points)

<u>Immersed Thom Conjecture</u>:

$$dH \in H_2(\mathbb{C}P^2;\mathbb{Z}) \quad ; \quad p \geq \frac{(d-1)(d-2)}{2}$$

<u>Pf</u>: Suppose not:

$$p = \frac{(d-1)(d-2)}{2} - 1 = \frac{d^2-3d}{2}$$

① Blow up: $\mathbb{C}P^2 \#_p \overline{\mathbb{C}P}^2 \#_n \overline{\mathbb{C}P}^2$

- $\Sigma = dH - 2\sum_{i=1}^{p} E_i$ rep by sphere:
- $\Sigma^2 < 0$ (may need to blow up a few more times for small p)

Consider $K = 3H - \sum_{i=1}^{p+n} E_i$; $\cdot$ K characteristic

$\cdot$ $\dim \mathcal{M}_K = 0$

$$\begin{aligned}\dim \mathcal{M}_{K-2\Sigma} &= \dim \mathcal{M}_K + \Sigma\cdot\Sigma - \Sigma\cdot K \\ &= \dim \mathcal{M}_K + \Sigma\cdot\Sigma - (3d-2p) \\ &= \dim \mathcal{M}_K + (d^2-4p) - (3d-2p) \\ &= \dim \mathcal{M}_K + (d^2-3d-2p) = \dim \mathcal{M}_K = \underline{\underline{0}}.\end{aligned}$$

So <u>do</u> elementary charge conjugation! Oops: $\mathbb{C}P^2$ has +'ve scalar curvature! So $SW(K,g) \rightsquigarrow SW(K-2\Sigma,g)$

$$\boxed{\begin{aligned} SW_X(K,g) &= \begin{cases} 0 & \text{if } K\cdot\omega_g > 0 \\ \pm 1 & \text{if } K\cdot\omega_g < 0 \end{cases} \\ SW_X(K-2\Sigma,g) &= \begin{cases} \pm 1 & K\cdot\omega_g > 0 \\ 0 & K\cdot\omega_g < 0 \end{cases}\end{aligned}}$$

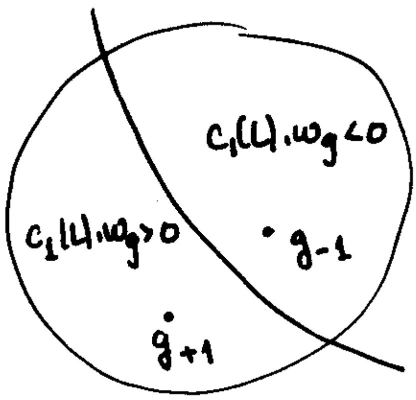
<u>Pf</u>: <u>Analyze chamber structure</u>:

$\Rightarrow \Leftarrow$!

$$SW(K,g) = \begin{cases} 0 & K\cdot\omega_g > 0 \\ \pm 1 & K\cdot\omega_g < 0 \end{cases}$$

$$SW(K-2\Sigma,g) = \begin{cases} \pm 1 & K\cdot\omega_g > 0 \\ 0 & K\cdot\omega_g < 0 \end{cases}$$

Metric g has ! <u>self-dual 2-form</u> ω_g



$c_1(L)$ admits g-a.s.d. connection iff $c_1(L)\cdot\omega_g = 0$ (since $\mathcal{H}^+_g \perp \mathcal{H}^-_g$).

So reducible solutions to SW equations can only occur if $\omega_g \in c_1(L)^\perp$

Thus $SW(L, g_t)$ any g_t takes on only two values

"Kuranishi model":

$$\dim SW_X(L, g_{-1}) = SW_X(L, g_{+1}) \pm 1.$$

In our <u>case</u>: g_+ has pos scalar curvature

$$SW(K, g_+) = SW(K-2\Sigma, g_+) = 0$$

But $K\cdot\omega_{g_+} \sim K\cdot H = 3 > 0$

$$(K-2\Sigma)\cdot\omega_{g_+} \sim (K-2\Sigma)\cdot H = 3-2d < 0 \quad (d \geq 2).$$

($\omega_{g_+} \sim H$ since Fubini-study has +'ve scalar curvature so does # sum. Take sequence of metrics $\{g_t\}$ shrinking necks in $\mathbb{CP}^2 \# \overline{\mathbb{CP}}^2 \# \cdots \# \overline{\mathbb{CP}}^2$ – get period point ω_{g_t} the self-dual to forms on $\mathbb{CP}^2 \amalg \overline{\mathbb{CP}}^2 \amalg \cdots \amalg \overline{\mathbb{CP}}^2$.)