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Cartan Geometry of Quantum Relativity
Witten's Laplacian and Ergodic Structures

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CARTAN GEOMETRY OF QUANTUM RELATIVITY WITTEN'S LAPLACIAN AND ERGODIC STRUCTURES

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Abstract - We present a geometrization of Relativistic Quantum Mechanics and a quantization of gravitation, in terms of the Riemann-Cartan-Weyl (RCW) geometries with Weyl-torsion and their associated diffusion processes. The central role is played by the Laplacian operator associated to the RCW geometries. We extend these diffusions on scalars to differential forms, and relate the RCW Laplacian with Witten's deformed Laplacian. The field equations for the RCW geometries are derived from a mean Cartan scalar curvature extremal principle which yields the coincidence between the quantum potential and $1/12R(g)$, $R(g)$ the metric scalar curvature. We introduce the quantum Perron-Frobenius semigroups of ergodic theory and the Lyapunov spectrum, associated to the flows generated by the RCW diffusions and give topological obstructions to the stability of these flows.

I Cartan Geometries and the Laplacian of Diffusion Processes

In this talk we shall deal with Markovian diffusion processes on a smooth space-time manifold M. For this, we need an invariant description of the most general second-order elliptic differential operator L acting on real functions on M.

Introduce an arbitrary Riemann-Cartan connection on M, whose covariant derivative we denote as ∇; assume ∇ is compatible with a Riemannian metric g on M. Consider the Laplacian operator on functions [2,3] $H(\nabla) = \frac{1}{2}tr(\nabla^2)$. Let $T^{\alpha}_{\beta\gamma} = \frac{1}{2}(\Gamma^{\alpha}_{\beta\gamma} - \Gamma^{\alpha}_{\gamma\beta})$ be the skew-symmetric torsion tensor of ∇; then:

$$H(\nabla) = \frac{1}{2}tr(\nabla^2) = \frac{1}{2}\triangle_g + \hat{Q}$$

with $Q = T^{\nu}_{\nu\beta}dx^{\beta}$, the trace-torsion one-form, $\hat{Q}$ the vector field conjugate to the 1-form Q: $\hat{Q}(f) =< Q, grad\ f > (x)$ and $\triangle_g$ the Laplace-Beltrami operator of g. Therefore,

$$L = \frac{1}{2}\triangle_g + \hat{Q} + L(1)$$

where 1 denotes the function on M constantly equal to 1. If we assume for a start that all the irreducible torsion components of ∇ vanish with the exception of the trace-torsion, we obtain, in general, a one-to-one correspondance between Cartan connections with torsion given by its trace component and Laplacians $H(\nabla)$. These restricted connections we call RCW (Riemann-Cartan-Weyl) connections, since Q plays the role of a Weyl 1-form, We are interested in the "trivial" case: $Q = d\ ln\psi, \psi : M \to R_{\geq 0}$; denote the corresponding Laplacian as $H(g,\psi) := \frac{1}{2}\triangle_g + \hat{Q}$. Assume the Markovian semigroups $\{P_\tau, \tau \geq 0\}$ generated by L preserves probability, then $L(1) = 0$, and we are left with a **geometrical** laplacian $H(g,\psi)$ for differential generator of the diffusion processes on M. This is the case we shall discuss.

The Ito representation for these diffusions is as follows [1].

By embedding M on R^d, with $d \leq 2n+1$, we can obtain a smooth section Y of $L(R^d, TM)$, so that if Y^* denotes the dual section of $L(T^*M, R^d)$, then for all $x \in M$,

$$g(x) = Y(x)Y^*(x). \tag{1}$$

Then, the representation is given by the stochastic differential eqt. (Ito):

$$dX_\tau = Y(x_\tau)dB_\tau + b(X_\tau)d\tau, b = grad\ ln\ \psi, \tag{2}$$

with B_τ a standard Wiener process on R^d. **Observation (Cartan Stochastic Method)**: The arbitrariness of the choice of Y, can be removed by constructing a random process on the orthogonal bundle, whose image by the bundle projection on M has $H(g,\psi)$ for differential generator. This is an extension to Wiener processes of Cartan's classical method.

II. Riemann-Cartan-Weyl Geometries and Quantum Mechanics

The transition density (heat kernel) $p_\psi(\tau, x, y)$ of the process generated by $H(g,\psi)$ is determined as the fundamental solution of the "heat" eqt. (τ is an internal time evolution parameter)

$$\frac{\partial u}{\partial \tau} = H(g,\psi)(x)u\ . \tag{3}$$

The semigroup $\{P_\tau : \tau \geq 0\}$ with differential generator $H(g,\psi)$ has a unique τ- independent **invariant** probability density determined as the fundamental weak solution (in the sense of the theory of generalized functions) of the τ-independent Fokker-Planck-Kolmogorov equation: $H(g,\psi)^\dagger(\rho) = 0$, where $H(g,\psi)$ is the formal $L^2(vol_g)$ adjoint of $H(g,\psi)$. Then, $H(g,\psi)^\dagger = 1/2\triangle_g - \text{div}_g$. One proves that $\rho = \psi^2\text{vol}_g$. It is important to remark that this density is a relaxation density for the Markov process, in the sense that when τ tends to infinity, $p_\psi(\tau, x, y)$ tends exponentially in τ, with x fixed on a compact set, to $\psi^2(y)$. Note that we must assume that $\psi \in L^2(vol_g)$, and thus the Markovian semigroup is defined on $C_c^\infty(M)$, the space of smooth compact supported functions on M, seen as a dense subset of the Hilbert space $L^2(M, \psi^2.vol_g)$. By Weyl's lemma [1], we take ψ to be everywhere positive and smooth.

We now associate with the diffusion process a Hamiltonean operator on the Hilbert space $L^2(\text{vol}_g)$ [2,11,16]. Denote still as $H(g,\psi)$ the unique self-adjoint Friedrichs extension of the RCW laplacian, defined by the choice of the domain $C_c(M)$. Define the inner product

$$< f_1, f_2 > = 1/2 \int g(\text{grad}f_1, \text{grad}f_2)\psi^2\text{vol}_g \tag{4}$$

Integration by parts yields

$$< f_1, f_2 > = -(f_1, H(g,\psi) f_2) \tag{5}$$

where $(.,.)$ denotes the inner product in terms of vol_g. Consider the quadratic form on $C_c(M)$, q associated to $< .,. >$, i.e. $q(f) =< f, f >$. q is then a Dirichlet form [16]. From eqt. (5) follows a unique Hamiltonean operator which generates q, $-H(g,\psi)$, which is then positive. Let us see how this construction is related to the usual formulation of Quantum Mechanics in terms of quadratic forms in $L^2(\mathrm{vol}_g)$. Consider the mapping $C_\psi : L^2(\mathrm{vol}_g) \to L^2(\psi^2 \mathrm{vol}_g)$ defined by multiplication by by ψ^{-1}; this mapping carries $C_c(M)$ in $C_c(M)$. For any f in $C_c(M)$ we have

$$q(\psi^{-1} f) = 1/2 \int f\{-\Delta_g + V\} f \mathrm{vol}_g = (f, Hf),, \tag{6}$$

with $b = \mathrm{grad} ln\psi$ and $H = C_{\psi^{-1}} \circ H(g,\psi) \circ C_\psi = -1/2\Delta_g + V$, where in the weak sense:

$$V = 1/2(\mathrm{div}_g\ b + g(b,b)) = \Delta_g \psi / 2\psi, \tag{7}$$

is the **relativistic quantum potential.** Thus we have proved that $-H(g,\psi)$ is unitarily equivalent to the Hamiltonean operator $H = -1/2\Delta_g + V$ defined on $L^2(\mathrm{vol}_g)$ and ψ is a generalized groundstate eigenfunction of H with 0 eigenvalue. Note the non-linear dependence of V on the invariant density introduced by ψ. We shall see below that this dependence is removed due to conformal invariance at the level of the field equations.

III. The Mean Curvature Extremal Principle

We start with a general Riemann-Cartan connection, ∇ and we consider its scalar curvature $R(\nabla)$ Consider the conformal invariant functional [3]

$$\int R(\Gamma) \phi^2 \mathrm{vol}_g. \tag{8}$$

We obtain

$$T^\gamma_{\alpha\beta} = \delta^\gamma_\alpha\ \partial_\beta\ ln\phi - \ \delta^\gamma_\beta \partial_\alpha\ ln\phi, \tag{9}$$

so that, up to normalization, $Q = d\ ln\phi$. The field equations are

$$G_{\alpha\beta}(g) = -6/\phi^2 T_{\alpha\beta}(\phi), \tag{10}$$

with $G_{\alpha\beta}(g)$ the Einstein **metric** tensor, and $T_{\alpha\beta}(\phi)$ the improved energy-momentum density of the field ϕ. Taking the trace in (10) we get

$$H\phi := (\Delta_g - 1/6 R(g))\phi = 0, \tag{11}$$

so that ϕ is a generalized groundstate of the conformal invariant wave operator defined on $L^2(\mathrm{vol}_g)$. Solving this equation for ϕ positive of class C^2, as a weak equation (we can assume

$R(g)$ is locally integrable and g of class C^1 [16]), we obtain a Dirichlet form with Hamiltonean given by $-H$, self-adjoint positive on $L^2(vol_g)$, which is conformally conjugate to $-H(g,\psi)$.

As a result of this, if we denote by $p(\tau,x,y)$ the heat kernel of the diffusion process generated by H, its relation with $p_\psi(\tau,x,y)$ of (3) is: $p_\psi(\tau,x,y)=\psi^{-1}(x)\psi^{-1}(y)p(\tau,x,y)$.
Observation: From (11) we conclude that the quantum potential is $1/12R(g)$ which certainly **does not** depend on the scalar field ϕ at all. This is fundamental for the above transformation of representation from $L^2(vol_g)$ to $L^2(\psi^2 vol_g)$.

IV. The Classical Most Probable Realizations of the Quantum Motions

We shall determine classical gravitational motions **from** the quantum motions $\{X(\tau):\tau\geq 0\}$ given by (2) [3]. Let $\{P_x\,,\ x\in M\}$ be the Markovian system on M determined by $\{P_\tau,\tau\geq 0\}:P_x[X(\tau)\in\mathcal{B}\}=p(\tau,x,\mathcal{B})$, $\mathcal{B}$ a Borel measurable set of M. Take an arbitrary **smooth** curve on M, $\varphi:[0,\infty)\to M$, $\varphi(0)=x$, and consider the "tube" of radius ε centered on φ, $\varepsilon>0$, of all quantum motions $X(\tau)$ starting at x, $\tau\in[0,T]$, $T>0$,

$$T_\varepsilon(\varphi)=\left\{X(\tau):X(0)=x,\ \sup_{\tau\in[0,T]}\ \|X(\tau)-\varphi(\tau)\|\ <\varepsilon\right\}\ , \tag{12}$$

where $\|\ \ \|$ denotes the Riemannian distance. We are interested in the asymptotic expression of $P_x[T_\varepsilon(\varphi)]$ as $\varepsilon\to 0^+$. This expression is derived in the theory of large deviations in probability theory [4] and is fundamental to the semiclassical approach to quantum gravity [6]. From an intuitive point of view, it has been developed as a kind of infinite dimensional version of the method of stationary phase for the evaluation of path integrals in Minkowski space quantum field theory [5]. We have

$$P_x[T_\varepsilon(\varphi)]\approx e^{-\lambda_1 T/\varepsilon^2}\ e^{-\int_0^T 1/2L(\varphi,\dot\varphi)(s)\ ds}\ , \tag{13}$$

where L is the so-called Onsager-Machlup lagrangian or probability functional on paths [4]

$$L(\varphi,\dot\varphi)=\|b-\dot\varphi\|^2+\mathrm{div}_g b-1/6R(g)\ . \tag{14}$$

From (11) we finally get that L takes the reduced modified kinetic energy form

$$1/2L(\varphi,\dot\varphi)=1/2\ \|\dot\varphi\|^2-<\dot\varphi,b>\ . \tag{15}$$

To obtain the classical realizations, φ, of $\{X(\tau):\tau\geq 0\}$, we extremize (15), so that they are the **most probable** realizations of the quantum motions. Note that whenever b is orthogonal to the classical velocity vector field $\dot\varphi$ we obtain the geodesic flow of g as the most probable approximation. Otherwise, we get a deviation of geodesic flow due to the torsion drift vector field $b=grad\ ln\ \psi$.

Observation : It is well known that relativistic neutral **spinless test** -particles are uninfluenced by the torsion of the background Cartan geometry [17] and follow the geodesic flow determined by the metric of the Cartan connection. Here we see that in contrast with this, the realization of the quantum motions deviate from the geodesic flow due to $b = \text{grad } ln\psi$. As shown above, this is due to the fact that from the conformal invariance of the field equations, the **relativistic quantum potential** $1/2\left(\|b\|^2 + \text{div}_g b\right) = \triangle_g\psi/2\psi$, which expresses the non-linear and non-local (in the sense of Einstein-Podolsky-Rosen) dependence of the quantum system on the square root of the invariant density, **equals** $1/12R(g)$, which no longer depends on the density at all!. This sheds new light on the phenomenology of quantum correlations produced by the quantum potential [10]: they are mediated by the metric scalar curvature.

V. The Perron-Frobenius Semigroups of Quantum Relativity

The construction of the ergodic theory of the quantum flows associated to a RCW geometry, resides in the fact that under weak analytical conditions on Y and $\hat{Q}$, the solution flow of (2) exists and is unique and it defines a **diffeomorphism** of M. We shall assume for simplicity that M is compact.

The flow of the s.d.e. (2), is a mapping $F_\tau : M \times \Omega \to M$, $\tau \geq 0$, such that for each $\omega \in \Omega = \{\omega : [0,\infty) \to R^n,\ \omega(0) = 0, \omega \text{ is continuous}\}$, $F(.\ ,\omega) : [0,\infty) \times M \to M$ is continuous and such that $\{F_\tau(x) : \tau \geq 0\}$ is a solution of (2) with $F_0(x) = x$, for any $x \in M$. If we now assume that all components of Y, and b, lie in appropiate Sobolev spaces, by the embedding theorem we obtain a diffeomorphism in M: $F_\tau(\omega) : M \to M$, $F_\tau(\omega)(x) = F_\tau(x,\omega)$, almost surely for $\tau \geq 0$ and $\omega \in \Omega$, of appropiate degree of differentiablity. In the following these analytical conditions are assumed [14].

Recall now the basic structures of the ergodic theory of dynamical systems [7]. Consider a dynamical system $\{\Theta_{\tau:\tau\geq 0\}}$ defined on a topological space Y, $\Theta_\tau : Y \to Y$, for any $\tau \geq 0$ verifying the semigroup composition property: $\Theta_{\tau+\tau'} = \Theta_\tau \circ \Theta_{\tau'}$. Example: The solution flows of ordinary differential equations on Y a smooth manifold, an instance of which is the geodesic flow of a metric on Y. In these cases, $\Theta_\tau(x)$ is the position at time τ of the integral curve of the ordinary differential equation, where the initial value has been fixed. We need more structure on Y, namely a σ algebra of sets $\mathcal{F}$ on Y such that the inverse image by Θ_τ of an arbitrary measurable set $F \in \mathcal{F}$ is a measurable set in $\mathcal{F}$. We further require the existance of a probability measure μ on Y, i.e.: $\mu(Y) = 1$, which is invariant by the flow $\{\Theta_\tau : \tau \geq 0\}$, i.e. $\mu(\Theta_\tau^{-1}(F)) = \mu(F)$, for any $F \in \mathcal{F}$, and $\tau \geq 0$. The role of this invariant measure is that of an **equilibrium** measure on Y. We consider the triple $(Y, \mathcal{F}, \mu)$, which in the language of the theory of dynamical systems is called the *phase space* of the dynamical system $\{\Theta_\tau, \tau \geq 0\}$. Then one introduces the following semigroups. Firstly, let $f \in L^\infty(Y)$; we define the Koopman operators: $(V_\tau f)(y) = f(\Theta_\tau(y))$, for any $\tau \geq 0$; then we have the semigroup property: $V_{\tau+\tau'} = V_\tau \circ V_{\tau'}$ on $L^\infty(Y)$. We finally introduce the Perron-Frobenius operators, $U_\tau, \tau \geq 0$: for any r a density on Y and $F \in \mathcal{F}$,

$$\int_F (U_\tau r)(y) d\mu(y) := \int_{\Theta_\tau^{-1}(F)} r(y) d\mu(y), \tag{16}$$

which also defines a semigroup: the *Perron-Frobenius (PF)* semigroup of operators. With respect to the pairing on Y defined by the measure μ, these semigroups are adjoint, i.e.: $U_\tau^\dagger = V_\tau$, for any $\tau \geq 0$.

For diffusions processes, in which quantum motions are described by s.d.eqts. (2) we wish to introduce similar structures in terms of the stochastic flow defined by integrating eqts. (2). Yet, for these flows, the usual composition rules are unvalid. To lift this problem, one needs to consider the enlarged space $Y = M \times \Omega$, and the mapping

$$\Theta_\tau : Y \to Y, \Theta_\tau(x,\omega) := (F_\tau(\omega)(x), \theta_\tau(\omega)), (x,\omega) \in M \times \Omega,$$

where $\Omega = \{\omega \in C^0([0,\infty) \to M), \omega(0) = x,\}$ is Wiener space provided with the Wiener measure P amd its σ-algebra, and $\theta_\tau(\omega)(s) = \omega(\tau + s) - \omega(s)$, for any $\omega \in \Omega$. Then, $\Theta_{\tau+s} = \Theta_\tau \circ \Theta_s, \tau, s \geq 0, a.s.$

Furthermore, if ρ is an invariant measure for the Markovian semigroup associated to the s.d.e., then the product measure $\mu = \rho \otimes P$ is **invariant** by the flow: $\mu(\Theta_\tau^{-1}(B \times \Lambda)) = \mu(B \times \Lambda)$, for any Borel measurable sets $B \in \mathcal{B}(M), \Lambda \in \mathcal{B}(\Omega)$.

Consider the triple $(Y, \mathcal{F}, \mu)$, where $\mathcal{F}$ is the σ-algebra $\mathcal{B}(M) \times \mathcal{B}(\Omega)$, the product of σ algebras of measurable sets on M and Ω respectively, $Y = M \times \Omega$ and $\mu = \rho \otimes P$. $(Y, \mathcal{F}, \mu)$ is a *stochastic phase space* with μ a Θ_τ-invariant measure representing an equilibrium measure. Introduce the *stochastic Koopman semigroup* of operators: $(V_\tau f)(y) = f(\Theta_\tau(y)), y \in Y, f \in L^\infty(Y)$ and a *stochastic PF semigroup* defined on the probability densities r on Y by (16). Then, $V_\tau^\dagger = U_\tau, \tau \geq 0$.

In the case of R.Q.M. as constructed in Section **I**, for a compact space-time manifold, M, and $H(g,\psi)$ is an elliptic operator with $\rho(B) = \int_B \psi^2 \mathrm{vol}_g$, $B \in \mathcal{B}(M)$, the unique invariant measure of the quantum flow defined by integrating eq. (2) we have a stochastic PF semigroup determined by this flow [11]. Therefore, from the fact that stochastic flows have the diffeomorphic property alike classical smooth flows, we have a stochastic covariant dynamics on M enlarged by the canonical shift of Ω. Yet, in this setting one is interested in the M-part of the stochastic flow while its Ω-part is considered to be inaccesible [11]. **Remark:** To resume, for the construction of an ergodic theory of the quantum flows generated by $H(g,\psi)$, all what is needed is the Wiener measure P and the "Born" measure ρ. on the M-part of the flow.

In our lecture, we shall introduce the notion of ergodicity of the system with respect to ρ, its Lyapunov spectrum, give a formula for the computation which is valid for the highest exponent, introduce the notion of stability [18] and study the relation between the latter and the contribution of the torsion. Yet, we shall need a stronger condition than stability which appears to be related to Witten's deformed laplacian in his proof of the generalized Morse inequalities.

V. Witten's deformed laplacian and the RCW stochastic flows

Assume in the following M is a smooth n-dimensional orientable compact manifold provided with a Riemannian metric, g and the canonical volume element, $\mathrm{vol}_(g)$; consider the Hilbert space of square summable ω of differential forms of degree q on M, with respect to vol_g. Denote this space as $L^{2,q}$ and the inner product as $< .,. >$

The de Rham-Kodaira operator on $L^{2,q}$ is $\triangle = -(d + \delta)^2 = -(d\delta + \delta d)$ where δ is the formal adjoint of d defined on $L^{2,q+1}$. For $q = 0$ this is the Laplace-Beltrami operator on functions encountered before; in the general case we have in addition to $tr(\nabla^g)^2$ the contribution of the Weitzenbock curvature term. Let us assume that the real valued function on M ψ is smooth and everywhere positive. We then have an induced smoooth density $\rho = \psi^2 \mathrm{vol}_g$ on M.

Introduce the Hilbert space $L^{2,q,\rho} = L^2\Omega^q(M,\rho)$, of differential forms on M of degree q, square integrable with respect to ρ, with inner product:

$$< \phi_1, \phi_2 >^\rho = \int_M < \phi_1(x), \phi_2(x) > \rho, \tag{17}$$

for $\phi_1, \phi_2 \in L^{2,q,\rho}$. Consider the quadratic form $q(\phi) =< \phi, \phi >^\rho$, with ϕ on the Hilbert space given by the completion of the space of all smooth q-forms under the $L^{2,\rho}$ inner product. In the case of exact one-forms, this is twice the quadratic form defined by $H(g,\psi)$.

The formal adjoint of d the operator δ^ψ $< \delta^\psi \omega, \phi >^\rho =< \omega, d\phi >^\rho$, (18) for any $\omega \in L^{2,q+1,\rho}$ and $\phi \in L^{2,q,\rho}$. Since $d^2 = 0$, we have $(\delta^\psi)^2 = 0$. Introduce the operator on $L^{2,q,\rho}$: $\triangle^{\psi,q} = -(d + \delta^\psi)^2$, which equals to $-(d\delta^\psi + \delta^\psi d)$. We have:

$$\triangle^{\psi,q} = \triangle^q + 2L_{\mathrm{grad}\ ln\psi}, \tag{19}$$

Define now the deformed exterior differential operator mapping $L^{2,q}$-forms in $L^{2,q+1}$-forms, by:

$$d^\psi = \psi d \psi^{-1}, \tag{20}$$

Then, $(d^\psi)^2 = 0$. This operator is the $\tau = -1$ version of Witten's deformed differential [12]. The deformed co-differential operator $(d^\psi)^*$ is the formal adjoint of d^ψ: $(d^\psi)^* = \psi^{-1}\delta\psi$. Finally, introduce the deformed Laplacian operator due to Witten [12], defined as:

$$L^\psi = -(d^\psi + d^{\psi *})^2, \tag{21}$$

which can still be written as

$$-(d^\psi d^{\psi *} + d^{\psi *} d).$$

We have the following relation between the two Laplacian operators on q-forms ($q = 0, \ldots, n$)

$$\triangle^{\psi,q} = \psi^{-1} L^{\psi,q} \psi, \tag{22}$$

so that these two operators are conformally equivalent under conjugation by ψ. Note that $\triangle^{\psi,0} = 2H(g,\psi)$.

Since $(d^\psi)^2 = 0$, we define the deformed de Rham complex: $H^q_\psi(M,R) = Ker(d^\psi : \Lambda^q \to \Lambda^{q+1})/Ran(d^\psi : \Lambda^{q-1} \to \Lambda^q)$; then, $H^q_\psi(M,R) \cong H^q(M,R)$, for any $q = 0, \ldots, n$. Therefore, by Hodge'theorem: $dim(Ker(\triangle^q)) = dim(Ker(L^{\psi,q})) = dim(Ker\triangle^{\psi,q})$. This is be essential for the proof of the Theorem below.

Define an average exponent of the flow F_τ generated by $H(g,\psi)$. For $p \in R$ define

$$\mu_x(p) = lim_{\tau\to\infty} \frac{1}{\tau} ln\, \mathbb{E}\, \|T_x F_\tau\|^p.$$

We shall say that the flow arising form (2) generated by $H(g,\psi)$ is **moment stable** if $\mu_x(1) < 0$, for ρ a.e. $x \in M$.

Theorem: Assume M compact with $H^1(M,R)$. Let g a smooth metric and a C^2 wave positive function ψ determining the RCW Laplacian $H(g,\psi)$. Then the flow generated by $H(g,\psi)$ with unique invariant density ρ, is not moment stable.

Remark This result links long time behavior of the flow with the eigenvalues of $\triangle^{\psi,1}$.

Conclusions

We have set a geometrical theory of diffusions which extends the usual association between diffusions and Riemannian geometry. In constructing the most general diffusion process on a manifold, torsion represents the average displacement of the Brownian process, and thus it cannot be neglected but for the standard Wiener process. When one goes from the formulation of the theory in the Hilbert space given by the groundstate measure to the usual Hilbert space given by the canonical volume form, onė loses the information on the torsion which now appears encoded in the quantum potential which equals $\frac{1}{12}R(g)$. Thus, the problem if whether the geometry of quantum gravitation is Cartanian or Riemannian, is here linked with the choice of Hilbert space, the latter corresponding to the conformal invariant (metric) wave operator. This presentation has allowed us to derive the classical realization of the diffusions from the lagrangian on the most probable paths; thus, our derivation is a kind of probabilistic extension of the derivation of the equations of motion from the Einstein lagrangian, as conceived originally by Einstein et al [9]. We have found that the group of diffeomorphisms is present still at the level of the quantum diffusions. We have introduced the basic structures for the development of the ergodic theory of these diffusions, and still linked their stability with a topological obstruction related to the kernel of Witten's deformed laplacian, or equivalently, to the kernel of its conformal conjugate. Yet, for doing this, no condition on ψ of being a Morse function is necessary. We can go one step further in relating the quantum diffusions with the properties of classical dynamical systems: If the adjoint semigroups to the $P_\tau : \tau \geq 0$ are exact semigroups, i.e. they are isometric, then they can be intertwined with classical dynamical systems which have the Kolmogorov property [15].

The core of this lecture's thesis is, **Quantization is Geometry**. A similar thesis has been proposed recently by Klauder [13]. Yet, in this approach the association between quantization and geometry is realized in phase-space by augmenting the symplectic structure with metrics on space-phase, which are rather arbitrary. Contrarily, in our presentation all the information is encoded into the RCW Laplacian operator, and thus the probabilistic features become dependant on it.

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