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CONFERENCE ON
TOPOLOGICAL AND GEOMETRICAL PROBLEMS
RELATED TO QUANTUM FIELD THEORY
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An invariant of Spin 4-manifolds and the 11/8-conjecture

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An Invariant of Spin 4-manifolds and the $11/8$-conjecture

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X : a closed <u>spin</u> 4-manifold with $b_1(X)=0$.

The intersection form of X:

$$2k E_8 \oplus \ell H$$

Assume $k \geq 1$.

(Donaldson 1984?) "Instanton"

- $k \geq 1 \Longrightarrow \ell \geq 3$

(strictly under $H_1(X, \mathbb{Z}/2)=0$)

(Kronheimer-Mrowka 1994) "monopole"

- $k \geq 2 \Longrightarrow \ell \geq 6$
- $k \geq 3 \Longrightarrow \ell \geq 9$

Remark

- $X = K3 \Rightarrow k=1, \ \ell=3$
- $X = (\#_k K3) \# (\#_a S^2 \times S^2)$

$$\Rightarrow \ell = 3k + a$$

$11/8$-Conjecture : $3k \leq \ell$

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OUTLINE

Define an invariant $\varphi(X)$ of X by using the Seiberg-Witten eq^n.

$$\boxed{\varphi(X) \in C(k.l)}$$

a certain set of stable homotopy classes of certain maps.

Lemma $\quad 2k \geq l \Longrightarrow C(k.l) = \phi$

Corollary If $2kE_8 \oplus lH$ is the intersection form of X,

$k \geq 1$ then $\boxed{1+2k \leq l}$

(Recall
$11/8$-conjecture $\Longleftrightarrow \boxed{3k \leq l}$)

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IDEA : Global Kuranishi Map

SW's monopole equation for the spin structure of X

$$L + Q : V \longrightarrow W$$

- $V = L^{s+1}_2 \left(\operatorname{Ker}(\Omega^1 \xrightarrow{d^*} \Omega^0) \oplus \Gamma(S_+) \right)$
- $W = L^s_2 \left(\Omega^+ \oplus \Gamma(S_-) \right)$
- $L = \begin{pmatrix} d^+ & 0 \\ 0 & D \end{pmatrix}$ linear
- $Q : \begin{pmatrix} a \\ \Phi \end{pmatrix} \longmapsto \begin{pmatrix} \sigma(\Phi.\Phi) \\ a\cdot\Phi \end{pmatrix}$ quad.

(s: large s.t. $s - \frac{4}{2} > 0$)

- $\Omega^1 \times \Gamma(S_+) \longrightarrow \Gamma(S_-)$

  $(a, \Phi) \longmapsto a\cdot\Phi$

  Clifford multiplication

- $\Gamma(S_+) \xrightarrow{\sigma} \Omega^+$

  $\Phi \longmapsto \sigma(\Phi, \Phi)$

$S_+ = P_+ \times_{Sp(1)} \mathbb{H}$, $\Omega^+ = P_+ \times_{Sp(1)} \mathbb{R}^3$

$$\begin{array}{ccc} \mathbb{H} & \xrightarrow{\sigma} & \mathbb{R}^3 \\ \wr\wr & & \wr\wr \\ \mathbb{C}\oplus\mathbb{C} & & \mathbb{C}\oplus\mathbb{R} \end{array}$$

$(z, w) \longmapsto (z\bar{w}, |z|^2 - |w|^2)$

$(Sp(1) \times \langle S^1, J\rangle$-equiv^t map$)$

Symmetry : S^1, J

	Ω^*	$\Gamma(S_\pm)$
S^1	trivial	weight 1
J (Kronheimer Mrowka)	-1	j (quaternion)

$G := \langle S^1, J\rangle \subset SU_2 = Sp_1$

(double cover of $O(2)$)

η : the nontrivial rep^n of G of dimension 1 over $\mathbb{R}$.

$$\boxed{\Omega^* = \eta^\infty, \quad \Gamma(S_\pm) = \mathbb{H}^\infty}$$

$$\mathcal{M} := (L+Q)^{-1}(0)/G$$

- A neighbourhood of $[0] \in \mathcal{M}$

has the model

$$\Psi^{-1}(0)/G,$$

$\exists\, \Psi :$ $\mathrm{Ker}\, L \to \mathrm{Coker}\, L$

(Kuranishi map).

$\boxed{\text{IDEA}}$: Describe the whole $\mathcal{M}$ by an extended version of Kuranishi map !

(Remark $\mathcal{M}$ is compact.)

$$\begin{array}{ccc} V & \xrightarrow{L} & W \\ \| & & \| \\ \bigoplus_{\lambda \geq 0} V_\lambda & & \bigoplus_{\lambda \geq 0} W_\lambda \end{array}$$

- $L^*L = \lambda^2$ on V_λ
- $LL^* = \lambda^2$ on W_λ
- $\|v\|_{L_2^{s+1}} = \lambda^{s+1}\|v\|_{L_2}$, $v \in V_\lambda$, $\lambda > 0$
- $\|w\|_{L_2^s} = \lambda^s \|w\|_{L_2}$, $w \in W_\lambda$, $\lambda > 0$

- $\bigoplus_{\lambda > 0} V_\lambda \xrightarrow{L_{\lambda > 0}} \bigoplus_{\lambda > 0} W_\lambda$ isometry

L_2^{s+1} $\quad$ L_2^s

- $P_{\lambda > \Lambda} : W \longrightarrow \bigoplus_{\lambda > \Lambda} W_\lambda$ projection

For each $\Lambda \geq 0$, we have

"Kuranishi map" $\Psi_\Lambda : \bigoplus_{\lambda \le \Lambda} V_\lambda \to \bigoplus_{\lambda \le \Lambda} W_\lambda$

s.t. $\Psi_\Lambda^{-1}(0) / G$

is a model of a nbd of $[0]$.

(Strictly Ψ_Λ is defined only on a nbd of $0 \in \bigoplus_{\lambda \le \Lambda} V_\lambda$)

Lemma For large Λ, the whole of $\mathcal{M}$ is described by the model.

(Key Point ① $\mathcal{M}$ compact, ② $L_2^{s+1} \hookrightarrow L_2^{s}$ compact ③ Hodge theory)

$$f_\Lambda : V \longrightarrow V$$
$$u \longmapsto u + L_{\lambda > \Lambda}^{-1} P_{\lambda > \Lambda} Q(u)$$

* $f_\Lambda \to \mathrm{id}$ on a ball in V, as $\Lambda \to \infty$.

* $f_\Lambda u = v \Rightarrow$

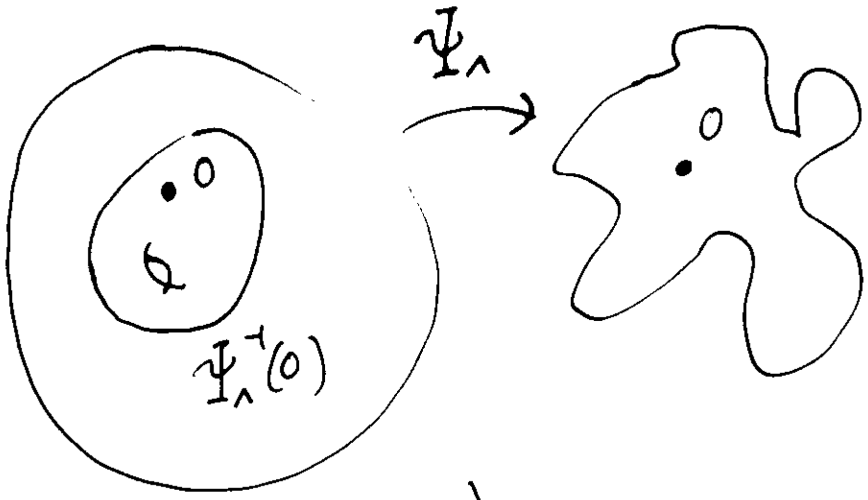
$$(L+Q)u = 0 \iff \begin{cases} v \in \bigoplus_{\lambda \le \Lambda} V_\lambda \\ \Psi_\Lambda v = 0 \end{cases}$$

where $\Psi_\Lambda : \bigoplus_{\lambda \le \Lambda} V_\lambda \longrightarrow \bigoplus_{\lambda \le \Lambda} W_\lambda$

$$v \longmapsto P_{\lambda \le \Lambda} (L+Q) f_\Lambda^{-1}(v)$$

Λ: large

$$\bigoplus_{\lambda \le \Lambda} V_\lambda \xrightarrow{\Psi_\Lambda} \bigoplus_{\lambda \le \Lambda} W_\lambda$$



sphere $S(\bigoplus_{\lambda \le \Lambda} V_\lambda)$

$\Rightarrow$ We have a map

$$\varphi_\Lambda : S\Big(\bigoplus_{\lambda \le \Lambda} V_\lambda\Big) \to S\Big(\bigoplus_{\lambda \le \Lambda} W_\lambda\Big)$$

G-equivariant

Definition.

$\boxed{\varphi(X) := \text{the stable homotopy class of the } G\text{-equiv}^t \text{ map } \varphi_\Lambda \text{ as } \Lambda \to \infty.}$

When the intersection form of X is $2k E_8 \oplus \ell H$,

$$\begin{cases} \bigoplus_{\lambda \le \Lambda} V_\lambda \cong \eta^x \oplus \mathbb{H}^{k+y} \\ \bigoplus_{\lambda \le \Lambda} W_\lambda \cong \eta^{\ell + x} \oplus \mathbb{H}^y \end{cases}$$

for some $x, y \ge 0$.

$(\dim_{\mathbb{R}} \eta = 1, \ \dim_{\mathbb{R}} \mathbb{H} = 4)$

Define

$$\boxed{C(k.l) := \varinjlim_{x,y} \left[S(\eta^{x} \oplus \mathbb{H}^{k+y}),\, S(\eta^{l+x} \oplus \mathbb{H}^{y}) \right]^{G}}$$

then $\boxed{\varphi(X) \in C(k.l)}$.

<u>Ex.</u>

① $\varphi(S^4) = id \in C(0.0)$

② $\varphi(S^2 \times S^2) = emb \in C(0.1)$

<u>Remark</u> $\varphi(X)$ is defined even when $b^+(X)$ is even.

(e.g. ~~Poincaré~~ homotopy 4-sphene)

<u>Ex</u> ③

$X = K3$

$$\begin{array}{ccc} \varphi(K3) & \in & C(1,3) \\ \uparrow & & \uparrow \\ \sigma & \in & [S(\mathbb{H}), S(\eta^3)]^G \end{array}$$

Recall

$$\begin{array}{ccc} \mathbb{H} & \xrightarrow{\sigma} & \eta^3 \\ \shortparallel & & \shortparallel \\ \mathbb{C} \oplus \mathbb{C} & & \mathbb{C} \oplus \mathbb{R} \\ (z.w) & \longmapsto & (z\bar{w}, |z|^2 - |w|^2) \end{array}$$

☹ Take a Calabi Yau metric.

Use the Weitzenböck formula.

To detect $\varphi(x)$ explicitly,
can use generalized cohomology theories.

e.g. (complex) K-theory.

$$\begin{cases} E_0 := (\eta^x \oplus \mathbb{H}^{l+y}) \otimes_{\mathbb{R}} \mathbb{C} \\ E_1 := (\eta^{l+x} \oplus \mathbb{H}^y) \otimes_{\mathbb{R}} \mathbb{C} \\ h := f_\Lambda \otimes \mathbb{C} : S(E_0) \to S(E_1) \end{cases}$$

(G-equivariant)

Define $\boxed{\varphi_K(x) \in K_G(pt) = R(G)}$

as follows

$$\begin{array}{ccc} K_G(B(E_1), S(E_1)) = K_G(pt)\,\tau(E_1) & \ni & \tau(E_1) \\ h^* \downarrow & & \downarrow \\ K_G(B(E_0), S(E_0)) = K_G(pt)\,\tau(E_0) & \ni & \boxed{\varphi_K(x)}\,\tau(E_0) \end{array}$$

(Here $\tau(E_0)$ and $\tau(E_1)$ are the Thom classes.)

For each multiplicative operation,
we have an equation for $\varphi_K(x)$.

Properties of $\varphi_K(X)$

① $\varphi_K(x)\, e(E_0) = e(E_1)$

② $\psi^i(\varphi_K(x))\, \theta_i(E_0) = \theta_i(E_1)\, \varphi_K(x)$ $(i = 1, 2, 3, \ldots)$

Here • $e(E_0) = \Lambda_{-1} E_0$, $e(E_1) = \Lambda_{-1} E_1$
(Euler classes)

• $\psi^i : R(G) \to R(G)$ $(i = 1, 2, 3, \ldots)$
the Adams operations

• θ_i is a characteristic class
defined by

$$\begin{array}{ccc} K_G(B, S) = K_G(pt)\,\tau & \ni & \tau \\ \psi^i \downarrow & & \downarrow \\ K_G(B, S) = K_G(pt)\,\tau & \ni & \theta_i\,\tau \end{array}$$

the Adams operation

$$C_k(k.l) \underset{\text{def}}{:=} \left\{ \alpha \in R(G) \;\middle|\; \begin{array}{l} ① \ \alpha\, e(E_0) = e(E_1) \\ ② \ \psi^i(\alpha)\, \theta_i(E_0) = \alpha\, \theta_i(E_1) \\ \quad i = 1, 2, 3, \cdots\cdots \end{array} \right\}$$

$$\cap$$

$$R(G)$$

$$\begin{array}{ccc} C(k.l) & \longrightarrow & C_k(k.l) \\ \ni & & \ni \\ \varphi(x) & \longmapsto & \varphi_k(x) \end{array}$$

Lemma

$$C_k(k.l) = \begin{cases} \phi & 2k \geq l \\ \{1 \text{ point}\} & 2k < l \end{cases}$$

$\Rightarrow$

- $\boxed{\text{Contradiction if } 2k \geq l}$

  This is what's wanted

- $\varphi_k(x)$ is determined by k and l if $2k < l$

- $\boxed{\text{Conj. } 3k > l \Longrightarrow C(k.l) = \phi}$

- Remark. the above conjecture

  $\Longrightarrow$ 11/8 conjecture.

- non spin case.

  (or general Spin^c-structure case)

  Similar constructions are still possible.

  $\Longrightarrow$ a refinement of the SW-invariants.

Refinement of SW invariants

- X (not assume it spin.)
- $b_1(X) = 0$
- $c \in H^2(X;\mathbb{Z})$ $\quad w_2 = c \bmod 2$
- Fix a spin^c-str for c.
- $d := \dfrac{c^2-\tau}{8} \in \mathbb{Z}$

① $b^+=0$ $\quad \varphi_{X,c} \in \varinjlim_{x,y} \left[S(\mathbb{C}^{d+x}\oplus\mathbb{R}^y), S(\mathbb{C}^x\oplus\mathbb{R}^y) \right]^{S^1}$ (d can be negative)

② $b^+\geq 2$ $\quad \varphi_{X,c} \in \varinjlim_{x,y} \left[S(\mathbb{C}^{d+x}\oplus\mathbb{R}^y)\cup\{0\}\oplus D(\mathbb{R}^y), S(\mathbb{C}^x\oplus\mathbb{R}^{y+b^+}) \right]^{S^1}$

③ $b^+=1$

with chamber $\varphi_{X,c} \in \varinjlim_{x,y} [$ the same as the case $b^+\geq 2]$

no chamber $\varphi'_{X,c} \in \varinjlim_{x,y} [$ the same as the case $b^+=0]$

(The sets of stable htpy classes in ① ② and ③ are abelian group.)

SPECULATIONS

(1) Math

(rough sketch!)	linear	nonlinear
∞-dim^l	e.g. Dirac op^r	e.g. monopole eq^n
finite dim^l geometric realization cf. polarization	det line metric connection η τ	(tower of) global Kuranishi maps (?)
invariants	index KO-char [Dirac] $\in K_*$	stable htpy class φ SW invariants $\boxed{?}$
theorems	· Rochlin · sc. curvatur $>0 \Rightarrow \hat{A}=0$	· Thom conj. "10/8" · sc. curvature $>0 \Rightarrow$ not symplectic ($b^+>1$)

higher (or secondary) SW invariants

(maybe) as

(2) "Physics"

"physics" of monopoles (expectation values by path integral etc) $\overset{??}{\subset}$ information of φ + $\boxed{?}$

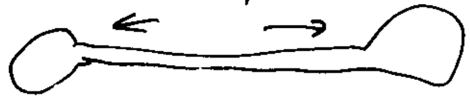
$\Uparrow$ over $\mathbb{R}$ $\qquad$ over $\mathbb{Z}$ $\Uparrow$ $\qquad$ $\Uparrow$ $\boxed{\text{operations related to cut \& paste or "Floer theory"}}$

∃ "physics" over $\mathbb{Z}$??
over p (prime) ??
over "K-theory" ??

cf

$\langle\phi\rangle = \dfrac{\langle 0\mid\phi\mid 0\rangle}{\langle 0\mid 0\rangle}$	$\theta_i = \dfrac{\psi^i\tau}{\tau}$, $\varphi_k(X) = \dfrac{\lambda^k\tau(E_1)}{\tau(E_0)}$
vacuum (or $\int e^S \mathcal{D}x$)	Thom class
$\mathbb{R}$	K-theory

(3) Problems

- $\varphi(X)$ really detects smooth str? cf. $\varphi_k(X)$ No!
- Poincaré conjecture? ∃ counterexample?
- Calculate the set $C(k,l)$
- ∃ Similar construction for instantons or pseudo holomorphic curves?
- Calculate the refinement of the SW invariants when X is Kähler.
- Find an analogue of the refinement in the world of $ch = p$.
- Analize 
- Suppose X_0, X_1 proj alg surfaces
$$X_0 \# m_0 \mathbb{CP}^2 \# n_0 \overline{\mathbb{CP}^2} \underset{\text{dif}}{\simeq} X_1 \# m_1 \mathbb{CP}^2 \# n_1 \overline{\mathbb{CP}^2}.$$
Use the refinement to distinguish these two.
- non spin case. Let $\Sigma \subset X$ be characteristic. Make a situation with symmetry $\langle S^1, J\rangle$.
cf. Prove Rochlin's theorem for non spin manifolds by using $\langle S^1, J\rangle$-symmetry.

SUMMARY

$$\begin{matrix} \mathcal{E} \\ \downarrow \\ \mathcal{B} \end{matrix} \Big) \, s \Leftarrow \text{monopole equation}$$

$$\dim \mathcal{E} = \dim \mathcal{B} + \dim \text{fibre}$$
$$+\infty = \frac{+\infty}{2} + \frac{+\infty}{2}$$

[cf. Problem Session FUKAYA]

- Two reasons why "the Euler class of $\mathcal{E}$" is { zero. / not well-defined

① $\mathcal{E}$ has $\mathbb{R}^{+\infty}$ as a summand,
or more generally $GL(\text{Hilbert space}) \simeq pt$ [cf. Braam's lecture]

② $\dim \mathcal{B} = +\infty$. (want something like $H^{*}_{cpt}(\mathcal{B})$ which is not good when $\dim \mathcal{B} = +\infty$)

$\boxed{\text{IDEA}}$ Use the "Fredholm" section s as a "symmetry" to CANCELL OUT ① & ② and define "the "Euler class" as a defect of the symmetry, or an "anomaly".

- the SW invariants "Euler class in ordinary cohomology" $\rightsquigarrow$ · a refinement "cohomotopy euler class"
- look only the space of solutions $\rightsquigarrow$ · look entire equation
- abstract cobordism argument for well-definedness $\rightsquigarrow$ · use "expectation value" to detect the refinement "at" generalized cohomology theory (with Thom isomorphism) instead of deRham theory which physics usually uses