UNITED NATIONS EDUCATIONAL, SCIENTIFIC AND CULTURAL ORGANIZATION
INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS
I.C.T.P., P.O. BOX 586, 34100 TRIESTE, ITALY, CABLE: CENTRATOM TRIESTE

SMR.847/3

CONFERENCE ON

TOPOLOGICAL AND GEOMETRICAL PROBLEMS

RELATED TO QUANTUM FIELD THEORY

(13 - 24 March 1995)

An algebraic geometry formulation of a model quantum field theory on a curve

A.K RAINA
Theoretical Physics Group
Tata Institute of Fundamental Research
Homi Bhabha Road
Bombay, 400 005
India

MAIN BUILDING STRADA COSTIERA, 11 TEL. 2240111 TELEFAX 224163 TELEX 460392 ADRIATICO GUEST HOUSE VIA GRIGNANO, 9 TEL. 224241 TELEFAX 224531 TELEX 460449
MICROPROCESSOR LAB. VIA BEIRUT, 31 TEL. 224471 TELEFAX 224600 TELEX 460392 GALILEO GUEST HOUSE VIA BEIRUT, 7 TEL. 22401 TELEFAX 2240310 TELEX 460392

An algebraic geometry formulation of a model quantum field theory on a curve*

A. K. Raina
Theoretical Physics Group
Tata Institute of Fundamental Research
Homi Bhabha Road, Bombay, INDIA 400 005
e-mail : raina@theory.tifr.res.in

Abstract

We describe an approach to the $b-c$ system on a compact Riemann surface, based on investigating the dynamical content of its *operator product expansion* (OPE). The analyticity constraints implicit in the OPE are brought out using techniques of modern algebraic geometry. Algebraic geometry is shown to provide a natural language to describe the system as well as powerful computational tools. The entire structure of the system finds a natural description in this way and all the correlation functions can be determined rigorously and explicitly. The current correlation functions are also obtained from the field correlation functions using *ringed spaces with nilpotent elements.* This provides a *global geometric formulation* of the problem of *normal-ordering* the product of the fields b and c. A geometric formulation of the energy-momentum tensor of the system is also provided. Algebraic geometry proofs of classical formulas in function theory on a Riemann surface are obtained in the course of this study.

*invited talk in the conference *Topological and Geometrical Problems related to Quantum Field Theory*, Trieste, Italy, 13-24 March, 1995.

1 Geometric formulation of the system

Our system consists of a pair of "quantum fields" b and c. The quotes indicate that we do not define what we mean by a quantum field, since it is a very singular object and extremely difficult to deal with mathematically with any rigour. Instead we shall always deal with certain well-behaved functionals of the fields, called the *correlation functions* of the system.

We denote the general correlation function $C(m,n)$ by the *symbolic* expression

$$C(m,n) \quad = \quad < b(Q_1)\dots b(Q_m)c(P_1)\dots c(P_n) > \tag{1.1}$$

where $Q_1,\dots,Q_m,P_1,\dots,P_n$ are points on M. Intuitively, of course, such a correlation function should give the expectation value of finding m particles of the field b and n of the field c at $Q_1,\dots,Q_m$ and $P_1,\dots,P_n$ respectively, in their ground state. It is difficult, however, to sustain a serious physical interpretation of the correlation functions in view of the apparent artificiality of the model. In fact, the model does arise naturally in intermediate steps in string theory as Faddeev-Popov ghost fields. The rather pedestrian origin, however, does not give any idea of the remarkable interest of this model, firstly from the viewpoint of mathematical physics and, secondly, for the study of conformal field theory, which has applications in "real physics" to the theory of phase transitions in two dimensional solids.

We shall, accordingly, regard the correlation functions $C(m,n)$ simply as a set of functionals which specify the model, if they can be determined. What is more, the $C(m,n)$ will be *sections of holomorphic line bundles* (though not necessarily holomorphic sections) rather than functions. The analyticity properties of these sections are constrained by the *operator product expansion* (OPE) of the b and c fields, which is given by the heuristic relation [1]

$$b(z)c(w) = \frac{I}{z-w} + \text{holomorphic terms}, \tag{1.2}$$

where I is the identity operator and this relation is supposed to make sense only inside one of the $C(m,n)$.

We shall now interpret $C(m,n)$ as a (meromorphic) section of some holomorphic line bundle α on M in each P-argument and of some holomorphic line bundle β on M in each Q-argument. Thus $C(m,n)$ is a meromorphic section of the line bundle

$$p_1^*(\beta)\otimes\cdots\otimes p_m^*(\beta)\otimes p_{m+1}^*(\alpha)\otimes\cdots\otimes p_{m+n}^*(\alpha) \tag{1.3}$$

on $M^{m+n}\equiv M_1\times\ldots\times M_{m+n}$, the cartesian product of $m+n$ copies of M, where $p_i : M^{m+n}\to M_i$ is the i-th canonical projection. The OPE (1.2) will now be interpreted as saying that the only singularities of $C(m,n)$ are simple poles when a Q and a P argument coincide. *We* *must emphasise that this global principle in no way restricts the possible singularities.* All it means is that we require that *any singularity, other than the physical ones coming from the OPE (1.2), must be forced on us by rigorous mathematical analysis.* (See the discussion of the spin $(1-J)-J$ system below). Our principle has a certain resemblance to the "principle of maximal analyticity" in vogue many years ago.

Our first conclusion from this is that the "one point functions" $C(1,0) = < b >$, $C(0,1) = < c >$ are simply holomorphic sections of β and α respectively. A physicist who was studying this system starting from a lagrangian would call these one point functions the *zero modes* of the system and would want to eliminate them. Starting from the OPE (1.2) as we are doing, however, there is no reason *up to now* to disallow them by putting constraints on the line bundles α and β.

The first nontrivial case is that of the "two point function" $C(1,1) = < b(Q)c(P) >$. The OPE (1.2) suggests the following definition :

Definition 1.1 *$< bc >$ is a meromorphic section of the line bundle $p_1^*(\beta)\otimes p_2^*(\alpha)$ whose only singularity is a simple pole along the diagonal Δ of $M\times M$.*

It is easy to see that a non-zero two point function exists if and only if the map i in the exact sequence

$$O \rightarrow H^0(M\times M, p_1^*(\beta)\otimes p_2^*(\alpha)) \xrightarrow{i} H^0(M\times M, p_1^*(\beta)\otimes p_2^*(\alpha)\otimes\mathcal{O}(\Delta)) \quad (1.4)$$

is not an isomorphism. So far the line bundles α and β have been completely arbitrary. We cannot, of course, expect to get anything interesting without some restriction, but it is important to make these restrictions as weak as possible. The optimum condition turns out to be to simply bound the sum of the *degrees* of the two line bundles by $2g-2$. We then have the following elegant characterisation of the $b-c$ system (see [2] for the proof):

Theorem 1.2 *Let $deg(\alpha)+deg(\beta)\leq 2g-2$. Then the two point function $< bc >$ exists if and only if :*
(i) $\beta\otimes\alpha = K \equiv$ the holomorphic cotangent bundle of M
(ii) $deg(\alpha) = g-1 = deg(\beta)$
(iii) neither α nor β have any holomorphic sections.
If these conditions hold then the two point function not only exists, but it is also unique (after normalisation).

Thus we see that simply requiring the existence of a non-zero two point function imposes very stringent conditions on the line bundles α and β, even though the degree condition we imposed is very much weaker than the conditions one would be led to impose in a field-theoretic approach. In fact *we can use Theorem 1.2 to understand the conventional formulation of the model.* Thus condition (i) is precisely the condition that the integrand of the standard action for the $b - c$ system, viz. $S \sim \int_M b\bar{\partial}c$, is indeed a volume form, as it must be for the integration over M to make sense. Condition (ii) means that we are in the case when b and c are fields of conformal spin 1/2, or rather a "twisted" version of it since we do not require $\beta = \alpha = \sqrt{K}$. Finally, condition (iii) says that zero modes must be absent. However, we now see that this is necessary in order to have a two point function whose singularity structure is determined by the OPE (1.2), rather than because some undefined functional integral will otherwise give trouble, as is usually argued !

It may seem that we are excluding from consideration the spin $(1-J)-J$ version of the $b - c$ system (J is a positive integer or half-integer) which is usually considered in the literature. However, that is not the case. For let us take $deg(\alpha) = 2J(g-1)$. The Riemann-Roch theorem tells us that α has holomorphic sections ("zero modes") if $J \geq 1$. Theorem 1.2 asserts that in that case *the two point function $< bc >$ must have extra singularities not coming from the OPE (1.2) and these extra singularities must be such that we obtain a new $b - c$ system which does satisfy the conditions of the theorem.* Thus one way is to introduce points $x_1, \ldots, x_I$, where $I = (2J-1)(g-1)$ and let D denote the divisor $x_1 + \cdots + x_I$. Define $\tilde{\alpha} \equiv \alpha \otimes \mathcal{O}(-D), \tilde{\beta} \equiv \beta \otimes \mathcal{O}(D)$. Then for $\tilde{\alpha}$ and $\tilde{\beta}$ we have the required properties $deg(\tilde{\alpha}) = g - 1 = deg(\tilde{\beta})$ and $\tilde{\alpha} \otimes \tilde{\beta} = \alpha \otimes \beta = K$. Condition (iii) of Theorem 1.2 will also be satisfied for a generic choice of the points $\{x_i, i = 1, \ldots, I\}$. This is *effectively* how physicists handle the spin $(1-J)-J$ case of the $b - c$ system. For further details we refer to [3].

Another interesting consequence of Theorem 1.2 is that it provides a proof of one of the folk theorems of the physics literature, viz. a kind of "charge conservation theorem" for "spin fields" (more generally, for "twist fields"). We are given pairs of points and rational numbers $\{x_i, \mu_i \mid 1 \leq i \leq N_+\}$, $\{y_j, -\nu_j \mid 1 \leq j \leq N_-\}$, where the x_i, y_j are points on M. The μ_i, ν_j are positive rational numbers which satisfy the constraint $\sum_{i=1}^{N_+} \mu_i - \sum_{j=1}^{N_-} \nu_j = \ell$ (ℓ is a positive or negative integer called the "total twist") and which describe the monodromy of the b and c fields near the corresponding points :

$$\begin{aligned} b(z) &\sim (z - x_i)^{-\mu_i} & c(z) &\sim (z - x_i)^{\mu_i} \\ &\sim (z - y_j)^{\nu_j} & &\sim (z - y_j)^{-\nu_j} \end{aligned} \qquad (1.5)$$

Then with the help of techniques from algebraic geometry we can reduce the problem of studying the $b - c$ system in the presence of such a "twist structure" to the generalised system of Theorem 1.2 on a finite cyclic covering $\tilde{M} \to M$, defined by a positive divisor D of M and a line bundle $\mathcal{L}$ such that $\mathcal{L}^{\otimes d} = \mathcal{O}_M(D)$, where d is the degree of the cyclic covering. *Theorem 1.2 then implies that the total twist ℓ must be zero.* In the case of spin fields this immediately implies that for a nonzero two point function we must have as many spin fields with a positive square-root behaviour as with negative, a well known folk theorem [4]. For details of the construction and proofs we refer to [2].

2 Field correlation functions

In the previous section we saw that algebraic geometry helped us to achieve a rather detailed qualitative understanding of the $b-c$ system from its OPE (1.2). In fact, algebraic geometry enables us to do much more and we will need no more input from physics (apart from the question of statistics). We shall from now on assume that the line bundles α and β satisfy the conditions of Theorem 1.2, i.e. that $\alpha \in Pic^{g-1}(M)$ and has no holomorphic sections and that $\beta = K \otimes \alpha^{-1}$. We shall not give any further discussion of the spin $(1-J)-J$ system, for which we refer to [3]. We can now obtain an explicit expression for the two point function $< bc >$ with the help of the following lemma :

Lemma 2.1 *Let $\mathcal{M}_\zeta(1,1) \equiv p_1^*(K \otimes \zeta^{-1}) \otimes p_2^*(\zeta) \otimes \mathcal{O}(\Delta)$, where $deg(\zeta) = g-1$. Then*
(i) if $g = 0$, $\mathcal{M}_\zeta(1,1)$ is the trivial line bundle on $M \times M$,
(ii) if $g \geq 1$, $\mathcal{M}_\zeta(1,1) = \pi_\zeta^*(\mathcal{O}(\Theta))$ where $\pi_\zeta : M \times M \to Pic^{g-1}(M)$ is given by $(Q,P) \mapsto \mathcal{O}(Q-P) \otimes \zeta$. Here Θ denotes the canonical theta divisor (in $Pic^{g-1}(M)$). (This becomes a translate of the usual theta divisor by the Riemann constant once a marking is chosen on M, which defines a Riemann matrix in canonical form).

We also need the concept of the "prime form" $E(Q,P)$ for which we have found it convenient to introduce a new *algebro-geometric* definition :

Definition 2.2 *We define the prime form to be the image of the canonical element $1 \in \mathcal{O}_{M\times M}$ in the exact sequence :*

$$0 \longrightarrow \mathcal{O}_{M\times M} \stackrel{1 \mapsto E(Q,P)}{\longrightarrow} \mathcal{O}_{M\times M}(\Delta) \tag{2.1}$$

This definition of the prime form $E(Q,P)$ can be related to the usual function-theoretic definitions in various genera to be found in the books of Fay[7] and Mumford[8] with the help of Lemma 2.1. Then by once again using Lemma 2.1, we can obtain the two point function $< bc >$ explicitly and for $g \geq 1$ it coincides with the *Szegö kernel for a compact Riemann surface*, which was introduced by Hawley and Schiffer [5]. We refer to [2] and [6] for details of our approach.

The question of determining the higher point field correlation functions of the system is, of course, not meaningful until we have specified the *statistics* of the system. In [2] we have analysed the possible statistics of the system from an axiomatic analysis of the OPE (1.2). We shall not go into that here but merely report the conclusion that the usual Fermi/Bose dichotomy holds (if we weaken our requirements then some more exotic possibilities do exist [2]). Of the two cases the fermionic one turns out to be more interesting and we shall confine our attention to that case, though the bosonic case requires only a simple modification.

Our claim that the fermionic case is more interesting than the bosonic one only holds if we implement the condition of fermionic statistics in a special way, viz. by adding to the OPE (1.2) the following OPE's for two b and two c fields:

$$b(z)b(w) \sim O(z-w), \qquad c(z)c(w) \sim O(z-w) \tag{2.2}$$

We can now write down simple axioms for all the field correlation functions $C(m,n)$:

Axioms 2.3 *Each field correlation function* $C(m,n) = < b(Q_1)\ldots b(Q_m)c(P_1)\ldots c(P_n) >$ *is a meromorphic section of the holomorphic line bundle*

$$\mathcal{F}_\alpha(m,n) \equiv p_1^*(K\otimes\alpha^{-1})\otimes\cdots\otimes p_m^*(K\otimes\alpha^{-1})\otimes p_{m+1}^*(\alpha)\otimes\cdots\otimes p_{m+n}^*(\alpha) \tag{2.3}$$

on M^{m+n} *having :*
($\mathcal{A}1$) *a simple zero for* $Q_i = Q_j$ *or* $P_i = P_j$,
($\mathcal{A}2$) *a simple pole for* $Q_i = P_j$,
($\mathcal{A}3$) *no singularities other than those required by the second axiom.*

Axioms ($\mathcal{A}1$) and ($\mathcal{A}2$) define *divisors* (formal sums with integral coefficients of codimension 1 subvarieties) of M^{m+n} which we respectively denote by $D_z(m,n)$ and $D_p(m,n)$ and the total divisor is $D(m,n) = D_z(m,n) - D_p(m,n)$, where we follow the usual convention of putting a plus sign for zeros and a minus sign for poles (see [9] for an introduction to this concept for physicists). Then by ($\mathcal{A}3$) we conclude that $C(m,n)$ defines a *holomorphic* section of the line bundle

$$\mathcal{M}_\alpha(m,n) \;=\; \mathcal{F}_\alpha(m,n)\otimes\mathcal{O}(-D(m,n)) \tag{2.4}$$

Thus, $C(m,n)$ defines an element of $H^0(M^{m+n},\mathcal{M}_\alpha(m,n))$ and we have posed a precise mathematical question, viz. what is the dimension of the space of holomorphic sections of $\mathcal{M}_\alpha(m,n)$? The answer is given by the following theorem (see [6] for the proof) :

Theorem 2.4 *(i) If* $m \neq n$, $dim H^0(M^{m+n},\mathcal{M}_\alpha(m,n)) = 0$
(ii) If $m = n$, $dim H^0(M^{2n},\mathcal{M}_\alpha(n,n)) = 1$

This theorem has significant implications for the physics of the $b-c$ system, for part (i) implies that $C(m,n) = 0$ for $m \neq n$, which is in perfect agreement with the physicist's argument that this must happen due to the requirement of *charge conservation.* This latter argument would normally be based on the lagrangian of the system, but here we see that the OPE's serve equally well. Part (ii) of the theorem is even more remarkable, as it implies the validity of the *Wick representation*

$$C(n,n) \;=\; \det(< b(Q_i)c(P_j) >)\Big|_{i,j=1}^{n} \tag{2.5}$$

since it is clear that the determinant of two point functions on the right-hand side of (2.5) satisfies all our axioms. This result shows that indeed our method of introducing the condition of fermionic statistics through the OPE's (2.2) was correct. Moreover, it also leads to - rigorous proofs of- interesting identities once we can write down the correlation functions explicitly. For this all we need is the following lemma and our previous observations on the prime form :

Lemma 2.5 *(i) If $g = 0$, then $\mathcal{M}_\alpha(n,n))$ is the trivial line bundle on M^{2n},*
(ii) for $g \geq 1$, $\mathcal{M}_\alpha(n,n)) = \pi_\alpha^n(\mathcal{O}(\Theta))$ where $\pi_\alpha^n : M^{2n} \to Pic^{g-1}(M)$ is given by $(Q_1, \ldots, Q_n, P_1, \ldots, P_n) \mapsto \mathcal{O}(\sum_1^n (Q_i - P_i)) \otimes \alpha$.

With the help of this lemma we can write down the $2n$-point function $C(n,n)$ directly as a product of the unique section of $\mathcal{M}_\alpha(n,n)$ (1 for $g = 0$ and a theta function for $g \geq 1$) and the canonical meromorphic section of $\mathcal{O}(D(n,n))$ (a ratio of products of prime forms). On the other hand, by the Wick representation (2.5), *which we have proved,* it can also be expressed as a determinant of two point functions. In this way we obtain, with complete mathematical rigour, interesting identities. These identities are usually referred to as the *bosonization identities* in the physics literature because of the way they first appeared in physics, viz. they provided a proof that two methods of obtaining the correlation functions of the $b-c$ system were consistent [10]. Our analysis, on the other hand, traces their physics origin to the OPE's of the system and is more powerful in that it provides a *proof* of the identities instead of using them to show consistency. Of course, the bosonization viewpoint can also be made rigorous in the Grassmannian formulation[11]. The identities take different forms in different genera and were first obtained by the mathematicians whose names are attached to them :

(i) **Cauchy's bialternant identity** ($g = 0$) :

$$\frac{\Pi_{1\leq i<j\leq n}(Q_i - Q_j)(P_j - P_i)}{\Pi_{1\leq i,j\leq n}(Q_i - P_j)} = \det\left(\frac{1}{(Q_i - P_j)}\right)\Big|_{i,j=1}^{n} \tag{2.6}$$

(ii) **Frobenius' identity** ($g = 1$) :

$$\frac{\sigma(\alpha + \sum_1^n (Q_i - P_i))}{\sigma(\alpha)} \frac{\Pi_{1\leq i<j\leq n}\sigma(Q_i - Q_j)\,\sigma(P_j - P_i)}{\Pi_{1\leq i,j\leq n}\sigma(Q_i - P_j)} = \det\left(\frac{\sigma(\alpha + Q_i - P_j)}{\sigma(\alpha)\,\sigma(Q_i - P_j)}\right)\Big|_{i,j=1}^{n} \tag{2.7}$$

(iii) **Fay's identity** ($g \geq 2$) :

$$\frac{\theta[\alpha](\sum_1^n (Q_i - P_j))}{\theta[\alpha](0)} \frac{\Pi_{1\leq i<j\leq n}E(Q_i\,,\,Q_j)\,E(P_j\,,\,P_i)}{\Pi_{1\leq i,j\leq n}E(Q_i\,,\,P_j)} =$$

$$\det\left(\frac{\theta[\alpha](Q_i - P_j)}{\theta[\alpha](0)\ E(Q_i\ ,\ P_j)}\right)\Big|_{i,j=1}^{n} \qquad (2.8)$$

Note that in (2.7) and (2.8) the conditions that $\sigma(\alpha) \neq 0$ and $\theta[\alpha](0) \neq 0$ hold if and only if α has no holomorphic sections, which is ($\mathcal{A}3$) of our axioms. The case $n = 2$ of (2.8) is usually known in the mathematics literature as the *trisecant identity* for geometrical reasons into which we shall not go here [8]. Detailed proofs of the results of this section can be found in [6] and [2]. An exposition of our proof for mathematicians, in which the connections with physics have been eliminated, has also appeared in the treatise [12].

Before we conclude our discussion of the field correlation functions of the $b-c$ system, let us expand on a remark in [2] concerning a variant of the system dicussed above, which sometimes appears in the literature (see the paper of the Verlindes in [10]). The only difference is that the defining line bundle α is now taken to be an *odd theta characteristic*. More generally, we take $deg(\alpha) = g - 1$, $h^0(M, \alpha) = 1$, i.e. α is a smooth point of the canonical theta divisor Θ in $Pic^{g-1}(M)$. Of course, Theorem 1.2 says that this system has no two point function in the sense of Definition 1.1, but in the physics context what is of interest is the question of the existence of higher point functions satisfying Axioms 2.3. By a careful analysis of the proof [6] of part (ii) of Theorem 2.4 we obtain :

Theorem 2.6 $dim H^0(M^{2n}, \mathcal{M}_\alpha(n,n)) = 1 \quad (n = 1, 2, \ldots) \Longleftrightarrow$ *either* $\alpha \in Pic^{g-1}(M) - \Theta$ *or* α *is a smooth point of* Θ.

With this theorem we can not only show that the system has $2n-$point functions (for $n \geq 2$) uniquely determined by Axioms 2.3, but with this we can also give a *direct proof* of a corollary to Fay's identity (2.8), which Fay [7] obtains by a limiting argument from (2.8) (see equation following eqn.43 on p.33 of [7]). Thus we see that this case of the $b - c$ system is also covered by our approach.

3 Currents

We shall now describe some recent work [13] on an algebraic geometry approach to the current correlation functions. The heuristic definition of the current $j(z)$ is through point-splitting and subtracting the leading singularity :

$$j(z) \ = \ \lim_{Q \to P = z} b(Q)c(P) \ - \ \frac{1}{Q - P} \qquad (3.1)$$

This definition does not lend itself in any obvious way to a geometric formulation, but we shall show (see [13]) that in fact the modern Grothendieck formulation of algebraic geometry provides us with the necessary concepts to achieve this. To understand the problem, let us consider the one point function $< j(z) >$. According to the heuristic definition (3.1), $< j(z) >$ should be identified with the coefficient of $(Q - P)$ in an expansion of $< b(Q)c(P) > E(Q,P) - 1$ about the diagonal Δ of $M \times M$. This definition suggests that $< j(z) >$ is a holomorphic one form on M, but this procedure does not offer a global geometric definition. It involves subtracting sections of (at least for $g \geq 1$) two different line bundles, viz. $\mathcal{M}_\alpha(1,1)$ and the trivial line bundle on $M \times M$, and performing a Taylor series expansion.

Our solution to this problem is based on the observation that we do not require $\mathcal{M}_\alpha(1,1)$ to be trivialisable on the whole of $M \times M$, which in any case is not true for $g \geq 1$, but only on the *first infinitesimal neighbourhood of* Δ *in* $M \times M$, which is defined through the concept of a *ringed space*. Since ringed spaces are not very familiar to physicists, let us first consider the variety Δ itself as a ringed space. It is defined as a pair $(\Delta, \mathcal{O}_\Delta)$ consisting of the topological space Δ and its structure sheaf of holomorphic functions $\mathcal{O}_\Delta$, which is the quotient of the sheaf of holomorphic functions on $M \times M$ modulo those vanishing on Δ. The restriction of $\mathcal{O}_\Delta$ to an affine open set $\mathcal{U}$ is of the form $k[x,y]/\Im(x-y)$, where $k[x,y]$ is the polynomial ring in the variables x and y and $\Im(x-y)$ is the ideal in $k[x,y]$ generated by $(x-y)$. This quotient is of the form of a polynomial ring in *one* variable $k[t]$, where $t = x + y$, which is as it should be for it to be the structure sheaf of the one dimensional variety Δ.

The first infinitesimal neighbourhood 2Δ of Δ consists of the pair $(\Delta, \mathcal{O}_{2\Delta})$, where Δ is, as before, the topological space but with a new structure sheaf $\mathcal{O}_{2\Delta}$. This latter is the quotient of the sheaf of holomorphic functions on $M \times M$ modulo those with a *double zero* on Δ. The restriction of this to an affine open set $\mathcal{U}$ is of the form $k[x,y]/\Im(x-y)^2$, where $\Im(x-y)^2$ is the square of the ideal generated by $\Im(x-y)$. This quotient is of the the form $k[t] \oplus k[t]dt$, where $t = x + y$ and $dt = x - y$ so that $(dt)^2 = 0$, i.e. *the ringed space* $(\Delta, \mathcal{O}_{2\Delta})$ *contains nilpotents.* The global geometric way of describing this is that

$$p_{1*}(\mathcal{O}_{2\Delta}) \;=\; \mathcal{O}_M \oplus K_M, \tag{3.2}$$

which says that the direct image of the structure sheaf of $\mathcal{O}_{2\Delta}$ to the first factor of $M \times M$ is the direct sum of the trivial line bundle and the cotangent bundle of M (or rather their associated sheaves). The important point is that the decomposition (3.2) is *canonical*. This means that we can in a natural way find the component of an element of the l.h.s. of (3.2) in each factor on the r.h.s.

4 Field-current correlation functions

We shall now see how the concepts introduced in the previous section enable us to compute not merely $< j(z) >$, but also the general field-current correlation function

$< b(Q_1)\dots b(Q_{n-1})c(P_1)\dots c(P_{n-1})j(z) >$. By (3.1), this field-current correlation function is the coefficient of $(Q_n - P_n)$ in an expansion of $C(n,n)E(Q_n,P_n)$ about the diagonal $Q_n = P_n = z$ of $M_n \times M_{2n}$. This suggests that the field-current correlation function is a meromorphic one form for fixed $\{Q_i,\ P_i,\ 1 \leq i \leq n-1\}$. As explained in the last section, our proposal is to study $C(n,n)E(Q_n,P_n)$ on *the first infinitesimal neighbourhood* 2Δ of Δ in $M_n \times M_{2n}$ and to use the *canonical global splitting* of (3.2) above to determine this meromorphic one form.

Now $C(n,n)E(Q_n,P_n)$ is a meromorphic section of $\mathcal{F}_\alpha(n,n)\otimes\mathcal{O}(D_{n,2n})$, where D_{ij} denotes the divisor of M^{2n} defined by the diagonal of $M_i \times M_j$. Since the only relevant variables of $C(n,n)E(Q_n,P_n)$ are Q_n and P_n, we can restrict $\mathcal{F}_\alpha(n,n)\otimes\mathcal{O}(D_{n,2n})$ to general position on each M_i for $i \neq n, 2n$ to get the line bundle $\mathcal{L} = \mathcal{M}_\gamma(1,1)\otimes\mathcal{G}$ on $M_n \times M_{2n}$, where $\mathcal{G} = p_n^*(\mathcal{O}(D))\otimes p_{2n}^*(\mathcal{O}(-D))$, $\gamma = \alpha\otimes\mathcal{O}(D)$ and $D = \sum_{i=1}^{n-1}(Q_i - P_i)$. As explained in the last section, our approach will only make sense if $\mathcal{L}$ is trivialisable on the ringed space 2Δ. While it is simple to see that $\mathcal{L}$ is trivialisable on Δ, this is a very subtle question on 2Δ, but nevertheless true :

Proposition 4.1 *The line bundles $\mathcal{M}_\gamma(1,1)$ and $\mathcal{G}$ are both trivialisable on 2Δ and thus $\mathcal{L}$ is so as well.*

We can now use the canonical splitting (3.2) to compute the image of $C(n,n)E(Q_n,P_n)$ in (3.2), where for $g \geq 1$ we take the classical trivialisation of $\mathcal{M}_\gamma(1,1)$ by Riemann's theta function $\theta[\alpha](\sum_1^{n-1}(Q_i - P_i) + \vec{u})/\theta[\alpha](\sum_1^{n-1}(Q_i - P_i))$. Here α in square brackets denotes theta characteristics determined by α (after having chosen a marking on M, a symplectic homology basis and the dual basis of holomorphic one forms $w_i, 1 \leq i \leq g$), $\vec{u}$ is in $\mathbb{C}^g$ and the sums are Abel sums. We also need to determine what happens to the natural meromorphic section of $\mathcal{G}$, which is a product of prime forms.

Lemma 4.2 *Let 1_D denote the the natural section of $\mathcal{O}(D)$ and now consider*
$p_n^*(1_D)\otimes p_{2n}^*(1_{-D})$, which can be written in the notation of prime forms as $(\Pi_1^{n-1}E(Q_n,Q_i)E(P_n,P_i))/(\Pi_1^{n-1}E(Q_n,P_i)E(P_n,Q_i))$. Then the restriction of this meromorphic section to 2Δ takes, in the canonical decomposition (3.2) above, the form $1 + w_D$, where w_D is a meromorphic one form on Δ (identified with M_n) having simple poles at $Q_1,\dots,Q_{n-1}$ with residue $+1$ and at $P_1,\dots,P_{n-1}$ with residue -1.

Lemma 4.2 is an algebro-geometric form of a well known formula for the prime form: $w_{a-b}(z) = d_z \ln(E(z,a)/E(z,b))$. Note that there is no canonical trivialisation of $\mathcal{G}$ on 2Δ. The effect of choosing a different trivialisation is to simply change w_{a-b} by the addition of a *holomorphic* one form, which does not matter since w_{a-b} was only defined up to the addition of such a one form. When we want to write the formulas for the correlation function in function- theoretic form, the correct choice will be made by the chosen marking of M and the representation of the prime form as a function on $\mathbf{U}\times\mathbf{U}$, where $\mathbf{U}$ is the universal covering space of M.

With these two results it is easy to obtain the field-current correlation function :

Theorem 4.3 *The field-current correlation function is given by :*

$$< b(Q_1)\dots b(Q_{n-1})c(P_1)\dots c(P_{n-1})j(z) >= C(n-1,n-1)\times$$
$$\left(w_D(z) + \sum_{j=1}^{g} w_j(z)\, \partial \ln \theta[\alpha](\sum_{i=1}^{n-1}(Q_i - P_i) + \vec{u})/\partial u^j \Big|_{\vec{u}=0} \right) \quad (4.1)$$

where the second term is absent for $g = 0$.

Alternatively, we can compute the field-current correlation function starting from the determinantal form of the field correlation function. For this we need an algebro-geometric analogue of a simple formula given by Mumford (see part (a) of the lemma on page 3.225 of [8]) for the prime form, viz. $d_z(E(z,a)/E(z,b))|_{z=a} = 1/E(a,b)$. From the proof it is clear that Mumford actually obtains $d_z E(z,a)|_{z=a} = 1$ from which this formula follows. Our algebro-geometric analogue of the latter is that we can find the image of the canonical section "1" of the trivial line bundle $\mathcal{O}_{M\times M}$ on $M \times M$ in $\mathcal{O}_{2\Delta}(\Delta)$ by traversing the following commutative diagram in two different ways :

$$\begin{array}{ccc} \mathcal{O}_{M\times M}|\Delta & \hookrightarrow & \mathcal{O}_{2\Delta}(\Delta) \\ \uparrow & & \uparrow \\ \mathcal{O}_{M\times M} & \hookrightarrow & \mathcal{O}_{M\times M}(\Delta) \end{array} \quad (4.2)$$

As a result of this calculation we obtain the following alternative form for the field-current correlation function :

Theorem 4.4 *From the Wick representation we obtain,*

$$< b(Q_1)\dots b(Q_{n-1})c(P_1)\dots c(P_{n-1})j(z) >= C(n-1,n-1) < j(z) >$$
$$- \sum_{k=1}^{n-1} < b(Q_1)\dots b(Q_{n-1})c(P_1)\dots c(P_k = z)\dots c(P_{n-1}) >< b(z)c(P_k) > \quad (4.3)$$

The two expressions (4.1) and (4.3) for the field-current correlation functions give us an identity, which is a simple generalisation of the "first corollary to Fay's identity" given by Mumford [8], valid in all genera.

5 The two point function of currents

The two point function of currents $< j(z_1)j(z_2) >$ is the most important current correlation function from the point of view of physics and so it is very important to see whether our techniques generalise to this case. From the OPE (1.2) we see that $< j(z_1)j(z_2) >$ is given in terms of the 4-point function $C(2,2)$ by the following heuristic double limit :

$$< j(z_1)j(z_2) > = \{C(2,2)E(Q_1,P_1)E(Q_2,P_2) \\ -1- < j(z_1) > - < j(z_2) >\}\Big|_{Q_i \to P_i = z_i (i=1,2)} \qquad (5.1)$$

It may look very unlikely that we can make mathematical sense out of (5.1), since we have to make sense first of the sum of 1, $< j(z_1) >$ and $< j(z_2) >$ and then of subtracting it from the first term in (5.1) !

We expect $< j(z_1)j(z_2) >$ to be a (meromorphic) one form in each variable, i.e. that it is a meromorphic section of the *canonical bundle* $\omega_{M\times M} \equiv p_1^*(K) \otimes p_2^*(K)$ of $M \times M$. Now $C(2,2)E(Q_1,P_1)E(Q_2,P_2)$, which appears in (5.1), is a meromorphic section of the line bundle

$$\mathcal{R} \equiv \mathcal{F}_\alpha(2,2) \otimes \mathcal{O}(D_{13} + D_{24}) = \mathcal{M}_\alpha(2,2) \otimes \mathcal{A}, \qquad (5.2)$$

where

$$\mathcal{A} \equiv \mathcal{O}(D_{12} + D_{34} - D_{14} - D_{23}) \qquad (5.3)$$

The natural generalisation of the procedure of the previous section is to consider the restriction of the line bundle $\mathcal{R}$ to the sub-*scheme* $Z \equiv 2\Delta_{13} \times 2\Delta_{24}$ of M^4. Our basic tool for studying this restriction is the following beautiful exact sequence

$$O \longrightarrow \omega_Y \longrightarrow \mathcal{O}_Z \longrightarrow \mathcal{O}_{2Y} \longrightarrow O \qquad (5.4)$$

where $Y \equiv \Delta_{13} \times \Delta_{24}$.

The exact sequence (5.4) does not seem to exist in the mathematics literature and its only *raison d'être* appears to be to enable us to make sense of the heuristic formula (5.1), for which it has just the right properties. For if pr_{12} denotes the projection $M^4 \to M_1 \times M_2$ and we take the direct image of (5.4) by it, we get a new exact sequence

$$O \longrightarrow \omega_{M_1\times M_2} \longrightarrow pr_{12*}(\mathcal{O}_Z) \longrightarrow pr_{12*}(\mathcal{O}_{2Y}) \longrightarrow O \qquad (5.5)$$

The remarkable feature of (5.5) is that $pr_{12*}(\mathcal{O}_{2Y})$ is a rank 3 vector bundle on $M_1 \times M_2$ which is *canonically* the direct sum of three line bundles :

$$pr_{12*}(\mathcal{O}_{2Y}) = \mathcal{O}_{M_1\times M_2} \oplus K_{M_1} \oplus K_{M_2} \qquad (5.6)$$

Note that this splitting enables us to make sense out of the formal sum $1 + < j(z_1) >$

$+ < j(z_2) >$ in (5.1). In addition, $pr_{12*}(\mathcal{O}_Z)$ is a rank 4 vector bundle on $M_1 \times M_2$ which is *canonically* the direct sum of four line bundles :

$$pr_{12*}(\mathcal{O}_Z) \;=\; \mathcal{O}_{M_1\times M_2} \;\oplus\; K_{M_1} \;\oplus\; K_{M_2} \;\oplus\; \omega_{M_1\times M_2} \qquad (5.7)$$

Now all we need is to know whether the line bundle $\mathcal{R}$ is trivialisable on Z.

Proposition 5.1 *The line bundles $\mathcal{M}_\alpha(2,2)$ and $\mathcal{A}$ are trivialisable on Z and hence so is $\mathcal{R}$.*

In order to be able to actually compute $< j(z_1)j(z_2) >$ we need to know what happens to $C(2,2)E(Q_1,P_1)E(Q_2,P_2)$ on restriction to Z. It is the product of the unique section of $\mathcal{M}_\alpha(2,2)$, which is easy to handle by a simple generalisation of the procedure for the case of the field-current correlation function, and the canonical meromorphic section of the line bundle $\mathcal{A}$, defined in (5.3), which we shall denote as $1_\mathcal{A}$. The answer in this case can be expressed in terms of what is sometimes called the *Bergmann kernel* ω_B [5] or, more appropriately, the *generalised Weierstrass $\wp$ function.* This concept was introduced for a compact Riemann surface in function-theoretic form by Hawley and Schiffer [5]. This way is not suitable for us and so we introduce it through an algebro-geometric definition: it is a symmetric meromorphic section of $\omega_{M\times M}$, defined by a holomorphic section of $\omega_{M\times M}(2\Delta)$ which is 1 on restriction to the diagonal Δ of $M\times M$.

Lemma 5.2 *Let $1_\mathcal{A}$ denote the canonical meromorphic section of the line bundle $\mathcal{A}$ defined in (5.3), which can be written as $\{E(Q_1,Q_2)E(P_2,P_1)\}/\{E(Q_1,P_2)E(Q_2,P_1)\}$ in the notation of prime forms. Then its restriction to Z has the following decomposition in the canonical decomposition of eqn.(5.7) :*

$$1_\mathcal{A}|Z \;=\; 1 \;+\; \omega_B(z_1,z_2) \qquad (5.8)$$

Since there is no canonical trivialisation of $\mathcal{A}$ on Z, $\omega_B(z_1,z_2)$ is defined only up to the addition of a holomorphic bidifferential. However, a definite one is automatically fixed when the appropriate choices have been made, just as for $w_D(z)$ in Lemma 4.2.

If we think about the meaning of restricting $1_\mathcal{A}$ to Z we easily realise that Lemma 5.2 is simply an algebro-geometric proof of the following well known formula $\omega_B(x,y) = \partial^2 \ln E(x,y)/\partial x\partial y$ (see [7] and [8]). When $g=1$ this gives the following well known formula linking two functions of Weierstrass, viz. $\wp(z) = -d^2 \ln\sigma(z)/dz^2$. The link between Lemma 5.2 and this equation is the elementary formula

$$\lim_{\substack{\epsilon\to 0\\ \delta\to 0}} \left(\frac{f(x+\epsilon,y+\delta)f(x,y)}{f(x+\epsilon,y)f(x,y+\delta)} \;-\; 1\right) = \frac{\partial^2}{\partial x\partial y}\ln f(x,y)dx\,dy. \qquad (5.9)$$

Lemma 5.2 gives a rather remarkable geometric interpretation of this formula for ω_B, for if we write down $1_{\mathcal{A}}$ on an affine subset of the Riemann sphere (case when $g = 0$), i.e. on the complex plane, we see that it is simply the *anharmonic ratio* of four points on the complex plane. *Thus* $1_{\mathcal{A}}$ *is a natural generalisation of the notion of the anharmonic ratio (or cross ratio) to a compact Riemann surface of arbitrary genus.* Thus the formula for the Weierstrass function $\wp(z)$ comes by coalescing two pairs of arguments in the generalised anharmonic ratio in the case of genus $g = 1$. This viewpoint accords an unexpected fundamental role to this well known formula as well as a new insight. In Lemma 5.2 we wrote down this generalised anharmonic ratio in terms of prime forms. By the dictionary we have earlier established for the prime form in [2] and [6], we find that this combination of prime forms coincides with a function-theoretic definition of a generalised anharmonic ratio proposed recently by Gunning [14].

We can now compute the image of $C(2,2)E(Q_1, P_1)E(Q_2, P_2)|Z$ in (5.7) to obtain :

Theorem 5.3 *The two point function of currents is given by*

$$< j(z_1)j(z_2) > \ = \ \omega_B(z_1, z_2) + \sum_{i,j=1}^{g} w_i(z_1)w_j(z_2)\frac{\partial^2\theta[\alpha](\vec{u}+\vec{v})}{\partial u_i \partial v_j \theta[\alpha](0)}\Big|_{\vec{u}=\vec{v}=0} \tag{5.10}$$

where the second term on the right-hand side is absent if $g = 0$.

Theorem 5.4 *From the Wick representation we obtain*

$$< j(z_1)j(z_2) > \ = \ < j(z_1) >< j(z_2) > - < b(z_1)c(z_2) >< b(z_2)c(z_1) > \tag{5.11}$$

Equating these two expressions for $< j(z_1)j(z_2) >$ gives us the "second corollary to the trisecant identity" [8] (it is tautological for $g = 0$).

It is important to note that our results for the field-current correlation functions (eqns.(4.1) and (4.3)) as well as for the two point function of the currents (eqns.(5.10) and (5.11)) are perfectly consistent with the standard field-current and current-current operator product expansions as found in [1] (with an adjustment for a difference of notation):

$$b(w)j(z) \ \sim \ \frac{b(w)}{z - w}, \qquad c(w)j(z) \ \sim \ -\frac{c(w)}{z - w} \tag{5.12}$$

$$j(z)j(w) \ \sim \ \frac{1}{(z - w)^2}. \tag{5.13}$$

It is straightforward to extend our calculations to the n-point function of the currents $< j(z_1)\dots j(z_n) >$: it is a meromorphic section of the canonical bundle of M^n which is given by the determinant of the $n \times n$ matrix $\mathbf{A} \equiv (a_{ij})$, where $a_{ij} =< b(z_i)c(z_j) >$ if $i \neq j$ and $a_{ii} =< j(z_i) >$. Now, there is a well known procedure by which one sees that the

OPE (5.13) is equivalent to the infinite oscillator algebra $\{c, j_n | [j_m, j_n] = mc\delta_{m+n,0}, [j_n, c] = 0\}$. It is then natural to regard the set $\{< j(z_1) \ldots j(z_n) > | n = 1, 2 \ldots\}$ as a *realisation* of the oscillator algebra with $c = 1$ on an arbitrary compact Riemann surface.

6 Concluding Remarks

We have seen how every detail of the structure of the $b - c$ system is a consequence of its OPE (1.2) and statistics, at least at the level of the field and current correlation functions. Algebraic geometry provides a rigorous mathematical language for describing the system and one which is as physically natural as the language of Hilbert space is for Quantum Mechanics.

The only aspect of the system that we have not discussed as yet is the energy-momentum tensor. For lack of time, I shall not describe it here in detail, but merely note that this approach suggests a rather remarkable picture of the energy-momentum tensor as an infinitesimal version of our generalised anharmonic ratio ! We can, moreover, show that the system has central charge 1 (defining the central charge as in the paper of Belavin-Polyakov-Zamolodchikov [15]). *This is an algebro-geometric counterpart on a compact Riemann surface of any genus of a well known result in the theory of the Virasoro algebra* (see case (i) of Remark 4.2 on page 46 of [16]), viz. the anomaly formula. Further developments are under study.

References

[1] Friedan, D., Martinec, E., Shenker, S. : *Conformal invariance, supersymmetry, and string theory.* Nuclear Phys. **B271** (1986) 93 - 165

[2] Raina, A.K. : *An algebraic geometry study of the $b-c$ system with arbitrary twist fields and arbitrary statistics.* Commun. Math. Phys. **140** (1991) 373-397

[3] Raina, A.K. : *Analyticity and chiral fermions on a Riemann surface.* Helv. Phys. Acta **63** (1990) 694-704

[4] Atick, J.J., Sen, A. : *Spin field correlators on an arbitrary genus Riemann surface and non-renormalization theorems in string theories.* Phys. Lett. **B186** (1987) 339-346

[5] Hawley, N.S., Schiffer, M. : *Half-order differentials on Riemann surfaces.* Acta Math. **115** (1966) 199-236

[6] Raina, A.K. : *Fay's trisecant identity and conformal field theory.* Commun. Math. Phys. **122** (1989) 625-641; *Fay's trisecant identity and Wick's theorem: an algebraic geometry viewpoint.* Expositiones Math. **8** (1990) 227-245

[7] Fay, J. : *Theta functions on Riemann surfaces.* LNM no.352. Springer-Verlag, 1973

[8] Mumford, D. : *Tata lectures on Theta. II* Springer-Verlag, 1984

[9] Raina, A.K. : *Chiral fermions on a compact Riemann surface and the trisecant identity.* In : *Proc. 3rd. Regional Conf. on Math. Phys.* eds. F. Hussain & A. Qadir, World Scientific, 1990

[10] Eguchi, T., Ooguri, H. : *Chiral bosonization on a Riemann surface.* Phys. Lett. **B187** (1987) 127-134; Verlinde, E., Verlinde, H. : *Chiral bosonization, determinants and the string partition function.* Nucl. Phys. **B288** (1987) 357-396; Alvarez-Gaumé, L., Bost, J-B, Moore, G., Nelson, P., Vafa, C. : *Bosonization on higher genus Riemann surfaces.* Commun. Math. Phys. **112** (1987) 503-552

[11] Alvarez-Gaumé, L., Gomez, C., Reina, C. : *New methods in string theory.* In : *Superstrings '87*, ed. L. Alvarez-Gaumé. World Scientific, 1988; Kawamoto, N., Namikawa, Y., Tsuchiya, A., Yamada, Y.: *Geometric realisation of conformal field theory on Riemann surfaces.* Commun. Math. Phys. **116** (1988) 247-308

[12] Lange, H., Birkenhake, C. : *Complex abelian varieties.* Springer-Verlag, 1992

[13] Raina, A.K. : *An algebraic geometry view of currents in a model quantum field theory on a curve.* C. R. Acad. Sci. Paris, **t. 318**, Série I (1994) 851-856

[14] Farkas, H.M. : *The trisecant formula and hyperelliptic surfaces.* Contemp. Math. **136** (1992) 161-169

[15] Belavin, A.A., Polyakov, A.M., Zamolodchikov, A.B. : *Infinite conformal symmetry in two-dimensional quantum field theory.* Nuclear Phys. B241 (1984) 333-380

[16] Kac, V.G., Raina, A.K. : *Highest weight representations of infinite dimensional Lie algebras.* World Scientific, 1987