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CONFERENCE ON
TOPOLOGICAL AND GEOMETRICAL PROBLEMS
RELATED TO QUANTUM FIELD THEORY
(13 - 24 March 1995)

Chern-Simons Perturbation Theory
about
an Instanton Moduli Space

LECTURE 2

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Talk 2 Outline

(1)

I) Some History

II) Basic Propagator $L = IK(D_m^{-1})$

III) Extended Propagator $\tilde{L} = IK(D_{m \times Met}^{-1})$

IV) Basic Blowup $M[2] = Bl(m^2, \Delta)$

V) Definition of $I_\ell \propto \int_{M[V]} Tr(\tilde{L}^{I}_{tot})$

VI) Feynman Rules $\frown \to \tilde{L}_s$; $Y \to \int_M Tr$

VII) Main Thm:

$$\tilde{I}_\ell([A_0]) = I_\ell - \beta_\ell CS_{grav} \text{ is an invt}$$

VIII) Proof:

$$d_{Met} I_\ell = \int_{\partial M[V]} Tr(\tilde{L}^{I}_{tot}) = d_{Met} \beta_\ell CS_{grav}$$

IX) $\ell = 2$ proof more explicitly

I) Some History

- Physicists (L. Alvarez-Gaume, A. Blasi, R. Collins, F. Delduc, F. Gieres, E. Guadagnini, J.M.F. Labastida, C. Lucchesi, M. Martellini, M. Mintchev, O. Piguet, A.V. Ramallo, S.P. Sorella, ...)

poked around:

On $\mathbb{R}^3$: Hidden SUSY + folklore

$\Rightarrow$ Finiteness, 2 loop Knot invts, shift in k, ...

- D. Bar-Natan: 2 loop Knot invts in $\mathbb{R}^3$
- S.A. & I.M. Singer: ("C.S. Pert. Thy", Proc. XXth DGM)

Superspace formulation $\Rightarrow$ finiteness on M. Defined pert series. Proved 2 loop invariance. Formal (Jacobi) argument to all orders.

- M. Kontsevich: outlined program...
- E. Witten: "Chern-Simons Gauge Theory as a String Theory"
- S. A. & I.M. Singer: "C.S. Pert Thy II" JDG 39 ('94)

II Basic Propagator

$$D_M : \Omega^*(M; \mathfrak{g}) \to \Omega^{*+1}(M; \mathfrak{g})$$

Choose $g \in \mathrm{Met}$

$$D_M^{-1} \equiv [\text{pseudo-inverse to } D_M] = D_M^{+} \circ \Delta_M^{-1}$$

$$\{D_M, D_M^{-1}\} = \{D_M, D_M^{+}\}\Delta_M^{-1} = \mathbb{1} - P_h$$

$$L \equiv [\text{integral Kernel for } D_M^{-1}]$$

$$\in \Omega^2(M^2; \mathfrak{g}\otimes\mathfrak{g})_{\text{sing on } \Delta}$$

$$(D_M^{-1}\Psi)_a(x) = \int L(x, y)_{ab} \wedge \Psi_b(y)$$

$\{T_a\} \equiv$ o.n. basis for $\mathrm{Lie}(G)$

$$\Rightarrow D_{M\times M} L(x,y)_{ab} = -\delta_\Delta(x,y)\,\delta_{ab} + P_h(x,y)_{ab}$$

Assume:

D_M acyclic $\Leftrightarrow$ $P_h = 0$

III) Extended Propogator

$$\tilde{L}^{(2,0)} \equiv L \in \Omega^{(2,0)}((M^2 \setminus \Delta) \times \mathrm{Met};\, \mathfrak{g} \otimes \mathfrak{g})$$

$$D_{M \times M}\, D_{\mathrm{Met}}\, \tilde{L}^{(2,0)} = -D_{\mathrm{Met}}\, P_h = 0$$

$$\Downarrow$$

$$D_{\mathrm{Met}}\, \tilde{L}^{(2,0)} = D_{M \times M}\, \tilde{L}^{(1,1)}$$

$$\vdots$$

$$\tilde{L} = \tilde{L}^{(2,0)} + \tilde{L}^{(1,1)} + \tilde{L}^{(0,2)}$$

$$\in \Omega^2_{ce}((M^2 \setminus \Delta) \times \mathrm{Met};\, \mathfrak{g} \otimes \mathfrak{g})$$

In Fact,

$$\tilde{L} = IK(\tilde{D}^{-1})$$

$$\tilde{D}^{-1} = D_M^{\perp} \circ \tilde{\Delta}^{-1}$$

$$\tilde{\Delta} = \{D_M^{\perp}\, D_{M \times \mathrm{Met}}\} = \{D_M^{\perp}\, D_M + \underbrace{x\{D_{\mathrm{Met}},\}}_{\text{0 order in Met}}\}$$

$$\{D_{M \times \mathrm{Met}},\, \tilde{D}^{-1}\} = \tilde{\Delta} \circ \tilde{\Delta}^{-1} = \mathbb{1} - \tilde{P}_h$$

VI) Thm: $\tilde{L}$ extends to

$$M[2] = BL(M^2, \Delta) \quad (\times \mathrm{Met})$$

with

$$\tilde{L}_{ab}\Big|_{\partial M[2]} = \delta_{ab}\, \tilde{\lambda} + (\tilde{b})^{*}(\tilde{\mathfrak{g}})_{ab}$$

$\tilde{b}$ smooth on $\Delta \cong M \quad (\times \mathrm{Met})$

$$\tilde{\lambda}(z, \omega) = -\frac{1}{8\pi}\sqrt{\det g(z)}\; \epsilon_{ijk}\, \omega^i \times [d_{vert}\, \omega^j\, d_{vert}\, \omega^k + \tilde{\Omega}^{jk}(z)]$$

$$= \tilde{\omega} + u \cdot \tilde{\Omega}$$

Volume form on $S(TM)_z$ (1800 Eq. M Greens Functions)

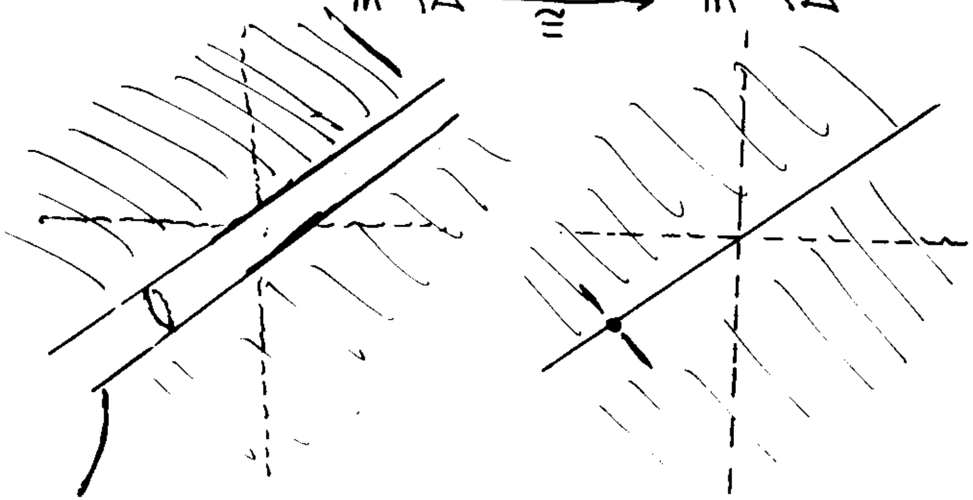
curvature 2-form for $TM \times \mathrm{Met} \to M \times \mathrm{Me}$

1) Basic Blowup

$M[2] = Bl(M^2; \Delta)$

$= (M^2 \setminus \Delta) \cup S(N/\Delta) \xrightarrow{b} M^2$

$M^2 \setminus \Delta \xrightarrow[\cong]{b} M^2 \setminus \Delta$



$\partial M[2] = S(N/\Delta) \xrightarrow{\partial b} \Delta$

$\quad\; \cong S(TM) \qquad\qquad\quad \cong M$

"Coords" near $\partial M[2]$:

$S(TM) \times [0, \epsilon) \dashrightarrow D_\epsilon(TM) \longrightarrow M \times M$

$((z, u), \epsilon) \longmapsto (z, \epsilon u) \mapsto (\exp_z \epsilon u,$ [illegible]$)$

2) Definition of I_ℓ

$$I_\ell \equiv C_{v,I} \int_{M[v]\times\{g\}} Tr^{(v)}\left(\tilde{L}^{I}_{tot,v}\right)$$

$v = 2(\ell - 1)$

$I = 3(\ell - 1)$

where

$M[v] \equiv \overline{M^v \setminus \{diags\}}$

(closure in $M^v \times \prod_{S \subset \{1,\dots,v\}} BL(M^S; \Delta_S)$)

$M[v]$ = submfld w/ corners of ↑

= sequence of blowups of M^v

$\tilde{L}_s = \tilde{L}_{ab}\, j^a_{(1)} \wedge j^b_{(2)} \in \Omega^2(M[2] \times Met; \Lambda^2(\mathfrak{g}_1 \oplus \mathfrak{g}_2))$

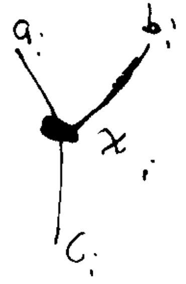
$\tilde{L}_{tot,v} = \sum_{i,j=1}^{v} (\pi_{ij})^* (\tilde{L}_s) \in \Omega^2(M[v] \times Met; A^*_v)$

$A^*_v = \Lambda^*(\mathfrak{g}_1 \oplus \dots \oplus \mathfrak{g}_v) = \bigotimes \Lambda^*(\mathfrak{g}_i)$

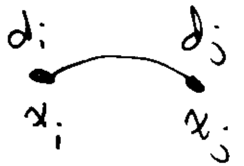
$Tr_v = \bigotimes_{i=1}^{v} Tr_i : A^*_v \to \mathbb{R}$

$Tr_i(j^{a_1}_{(i)} \wedge \dots \wedge j^{a_n}_{(i)}) = \delta_{n,3} f_{a_1 a_2 a_3}$

VI) Feynman Rules



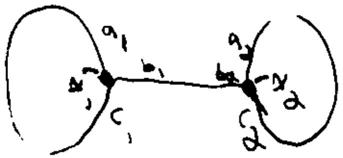
$\longrightarrow \pm \int_{x_i \in M} \sum_{a_i b_i c_i} f_{a_i b_i c_i}$



$\longrightarrow \tilde{L}(x_i, x_j)_{d_i d_j}$

E.g.:

$\rightarrow \pm \frac{1}{12} \int_{M^2} f_{a_1 b_1 c_1} f_{a_2 b_2 c_2} \times \tilde{L}(x_1, x_2)_{a_1 b_1} \tilde{L}(x_1, x_2)_{a_2 b_2} \tilde{L}(x_1, x_2)_{a_3 b_3}$



$\rightarrow \pm \frac{1}{8} \int_{M^2} f_{a_1 b_1 c_1} f_{a_2 b_2 c_2} \times \tilde{P}(x_1)_{a_1 b_1} \tilde{L}(x_1, x_2)_{b_1 b_2} \tilde{P}(x_2)_{a_2 c_2}$

(2a)

VII) Main Theorem:

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1) $\exists$ constants β_ℓ depending only on $(\mathrm{Lie}(G), \mathrm{Tr})$ so that

$$d_{Met}\, I_\ell^{conn}([A^{(0)}], g) = d_{Met}\, \beta_\ell\, CS_{grav}(g, s)$$

where $CS_{grav}(g, s)$ is the Chern-Simons action of the metric connection w.r.t. the canonical framing s.

I.e.

$$\tilde{I}_\ell^{conn}([A_0]) \equiv I_\ell^{conn}([A_0], g) - \beta_\ell\, CS_{grav}(g, s)$$

is a differential invariant.

2) Furthermore $\beta_\ell = 0$ for ℓ odd and β_2 = value expected from exact solutio

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Conjecture: $\beta_\ell = 0$ for $\ell > 2$

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Proof:

$$C_{v,\mathcal{I}}^{-1}\, d_{Met}\, I_\ell = \int_{M[v]} d_{Met}\, Tr^{(v)}\left(\tilde{L}_{tot}^{\mathcal{I}}\right)$$

$$= -\int_{M[v]} d_{M[v]}\, Tr^{(v)}\left(\tilde{L}_{tot}^{\mathcal{I}}\right)$$

$$= -\int_{\partial M[v]} Tr^{(v)}\left(\tilde{L}_{tot}^{\mathcal{I}}\right)$$

Use:

$$\partial M[v] = \bigcup_{S\subset\{1,\dots,v\}} [\partial BL(M^S_{\smile}\Delta_S)]\times M^{\{1,\dots,v\}\setminus S}$$

(mod higher codimension)

Integral over $\partial BL(M^S_{\smile}\Delta_S)$ is explicit in terms of $\tilde{\omega}+v\cdot\tilde{\Omega}$.

Form counting, Lie algebra cohomology, symmetries $\Rightarrow$ Result

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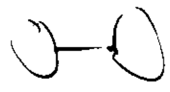
IX) $\ell=2$ proof more explicitly

Recall: $\tilde{L}_{ab} = \tilde{\rho}_{ab} + \delta_{ab}(\tilde{\omega}+v\cdot\tilde{\Omega})$ on ∂M

$d_{Met} I_2 = d_{Met}\Big(\tfrac{1}{8}$  $+ \tfrac{1}{12}$  $\Big)$

$$= \int_{S(TM)} f_{a_1b_1c_1} f_{a_2b_2c_2}\left(\tfrac{1}{8}\tilde{\rho}_{a_1c_1}\tilde{L}_{b_1b_2}\tilde{\rho}_{a_2c_2} + \tfrac{1}{12}\tilde{L}_{a_1a_2}\tilde{L}_{b_1b_2}\tilde{L}_{c_1c_2}\right)$$

$$= (\tilde{\rho}^2 \text{ piece}) + (\tilde{\rho}^1 \text{ piece}) + (\tilde{\rho}^0 \text{ piece})$$

$\tilde{\rho}^2$ piece:

$$\sum f_{a..} f_{a..}\, \tilde{\rho}_{..}\, \tilde{\rho}_{..} = 0$$

by Jacobi

$f_{abc} f_{ade} + \text{perm} = 0$

or:

$\bar{\Lambda}^4(\mathrm{ad}^{\mathrm{diag}})^G \simeq H^4(\mathcal{G}) = 0$

$\tilde{\rho}^1$ piece:

$$\int_{S(TM)} (\tilde{\omega}+v\cdot\tilde{\Omega})^2 = 2\int_{S(TM)_z} \omega\, \sigma^*|_z\, \tilde{\Omega} = 0$$

by symmetry

$\tilde{\rho}^0$ piece:

$$\int_{S(TM)} (\tilde{\omega}+v\cdot\tilde{\Omega})^3 = 3\int_{S(TM)} \tilde{\omega}\,(v\cdot\tilde{\Omega})^2 = \int_M \langle\tilde{\Omega},\tilde{\Omega}\rangle = d_{Met}\, CS_{grav}(g)$$