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# Intersection pairings on moduli spaces of holomorphic bundles on Riemann surfaces

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# I. Introduction

$\Sigma$ cpt Riemann surface, genus $g$

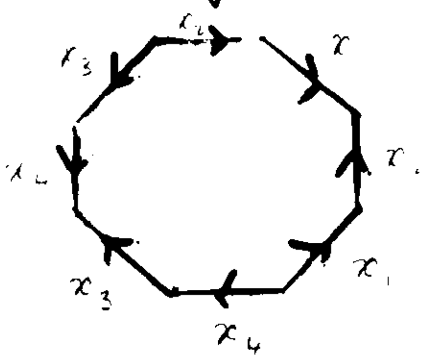
Prototype: $\mathrm{Jac}(\Sigma) \cong U(1)^{2g}$

$= \{(e^{i\theta_1}, \dots, e^{i\theta_{2g}})\}$

(a) 3 descriptions:

(i) $\pi = \pi_1(\Sigma) = \langle x_j,\ j = 1, \dots, 2g \mid R = 1 \rangle$

$$R = \prod_{j=1}^{g} x_{2j-1} x_{2j} x_{2j-1}^{-1} x_{2j}^{-1}$$



$H_1(\Sigma) = \mathbb{Z}^{2g}$

$\mathrm{Jac}(\Sigma) = \mathrm{Hom}(\pi, U(1))$

$= \mathrm{Hom}(H_1(\Sigma), U(1))$



(ii) $P = \Sigma \times U(1)$ trivial principal $U(1)$ bundle over $\Sigma$

Connections: $\mathcal{A} = \Omega^1(\Sigma) \otimes \mathbb{R}$

Flat connections: $\mathcal{A}_F = \{A \in \mathcal{A} : dA = 0\} = Z^1(\Sigma)$

Gauge group $\mathcal{G} = \mathrm{Maps}(\Sigma, U(1))$ acts on $\mathcal{A}$ by $(u : \Sigma \to U(1))$

$u \in \mathcal{G}: A \longmapsto A + u^{-1} du$

$\mathrm{Jac}(\Sigma) \cong \mathcal{A}_F / \mathcal{G}$ [via holonomy map]

(iii) A choice of complex structure on $\Sigma$ endows $\Sigma$ with the structure of complex manifold or Riemann surface.

· Then a flat connection $A$ allows one to specify a $\bar\partial$ operator $\bar\partial_A$ (or structure of holo. line bundle on the product bundle $\Sigma \times \mathbb{C}$).

A section is holo. $\iff$
$$\bar\partial_A s = 0 \quad (\text{fn } s : \Sigma \to \mathbb{C}).$$

· In fact the Jacobian $\mathrm{Jac}(\Sigma)$ parametrizes holo. line bundles on $\Sigma$.

(b) $\mathrm{Jac}(\Sigma)$ is a SYMPLECTIC MANIFOLD: equipped with a nondegenerate closed 2-form
$$\omega = \sum_{j=1}^{g} d\theta_{2j-1} \wedge d\theta_{2j}$$



(c) There is a distinguished line bundle (Poincaré line bundle)

$$\begin{array}{c} \mathbb{L} \\ \downarrow \\ \mathrm{Jac}(\Sigma)\times\Sigma \end{array} \quad ; \quad \mathcal{L} = \mathbb{L}|_{\mathrm{Jac}\times(pt)}$$

$$c_1(\mathcal{L}) = [\omega]$$

(d) $\dim H^0(\mathrm{Jac}(\Sigma); \mathcal{L}^k)$
$$= ch(\mathcal{L}^k)\, Td(\Sigma)[\Sigma]$$
$$= \frac{(k\omega)^g}{g!}[\mathrm{Jac}] = k^g$$

## II. Nonabelian analogue

$G$ compact Lie group eg. $SU(n)$

(a) Three descriptions of a space $\mathcal{M}$:

(i) $\mathcal{M} = Hom(\pi, G) \big/ \substack{\text{conjugation} \\ \text{by } G}$

$$= \left\{ (h_1, \dots, h_{2g}) \in G^{2g} : \prod_{j=1}^{g} h_{2j-1} h_{2j} h_{2j-1}^{-1} h_{2j}^{-1} = 1 \right\} \Big/ G.$$

$\mathcal{M}$ is a compact variety of (real) dimension $(2g-2)\dim G$.

(ii) $P = \Sigma \times G$: trivial principal $G$ bundle over $\Sigma$

Connections: $\mathcal{A} \cong \Omega^1(\Sigma) \otimes Lie(G)$



Flat connections:

$$\mathcal{A}_F = \{ A \in \mathcal{A} : dA + A \wedge A = F_A = 0 \}$$

$$A = \sum_{j=1}^{2} A_j \, dx_j \, , \quad A_j \in Lie(G);$$

$$F_A = \sum_{i,j} \left( \partial_i A_j - \partial_j A_i + [A_i, A_j] \right) dx_i \wedge dx_j$$

Gauge group

$$\mathcal{G} = Maps(\Sigma, G)$$

acts on $\mathcal{A}$ by $(g : \Sigma \to G)$

$$A \mapsto g^{-1} A g + g^{-1} dg$$

$\mathcal{M} \cong \mathcal{A}_F / \mathcal{G}$ via holonomy map

(iii) $\mathcal{M} \cong$ (semistable) rank $n$ degree 0 holomorphic vector bundles over $\Sigma$

A flat connection $\rightsquigarrow \bar{\partial}_A$ defines holomorphic bundle structure on $\Sigma \times \mathbb{C}^n$

( $s : \Sigma \to \mathbb{C}^n$ is holo $\Leftrightarrow \bar{\partial}_A s = 0$ )

Narasimhan - Seshadri 1965: this correspondence is bijective

(b) $\mathcal{M}$ is a symplectic variety (symplectic form $\omega$)

(c) Universal vector bundle

$$\begin{array}{ccc} \mathbb{C}^n & \longrightarrow & \mathbb{U} \\ & & \downarrow \\ & & \mathcal{M} \times \Sigma \end{array} \quad \text{for } G = U(n) \text{ or } SU(n)$$

and line bundle $\mathcal{L}$, $\mathcal{L}^n = \det \mathbb{U}|_{\mathcal{M} \times \{pt\}}$



$c_1(\mathcal{L}) = [\omega]$

d. <u>Verlinde formula</u> (E. Verlinde, 1988)

- formula for $\dim H^0(\mathcal{M}, \mathcal{L}^k)$ as a function of $k \in \mathbb{Z}^+$.

e. $\mathcal{M}$ is singular

BUT: smooth analogue ($G = SU(n)$)

$$\mathcal{M}(n,d) = \left\{ (h_1, \dots, h_{2g}) \in G^{2g} : \prod_{j=1}^{g} [h_{2j-1}, h_{2j}] = e^{\frac{2\pi i d}{n}} \mathbb{1} \right\} / G$$

$$e^{\frac{2\pi i d}{n}} \mathbb{1} \in Z(G) \cong \mathbb{Z}_n$$

If $(d, n)$ coprime, $\mathcal{M}(n,d)$ is a smooth manifold.

<u>Ex</u>. $\mathcal{M}(2,1)$: $e^{\frac{2\pi i d}{n}} \mathbb{1} = -\mathbb{1}$

## III. Generators of cohomology ring

Atiyah–Bott 1983: • determined Betti numbers of $\mathcal{M}(n,d)$

[rank of $H^r(\mathcal{M}(n,d))$ as a vector space]

• determined a set of generators for $H^*(\mathcal{M}(n,d);\mathbb{R})$ as a RING

$$\begin{array}{ccl} \mathbb{U} & & \text{universal vector bundle} \\ \downarrow & & \\ \mathcal{M}(n,d) \times \Sigma & & \end{array}$$

$$c_r(\mathbb{U}) = a_r \otimes 1 + \sum_{j=1}^{2g} b_r^j \otimes \sigma_j + f_r \otimes [\Sigma]$$

$$H^{2r}(\mathcal{M}\times\Sigma) = H^{2r}(\mathcal{M}) \oplus H^{2r-1}(\mathcal{M})\otimes H^1(\Sigma) \oplus H^{2r-2}(\mathcal{M})\otimes H^2(\Sigma)$$



• Atiyah–Bott: the $a_r$, $b_r^j$, $f_r$ $(r=2,\dots,n;\ j=1,\dots,2g)$ generate $H^*(\mathcal{M})$ as a ring.

• Note: $f_2 = [\omega]$

To know structure of ring, must know relations

__Ex.__ $\mathcal{M}(2,1) = \mathcal{M}$

$H^*(\mathcal{M}(2,1))$ is generated by

$a_2 \in H^4(\mathcal{M})$

$f_2 \in H^2(\mathcal{M})$

$b_2^j \in H^3(\mathcal{M})$ $(j=1,\dots,2g)$

__Conj__ [Newstead ] $a_2^g = 0$

(Kirwan 1991)

<u>Conj</u>: [Mumford ] : gave a complete set of relations for $\mathcal{M}(2,1)$

[Kirwan 1991]

Since $\mathcal{M}(n,d)$ satisfies Poincaré duality, knowing <u>relations</u> (all polyns $P$ s.t.

$$P(\{a_r, b_r^j, f_r\}_{\substack{r=2,\dots,n \\ j=1,\dots,2g}}) = 0 \quad )$$

is equiv. to knowing all the <u>intersection pairings</u>

$$\prod_{r,j_r} a_r^{m_r}(b_r^{j_r})^{n_{r,j_r}} f_r^{p_r}[\mathcal{M}(n,d)]$$



$$P(\{a_r, b_r^j, f_r\}) = 0 \iff$$

$$P(\{a_r, b_r^j, f_r\})\, Q(\{a_r, b_r^j, f_r\})\, [\mathcal{M}(n,d)] = 0$$

for all $Q$.

For $\mathcal{M}(2,1)$, the pairings were determined by Thaddeus (1991) and Donaldson (1992):

$$a_2^m f_2^n [\mathcal{M}(2,1)]$$

$$= (-1)^g \frac{n!\, 2^{2g-2}}{(n-g+1)!} (2^{n-g+1}-2) B_{n-g+1}$$

when $2m+n = 3g-3 = \frac{1}{2}\dim \mathcal{M}(2,1)$

and $B_{2N} = \dfrac{(-1)^{N-1}(2N)!\, \zeta(2N)}{2^{2N-1}\pi^{2N}}$ <u>BERNOULLI NUMBERS</u>

(Note $n-g+1 = 2(g-1-m)$)

## IV. Witten formulas

Convention: $x[\mathcal{M}] = 0$ if $\deg x \neq \dim \mathcal{M}$.

$$\prod_{r\geq 2} a_r^{m_r} \exp f_2 \, [\mathcal{M}(n,d)]$$
$$= \mathcal{K} \sum_{\lambda \in \Lambda^{\text{weight}} \cap FWC} e^{2\pi i (\lambda, \tilde{c})} \frac{\tau_r(\lambda)^{m_r}}{\mathcal{D}(\lambda)^{2g-2}}$$

Notation:

$$\mathcal{K} = \left(\frac{\text{vol } SU(n)}{(2\pi)^{\dim SU(n)}}\right)^{2g-2}$$

$\lambda = (\lambda_1, \dots, \lambda_n) \in \Lambda^{\text{weight}} \cap FWC \iff$

$$\lambda = \sum_{j=1}^{n-1} m_j \alpha_j \qquad m_j \in \mathbb{Z}^+$$

$$\alpha_j = (\underbrace{1, \dots 1}_{j}, 0, \dots 0) - \frac{j}{n}(1, \dots, 1)$$

(fundamental weights)



$\tau_r$ : elementary symmetric polyns (permutation invt. polyns on $\mathbb{R}^n$) corresp. to $c_r$

$$\mathcal{D}(\lambda) = \prod_{i<j} (\lambda_i - \lambda_j) \quad \text{VANDERMONDE DETERMINANT}$$

$$\tilde{c} = \frac{d}{n}(1, \dots 1, -(n-1))$$

Ex. $\mathcal{M}(2,1)$ $\qquad \tau_2(x) = -\frac{x^2}{4}$

$$a_2^m \frac{f_2^n}{n!} [\mathcal{M}(2,1)]$$
$$= \frac{1}{2(8\pi^2)^{g-1}} \left(\frac{-1}{4}\right)^m \sum_{\ell \in \mathbb{Z}^+} \frac{(-1)^{\ell+1}}{\ell^{2g-2-2m}}$$
$$= \frac{1}{2(8\pi^2)^{g-1}} \left(\frac{-1}{4}\right)^m \left(1 - \frac{1}{2^{N-1}}\right) \zeta(N)$$

$N = 2(g-1-m)$

This is Donaldson - Thaddeus result.

- Witten gives more complicated formulas (of same type) for

$$\prod_{r, j_r} a_r^{m_r} (b_r^{j_r})^{n_{r,j_r}} f_r^{p_r} [\mathcal{M}(n,d)].$$

- <u>Special cases previously known for general $n$:</u>

## 1. Symplectic volume

$$\mathrm{vol}_\omega \mathcal{M}(n,d) = \frac{f_2^N}{N!} [\mathcal{M}(n,d)]$$

$$= \left( \frac{\mathrm{vol}\, G}{(2\pi)^{\dim G}} \right)^{2g-2} \sum_{\lambda \in \Lambda^{\text{weight}} \cap FWC} \frac{e^{2\pi i (\lambda, \hat{c})}}{\mathcal{D}(\lambda)^{2g-2}}$$

$$\mathrm{vol}_\omega \mathcal{M}(2,1) = \frac{1}{2(8\pi^2)^{g-1}} \sum_{l=1}^{\infty} \frac{(-1)^{l+1}}{l^{2g-2}}$$



Rigorous proof (Witten 1991) using Reidemeister - Ray - Singer torsion.

## 2. VERLINDE FORMULA

(Beauville - Laszlo 1993
Faltings 1993
Kumar - Narasimhan - Ramadas 1994)

Formula for $\dim H^0(\mathcal{M}(n,d), \mathcal{L}^k)$ for $k \in \mathbb{Z}^+$

This follows from Witten's formulas (since $\mathrm{ch}\, \mathcal{L}^k = \exp k f_2$ and $Td\, \mathcal{M}(n,d)$ is a known polynomial in $a_2, \dots, a_n$)

(For $\mathcal{M}(2,1)$ ONLY, Witten - Donaldson - [illegible] formula for intersection pairings follows from Verlinde formula : proved by Szenes 1991, Daskalopoulos - Wentworth [illegible] Narasimhan - Ramadas 1992.)

## V. Symplectic geometry background

Nonabelian localization theorem

(Witten 1992; Jeffrey-Kirwan 1993)

Relates pairings on the symplectic quotient $M_{red}$ of a symplectic manifold $M$ to data on $M$ itself.

$(M, \omega)$ sym. mfd acted on by compact Lie group $G$ preserving $\omega$

$\mu$: moment(um) map

$$M \xrightarrow{\mu} \mathrm{Lie}(G)^*$$

e.g. $G = U(1)$

$$M \xrightarrow{\mu} \mathbb{R}$$



- $d\mu_X(Y) = \omega(X^\#, Y)$ if $X^\#$ is the vector field gen. by $X \in \mathrm{Lie}(G)$
- $\mu$ is $G$-equivariant ($G$ abelian: $\mu$ $G$-invariant)

Symplectic quotient:

$$M_{red} = \mu^{-1}(0)/G$$

- $M_{red}$ is a symplectic orbifold (locally a quotient of smooth manifold by finite group action) if $0$ regular value of $\mu$

$\omega_{red}$ : symplectic form on $M_{red}$

- If $G$ is abelian (torus), can reduce at any regular value $t$, not just $0$:

$$M_t := \mu^{-1}(t)/G.$$

NOTE: Atiyah-Bott (1983:)

$\mathcal{M}(n,d)$ is an (infinite dim'l) symplectic quotient:

$\Omega^1(\Sigma) \otimes \mathrm{Lie}(G) = \mathcal{A}$ is sym. vector space

$\mu \quad A \mapsto F_A = dA + A \wedge A$

is moment map for action of $\mathcal{G}$ (gauge group).



## VI. Background on equivariant cohomology

Equivariant cohomology

$$H^*_G(M) = H^*(\Omega^*_G(M), D)$$

$$(\cong H^*(M \times_G EG))$$

Cartan model

$$\Omega^*_G(M) = \left(S(\mathfrak{g}^*) \otimes \Omega^*(M)\right)^G$$

$$f : \underset{\mathfrak{g}}{X} \longmapsto f(X) \in \Omega^*(M)$$

interior product

$$D_X(f(X)) = d(f(X)) + \iota_{X^\#} f(X)$$

$$D_X \circ D_X = 0$$

Ex. (a) $S(\mathfrak{g}^*)^G$ (invt polyns)

(b) $\overline{\omega}(X) = \omega + (X, \mu) \in \Omega^2_G(M)$ if $M$ symplectic.

$D_X \bar{\omega} = 0$

<u>Kirwan map</u>

$$\Phi : H^*_G(M) \to H^*_G(\mu^{-1}(0)) \xrightarrow{\simeq} H^*(\mu^{-1}(0)/G)$$

$$= H^*(M_{red})$$

If 0 reg. value of $\mu$, $\Phi$ is surjective.



## VII. Residue formula (J - Kirwan 1993)

$$\Phi(\eta) e^{i\omega_{red}} [M_{red}]$$

$$= (\text{const.}) \operatorname{Res} \left( \nu^2 \sum_{F \in \mathcal{F}_+} \int_F \frac{\eta e^{i\overline{\omega}}}{e_F} \right)$$

Here, • $\nu^2 \in H^*_G(pt) = S(\mathfrak{g}^*)^G$

$$\nu^2(X) = \prod_{\text{roots } \gamma > 0} \gamma(X)^2$$

- $F$ : component of fixed point set of maxl torus $T$
- $e_F$ : equivariant Euler class of normal bundle to $F$ in $M$.

Res : (meromorphic forms on $Lie(T) \otimes \mathbb{C}$) $\to \mathbb{C}$

<u>Special case</u>: $T = U(1)$

$G = U(1)$ or $SU(2)$ or $SO(3)$

$\mu^2(X) = X^2 \quad (X \in \mathbb{R} = Lie(T))$

$$\int_{M_{red}} \Phi_0(\eta)\, e^{i\omega_{red}} = (\text{const}) \sum_{\substack{F \in \mathcal{F}: \\ \mu(F) > 0}} \operatorname{Res}_{X=0} \left( \frac{X^2 \int_F \eta\, e^{i\omega} e^{i\mu X}}{e_F(X)} \right),$$

where $e_F(X) = \left( \prod_{j=1}^{N} n_j \right) X^N$ if

$\dim M = 2N$ and

$u \in T = U(1)$ acts on

$\mathbb{C}_{(j)} \subseteq T_F M \cong \mathbb{C}^N$ as

multiplication by $u^{n_j}$

($G = U(1)$: omit $X^2$ factor)



Residue formula is derived from

<u>Duistermaat-Heckman Thm</u> (DH 1982)

$X \in Lie(G)$

$$\int_M e^{i\omega} e^{i(\mu, X)} = \sum_{F \in \mathcal{F}} \int_F \frac{e^{i\omega} e^{i(\mu(F), X)}}{e_F(X)}$$


Special case: $M = S^2$

$G = U(1)$ acts by rotation around $z$ axis

$\omega = dz\, d\phi$

$X^\# = \frac{\partial}{\partial \phi} \Rightarrow \mu(z, \phi) = z$

$$\int_{S^2} e^{i\omega} e^{i(\mu, X)} = \int_{S^2} i\, dz\, d\phi\, e^{izX}$$

$$= i \int_{z=-1}^{1} dz\, e^{izX} = \Bigg( \underbrace{\frac{e^{iX}}{X}}_{F = P_+} + \underbrace{\frac{e^{-iX}}{-X}}_{F = P_-} \Bigg)$$

D. H. Thm is a special case of Abelian localization thm (Berline - Vergne 1982; Atiyah - Bott 1984)

$$\int_M \eta(X) = \sum_{F \in \mathcal{F}} \int_F \frac{i_F^* \eta(X)}{e_F(X)}$$

for $\eta \in \Omega^*_G(M)$, $D\eta = 0$.

LH: $\eta(X) = \exp i\,\bar{\omega}(X)$



Lemma (Szenes)

$$\sum_{\substack{\lambda \in \Lambda^{weight} \\ \cap FWC}} f(\lambda) e^{2\pi i \lambda(\tilde{c})} / \mathcal{D}(\lambda)^{2g-2}$$

$$= \operatorname{Res} \left( \frac{e^{2\pi i X(\tilde{c})} f(X)}{\mathcal{D}(X)^{2g-2} \prod_{\alpha \text{ simple roots}} \cot(\alpha, X)} \right)$$

Special case $M(2,1)$:

$$\sum_{n \in \mathbb{Z}^+} \frac{(-1)^n}{n^{2g-2-2m}} = \pi \operatorname{Res}_{X=0} \frac{e^{i\pi X}}{X^{2g-2-2m}(1-e^{2\pi i X})}$$

$$= 2\left(1 - \frac{1}{2^{2g-3-2m}}\right) \zeta(2g-2-2m)$$

## VIII Extended moduli space [J, '93]

- There is a symplectic space $M(c)$ equipped with a Hamiltonian action of $G$, for which the symplectic quotient $M(c)_0$ is $M(n,d)$
- $G = SU(n)$ $\qquad c = e^{2\pi i d/n}\,\mathbb{1}$

$$\begin{array}{ccc} M & \xrightarrow{\mu} & \mathfrak{g} \\ \downarrow & & \downarrow \exp \\ G^{2g} & \longrightarrow & G \end{array}$$

$$(h_1, \dots, h_{2g}) \longrightarrow \prod_j h_{2j-1} h_{2j} h_{2j-1}^{-1} h_{2j}^{-1}$$

$$M(c) = \Big\{ \big((h_1, \dots, h_{2g}), \Lambda\big) \in G^{2g} \times \mathfrak{g} : \prod_j h_{2j-1} h_{2j} h_{2j-1}^{-1} h_{2j}^{-1} = c \exp \Lambda \Big\}$$



- Origin: $\mathcal{M}(n,d)$ is symplectic quotient of $\mathcal{A}$ w.r.t. gauge group $\mathcal{G}$ while $M(c)$ is symplectic quotient w.r.t. <u>based</u> gauge group $\mathcal{G}_0$, and $\mathcal{G}/\mathcal{G}_0 \cong G$.

<u>Properties</u>: (a) There is a $G$-invt. 2-form $\omega$ on $G^{2g} \times \mathfrak{g}$ which restricts to a symplectic form on $M(c)$

(b) There is a moment map for the action of $G$, given by
$$\mu\big((h_1, \dots, h_{2g}, \Lambda)\big) = \Lambda$$

(c) $M(c)$ is smooth near $\mu^{-1}(0)$ and $G$ acts (locally) freely there

(d) $M(c)_0 = \mu^{-1}(0)/G = \mathcal{M}(n,d)$.

Equivariant cohomology of $M(c)$:

- There are classes $\tilde{a}_r, \tilde{b}_r^j, \tilde{f}_r \in H_G^*(M(c))$ which give rise to the generators $a_r, b_r^j, f_r$ of $H^*(M(n,d))$ under the Kirwan map $\Phi$.
  [L.J., Duke Math J., in press]
- In fact $\tilde{a}_r = \tau_r \in H_G^*(pt) \to H_G^*(M(c))$



Want to apply residue formula to $M(c)$ to get Witten's formulas:

Formally: Fixed point set of $T$ in $M(c)$

$$\begin{array}{ccc} M^T & \longrightarrow & \exp^{-1}(c) \subseteq \mathfrak{t} \\ \downarrow & & \downarrow \\ T^{2g} & \longrightarrow & c^{-1} \in T \end{array}$$

$e_F(X) = \nu^{2g}(X)$

- $M^T = T^{2g} \times (\mathrm{Ker}(\exp) \subseteq \mathfrak{t} \cong \mathbb{Z}^{n-1})$

$e_F(X)$
$= e_\circ(T^{2g} \hookrightarrow \mathbb{C}^{2g})$
$= \nu(X)^{2g}$.

For simplicity $\mathcal{M}(2,1)$ : $c = -1$

$e_F(X) = X^{2g}$

- Values of moment map:

$1 = c \exp \Lambda = -\exp \Lambda$

$\Longleftrightarrow \boxed{\Lambda = (2k+1)\pi}$

$$\int_{M(2,1)} a^j e^f$$

$$= \int_{M(2,1)} \Phi_G\left(X^{2j} e^{i\bar{\omega}}\right) = \mathrm{Res}_0 \sum_{k\geq 0} \frac{e^{(2k+1)i\pi X}}{X^{2g-2-2j}}$$

$$= \mathrm{Res}_0 \frac{1}{X^{2g-2-2j}} \frac{e^{i\pi X}}{1-e^{2i\pi X}} = \mathrm{Res}_0\left(\frac{-1}{2i X^{2g-2-2j} \sin \pi X}\right)$$

$$= K\zeta(2g-2-2j) \text{ as desired.}$$

- Application of residue formula to $M(c)$ is unjustified since $M(c)$ is noncompact and singular.
- We recover Witten's formula by desingularizing $M(c)$ then applying some additional properties:



## IX. Extra steps in proof

(a) Reduction to maximal torus (Shaun Martin)

$$\int_{\mu_G^{-1}(0)/G} \Phi_G\left(\eta e^{i\bar{\omega}}\right) = \int_{\mu_G^{-1}(0)/T} \Phi_G\left(\nu \eta e^{i\bar{\omega}}\right)$$

$$= \int_{\mu_T^{-1}(0)/T} \Phi_G\left(\nu^2 \eta e^{i\bar{\omega}}\right)$$

(b) Change under $t_0 \to t_1$ (Guillemin–Kalkman; S. Martin)

(special case $T = U(1)$)

$$\int_{M_{t_0}} \Phi_T\left(\eta e^{i\bar{\omega}}\right) - \int_{M_{t_1}} \Phi_T\left(\eta e^{i\bar{\omega}}\right)$$

$$= \sum_{\substack{F \in \mathcal{F}: \\ t_0 < \mu(F) < t_1}} \mathrm{Res}_0\left(\int_F \frac{\eta(X) e^{i\omega} e^{i\mu_T(F)X}}{e_F(X)}\right)$$

if $T$ acts freely on $\mu_T^{-1}(t_0), \mu_T^{-1}(t_1)$

(c) Periodicity

$M_{\frak{t}}(c) \stackrel{\text{def}}{=} \mu^{-1}(\frak{t}) \subseteq M(c)$

is acted on by $T$:

Lemma There is a homeo. $S_m : M_{\frak{t}}(c) \to M_{\frak{t}}(c)$

$(\underline{h}, \Lambda) \to (\underline{h}, \Lambda + 2\pi m)$

for every $m \in \mathbb{Z}$

$(2\pi i m \in \mathrm{Ker}(\exp))$



Desingularization of $M(c)$:

For generic $\xi \in \mathrm{Lie}(T)$,

$M(c \exp \xi) \stackrel{\text{def}}{=} \{ (h_1, \dots, h_{2g}, \Lambda) \in G^{2g} \times \mathrm{Lie}$

s.t. $\prod_{j=1}^{g} h_{2j-1} h_{2j} h_{2j-1}^{-1} h_{2j}^{-1} = c \exp \xi \exp \Lambda \}$

is smooth.

We may reformulate the problem so we can work with $M(c \exp \xi)$ rather than with $M(c)$.

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