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CONFERENCE ON

TOPOLOGICAL AND GEOMETRICAL PROBLEMS

RELATED TO QUANTUM FIELD THEORY

(13 - 24 March 1995)

Chern-Simons Perturbation Theory about an Instanton Moduli Space

LECTURE 1

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Chern-Simons Perturbation Theory

about

on Instanton Moduli Space

talks at Trieste Conference on Topological and Geometrical Methods Related to Quantum Field Theory.
March 13-24 1995.

- Scott Axelrod
MIT Math Department

Basic Object of Study

$$Z_k = \frac{1}{\mathrm{Vol}(\mathcal{G})} \int_{A \in \mathcal{F}} \mu\, e^{ikS(A)}$$

where $S : \mathcal{F} \to \mathbb{R}$, $\mu \in \Omega^{|\mathcal{F}|}(\mathcal{F})$ are $\mathcal{G}$-invt.

Goal:

Formulate perturbation theory (stationary phase approx. computing large k asymptotics) in finite dim so that result and proof of independence of arbitrary choices carries over to QFT where $\mathcal{F} \to \infty$-dim'l.

For Chern-Simons Theory:

Fields: $\mathcal{F} = \mathrm{Conn}\left(\begin{matrix} G \to P \\ \downarrow \\ M \end{matrix} \right)$ — G cp… ; M^3 clos… ori…

gauge trsfs: $\mathcal{G} = \mathrm{Aut}(P)$

action: $S(A) = \frac{1}{2\pi} \int_M \mathrm{Tr}\left(\frac{1}{2} A \wedge dA + \frac{1}{3!} A[A \wedge A \right.$

"$\frac{1}{\hbar}$" : $k \in \mathbb{Z}_{>0}$ "level"

Recall:

$\mathcal{F} = \{\text{crit. pts of } S\} = \{\text{sol'ns of E.O.M.}\} = \{\text{flat connections on } P\}$

$\mathcal{M} = \mathcal{F}/\mathcal{G}$ = instanton moduli space $\subset \mathrm{Hom}(\pi_1(M), G)/G$

Theorem:

Chern-Simons Perturbation Theory about a smooth component $\mathcal{M}_0$ of the moduli space of flat connections is well defined and gives a series of differential invariants of $(M, \mathcal{M}_0)$.

Talk 1: Derive Formulation by General Method (MSRI Jan '94 Conjecture)

Talk 2: Thm. & Pf. for $\mathcal{M} = \{[A_0]\}$ (A. & Singer, C.S. Pert. I & II)

Talk 3: Thm. & Pf. for general $\mathcal{M}$ (write up in progress)

Philosophy, future directions

Talk 1 Outline

I) Stationary Phase About Crit. Poin[t]

Feynman Rules

II) Stationary Phase About Crit. Manifold

Evaluation Map $N \times \mathrm{Met} \to \mathcal{F}$

Integrate over $\mathcal{G}$-orbits by invarian[ce]

$$\to \quad Z_K^{pert} = \int_{\mathcal{M} \times \{g\}} \hat{Z}_K^{pert}$$

$$\hat{Z}_K^{pert} \in \Omega_{el}^{|\mathcal{M}|}(\mathcal{M} \times \mathrm{Met})$$

III) Trivial Example

$\to$ Explicit formulas to generalize

III) Final Formulation

Structures on Def. Cplx & Supermanifolds

$$\to \quad \hat{Z}_K^{he} = \int_{e\mathcal{M}} \mathrm{Tr}\left(e^{\Phi}\right)$$

I) Stationary Phase About a Non-Degenerate Critical Point (Feynman Rules)

for μ supported near crit. pt. $A_0 \in \mathcal{F}^N$

$$Z_k = \int_{B\in T\mathcal{F}_{A_0}} d^N B\, \chi(B)\, \exp ik\left[S(A_0)+\tfrac{1}{2}(B,HB)+V(B)\right]$$

$$\sim e^{ikS(A_0)}\, e^{ikV(\partial/\partial J)}\Big|_{J=0} \int d^n B\, \chi\, e^{\frac{ik}{2}(B,HB)+ikJ\cdot B}$$

... complete square, integrate by parts...

$$Z_k \sim Z_k^{pert} = Z_k^{nl}\, Z_k^{sc} \in PS(k) = \Big\{\sum_{\alpha\in N} a_{S_\alpha \alpha}\, e^{ikS_\alpha} k^{\alpha}\Big\}$$

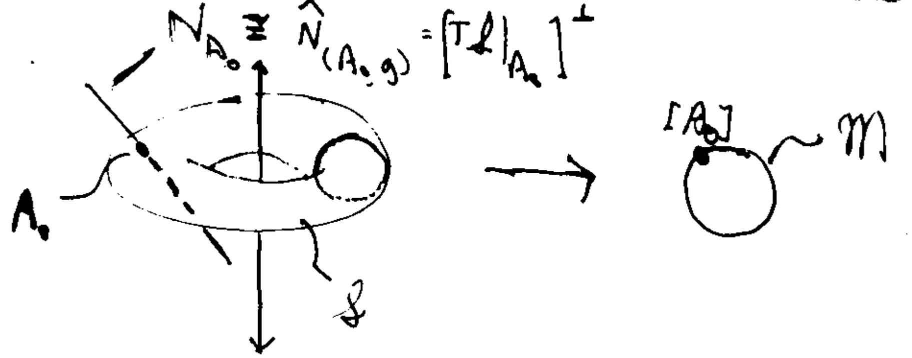
$$Z_k^{sc} = \frac{e^{\frac{\pi i}{4}\operatorname{sign}(H)}}{\sqrt{|\det(H)|}}\left(\frac{2\pi}{k}\right)^{N/2} e^{ikS(A_0)}$$

$$Z_k^{nl} = e^{ikV(\partial/\partial J)}\Big|_{J=0}\, e^{-\frac{1}{2ik}\langle J, H^{-1}J\rangle}$$

$$= 1 + \sum_{\ell\geq 2}(ik)^{1-\ell}\, I_\ell\,, \quad I_\ell = \sum_{\substack{\Gamma \text{ graph} \\ \beta_1(\Gamma)-\beta_0(\Gamma)=\ell-1}} \mathcal{I}(\Gamma)$$

Principle: Stationary phase algorithm does not involve (path) integrals.

II) Stationary Phase about Critical Manifold



Choice of $g \in Met = \{\mathcal{G}\text{-inv't metrics}\}$ gives exponential map:

$$\exp: \mathring{N} \xrightarrow{\cong \text{ near } \ell} \mathcal{F}$$

Non-degeneracy (Morse-Bott condition) says Hessian of S on $\hat{N}_{(A_0,g)}$ is invertible.

$$Z_k \sim \int_{[A_0]\in\mathcal{M}} \frac{1}{vol(\mathcal{G})} \int_{A_0\in\ell_{[A_0]}} \underbrace{\int^{pert}_{B\in\hat{N}_{(A_0,g)}} \exp^*(\mu\, e^{ikS})}_{\text{critical point algorithm}}$$

eed to:

- Handle Differential Forms
- make explicit
- Prove g independence in a way that will carry through to C.S. QFT

ntroduce:

$\mathcal{H} \to \hat{\mathcal{S}} = \mathcal{S} \times \mathrm{Met} \dashrightarrow \hat{\mathfrak{M}} = \mathfrak{M} \times \mathrm{Met}$

$\hat{N} = N \times \mathrm{Met} \longrightarrow \hat{\mathcal{S}} = \mathcal{S} \times \mathrm{Met}$

$E : \hat{N} \longrightarrow \mathcal{F}$

$((A_0, g), B) \longmapsto \exp^g_{A_0}(B)$

$\triangleright\ \hat{\mathcal{S}} \to \hat{\mathfrak{M}}$

- trivial in Met directions
- $T_{hor}(\hat{\mathcal{S}}_g \to \hat{\mathfrak{M}}_g) = T_{A_0}(\mathrm{Lie}(\mathcal{G}))^{\perp_g}$

$\triangleright\ \hat{N} \to \hat{\mathcal{S}}$

- // transport in $\mathcal{G}$ directions from group action
- projected connection otherwise $\hat{N} \subset \widehat{T\mathcal{F}|_{\mathcal{S}}}$

$$\Omega^{|\mathcal{S}|}_{cl}(\mathcal{F})^{\mathcal{G}} \xrightarrow{E^*} \Omega^{|\mathcal{F}|}_{cl}(\hat{N})^{\mathcal{G}}$$

$\swarrow$ ue^{ikS}

$$\xrightarrow{\mathcal{F}} \Omega^{|\mathcal{S}|}_{cl}(\hat{\mathcal{S}}, \Omega^{|N|}_{ver}(\hat{N}))^{\mathcal{G}}$$

$$\xrightarrow{\int^{pert}_{\hat{N}\to\hat{\mathcal{S}}}} \Omega^{|\mathcal{F}|}_{cl}(\hat{\mathcal{S}})^{\mathcal{G}} \otimes PS(k)$$

$$\xrightarrow{[\mathrm{vol}(\mathcal{G})\, u_{\mathcal{G}}]^{-1}} \Omega^{|\mathfrak{M}|}_{cl}(\hat{\mathcal{S}})_{basic} \otimes PS(k)$$

$$\xrightarrow{\tilde{\Xi}} \Omega^{|\mathfrak{M}|}_{cl}(\hat{\mathfrak{M}}) \otimes PS(k) \ni \hat{Z}^{pert}_k$$

$$\xrightarrow{\int_{\mathfrak{M}\times\{g\}}} PS(k) \ni Z^{pert}_k$$

$$Z_k = \frac{1}{\mathrm{vol}(\mathcal{G})} \int_{\mathcal{F}} ue^{ikS} \sim Z^{pert}_k = \int_{\mathfrak{M}\times\{g\}} \hat{Z}^{pert}_k$$

Remarks:

(i) In QFT $\mathfrak{M}$ finite dim'l $\Rightarrow$ no functional $\int$

(ii) Z^{pert}_k independent of g because $\hat{Z}^{pert}_k$ closed

(iii) Use supermanifolds below to be explicit

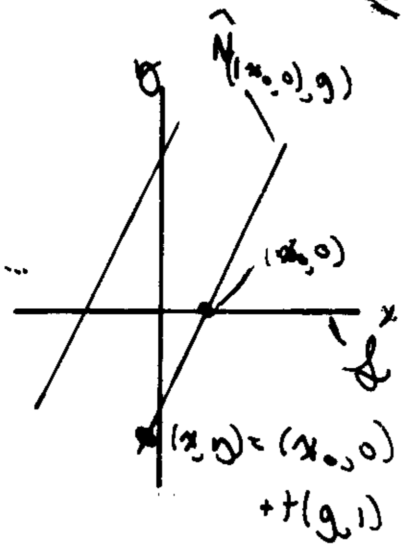
III) Trivial Example

$\mathcal{G} = \{1\}$

$F = S^1 \times \mathbb{R} = \{A = (x, y)\}$

$S(x,y) = \frac{1}{2} H(x) y^2 + \frac{1}{3!} V_3(x) y^3 + \dots$

$\mathcal{L} = \mathfrak{M} = S^1 = \{(x_0, 0)\}$



$$\mu = dx \wedge dy = [dx_0 + dg\, t + \cancel{g\, dt}] \wedge dt$$

$$= [1 + t\, dg\, I(\tfrac{\partial}{\partial x_0})] \lrcorner (dx_0 \wedge dt)$$

interior prod. + wedge prod.

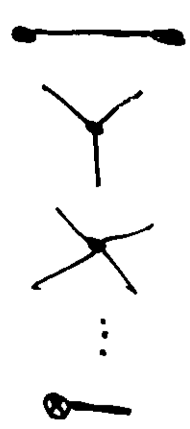
$$S(x,y) = \tfrac{1}{2} H(x_0) t^2 + \left(\tfrac{g}{2} H'(x_0) + \tfrac{1}{3!} V_3(x_0)\right) t^3 + \dots$$

$$\hat Z_K^{pert} = \int^{pert}_{t \in \mathbb{R}} \mu \exp iK S(x,y) = \hat Z_K^{bl} \lrcorner \hat Z_K^{sc}$$

$$\hat Z_K^{sc} = \sqrt{\frac{2\pi i}{K}}\, \frac{1}{\sqrt{H(x_0)}}\, dx_0$$

$$\hat Z_K^{bl} = 1 + \sum_{\ell \geq 1} \hat I_\ell (iK)^{-\ell} = \sum_{\text{graph } \Gamma} \frac{1}{S(\Gamma)} \mathfrak{F}(\Gamma)$$

Feynman Rules:



$\frac{1}{iK} H(x_0)^{-1}$

$iK \frac{\partial^3 S}{\partial t^3}\Big|_{x_0} = iK\left[3H'(x_0) g + V_3(x_0)\right]$

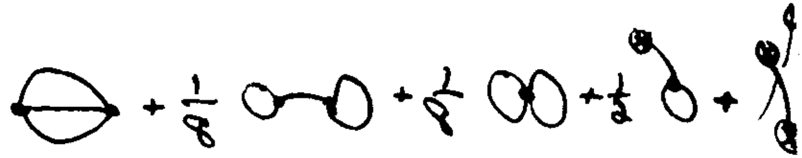
$iK \frac{\partial^4 S}{\partial t^4}\Big|_{x_0} = iK\left[6H'' g^2 + 4V_3' g + V_4\right]$

$\vdots$

$dg\, I(\frac{\partial}{\partial x_0})$ ← at most one since $\dim(\mathfrak{M}) = 1$

E.g.:

$\hat I_2 = \frac{1}{12}$ (graph) $+ \frac{1}{8}$ (graph) $+ \frac{1}{8}$ (graph) $+ \frac{1}{2}$ (graph) $+$ (graph)



$$= \left[\tfrac{1}{12} + \tfrac{1}{8}\right] H^{-3}(3H'g + V_3)^2 + \tfrac{1}{8}(\dots) + \tfrac{1}{2} H^{-2}(3H'g + V_3)\, dg\, I \dots$$

$$8\, \hat I_2 \lrcorner \frac{dx_0}{\sqrt{H}} = \left[H^{-7/2} V_3^2 + H^{-5/2} V_4\right] dx_0$$ ← indep. of g, closed

$$+ \left[dx_0 \tfrac{\partial}{\partial x_0} + dg \tfrac{\partial}{\partial g}\right]\left(-6H^{-5/2} H' g^2 - 4H^{-5/2} V_3 \dots\right)$$

← exact

Summary

$$\hat{Z}_K^{sc} \in \Omega_{cl}^{|m|}(\mathfrak{M}) \subset \Omega_{cl}^{|m|}(\hat{\mathfrak{M}}) \otimes PS(K)$$

$$\hat{Z}_K^{hl} \in \bigoplus_K \Omega_{cl}^{K}(\hat{\mathfrak{M}}, \Lambda^K(T\mathfrak{M})) \otimes \mathbb{C}[[\tfrac{1}{K}]]$$

$$\hat{Z}_K^{pert} = \hat{Z}_K^{hl} \lrcorner \hat{Z}_K^{sc} \in \Omega_{cl}^{|m|}(\hat{\mathfrak{M}}) \otimes PS(K)$$

$$Z_K^{pert} = \int_{\mathfrak{M}} \hat{Z}_K^{pert} \in \Omega_{cl}^{0}(Mot) \otimes PS(K) \cdot PS(K)$$

$$[\hat{Z}_K^{pert}] \in H^{|m|}(\hat{\mathfrak{M}}) \cong H^{|m|}(\mathfrak{M})$$

$$Z_K^{pert} = \langle [\hat{Z}_K^{pert}], [\mathfrak{M}] \rangle$$

Note: $\lrcorner$ acts on $\Lambda^K(T\mathfrak{M})$ by divergence w.r.t. $\hat{Z}_K^{sc}$

Problem:

Derive general definition of $\hat{Z}_K^{hl}$.
Prove its well defined and closed even for field theory.

IV) Final Formulation

Supermanifold trick (Mathai-Quillen):

$$\int_{A_0 \in \mathcal{F}} \mu e^{iKS} = \int_{A \in \mathcal{F}} \int_{x \in T^*\mathcal{F}} \int_{\delta A \in T\mathcal{F}} e^{i[\langle x, \delta A\rangle + KS(A)]}$$

$$C^\infty(\{A, x, \delta A\}) = \Omega^*(\mathcal{F}; \Lambda^*(T\mathcal{F}))$$

$$\delta A^\mu \longleftrightarrow dx^\mu$$

$$x_\mu \longleftrightarrow I\left(\tfrac{\partial}{\partial a^\mu}\right)$$

$$\langle x, \delta A\rangle \in \Omega^1(\mathcal{F}; \Lambda^1(T\mathcal{F}))$$

$$\begin{array}{ccc} \Omega^*(\mathcal{F}) \to \cdots & \to & \Omega^*(\hat{\mathfrak{M}}) \\ \Downarrow & & \Downarrow \\ \mu e^{iKS} & \longmapsto & \hat{Z}_K^{pert} \end{array}$$

comes from sequence of change of vars. & fiber integrals defined using $\nabla^{\hat{N} \to \hat{\mathcal{L}}}$, $\nabla^{\hat{\mathcal{L}} \to \hat{\mathfrak{M}}}$ and deformation chain complex:

$$0 \to Lie(\mathcal{G}) \xrightarrow{T_A} T\mathcal{F} \xrightarrow{H(A)} T^*\mathcal{F} \xrightarrow{T_A^*} Lie(\mathcal{G})^* \to 0$$

$$0 \to \Omega^0_m \xrightarrow{D^0} \Omega^1_m \xrightarrow{D^1} \Omega^2_m \xrightarrow{D^2} \Omega^3_m \to 0$$

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For C.S.: $\Omega^*_M = \Omega^*(M; \mathfrak{g})$

Generally:

$$\begin{array}{c}\hat{\Omega}^*_M = \hat{\Omega}^*_h \oplus \hat{\Omega}^*_d \oplus \hat{\Omega}^*_\delta \supset \mathcal{G} \\ \downarrow \\ \hat{\mathcal{A}}\end{array}$$

Bundle of D.G.A's w/ Hodge Decomp. $\mathcal{G}$ action and connection

Affine Case:

$$A = A_0 + B$$

$$x = x_h + x_d + x_\delta$$

$$\delta A = \delta A_{0,h} + T_A(c) + \delta_{vert} B$$
$$+ (\delta_{\delta \hat{A}_{0,h}} P_N) B \leftarrow \text{a connection 1-form}$$

$$c \in \hat{\Omega}^0_\delta = \mathrm{Lie}(\mathcal{G})^\perp \subset \mathrm{Lie}(\mathcal{G})$$

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Result in terms of:

$$\mathcal{A} = c + B + x_\delta \in \hat{\Omega}^*_\delta$$

$$\mathcal{A} \to \frac{\partial}{\partial \mathcal{J}}, \quad \mathcal{J} \in \hat{\Omega}^*$$

$$x_h \in \hat{\Omega}^2_h$$

$$x_h = \Sigma x_\alpha \psi^\alpha$$

$\{\psi_\alpha\}$ basis for $\hat{\Omega}^1_h \cong TM_{[A]}$

$\{\psi^\alpha\}$ dual basis for $T^*M_{[A]}$

$$x_\alpha \to I(\psi_\alpha)$$

$\circ$

$\circ$

$\circ$

$\Omega^{*}(\hat{\mathfrak{d}} ; \Lambda^{*}(\mathbb{R}_{n}^{\geq 2}))\underset{\text{basic}}{\otimes} C^{\infty}(\mathfrak{d})$ 14

$$\hat{Z}_{k}^{hl}\left((A_0, \delta A_{0,h}, g\,\delta a), x_h ; \mathfrak{d}\right)$$

$$= e^{-ik\langle x_h, \mathfrak{d}\rangle} e^{ik\left[\frac{1}{3!}\langle \frac{\partial}{\partial \mathfrak{d}}, [\frac{\partial}{\partial \mathfrak{d}}, \frac{\partial}{\partial \mathfrak{d}}]\rangle V^{\geq 4}\right]}$$

$$e^{\frac{1}{2ik}\Phi}$$

$$\Phi = \langle \mathfrak{d}, \hat{D}^{-1}\mathfrak{d}\rangle + \langle (\mathbb{1} - \hat{D}^{-1}\delta P_h) x_h, \mathfrak{d}\rangle$$

acting on Ω_M^{*}

$$+ \langle x_h, (\delta P_h)\hat{D}^{-1}(\delta P_h) x_h\rangle$$

$$\hat{D} = P_d(D_M + D_{\hat{M}}) P_\delta = D_M + P_d(\delta P_\delta) P_\delta$$

$\hat{D}^{-1}$ = pseudo-inverse to $\hat{D}$

$$\xrightarrow{CS} \hat{Z}_k^{hl} = \int_{e^M} \mathrm{Tr}\left(e^{\Phi}\right)$$