INTERNATIONAL ATOMIC ENERGY AGENCY
UNITED NATIONS EDUCATIONAL, SCIENTIFIC AND CULTURAL ORGANIZATION
INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS
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SMR.847/8

CONFERENCE ON

TOPOLOGICAL AND GEOMETRICAL PROBLEMS

RELATED TO QUANTUM FIELD THEORY

(13 - 24 March 1995)

Moduli of vector bundles on curves and integrable systems

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MODULI OF VECTOR BUNDLES ON CURVES AND INTEGRABLE SYSTEMS

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The purpose of this talk is to show that the multi-component KP equations generate all the Hamiltonian systems defined by Hitchin on the cotangent bundle of the moduli spaces of stable vector bundles over an algebraic curve. In order to compare Hitchin's Hamiltonian systems and the multi-component KP equations, we give a natural mapping of the cotangent bundle of the moduli spaces of vector bundles into an infinite-dimensional Grassmannian on which the KP equations are defined. Through this mapping the Hamiltonian vector fields of the Hitchin system are sent exactly to the multi-component KP flows.

Hitchin's discovery [5] of the *abelianization* is motivated by the mysterious equality of the dimensions

$$\dim U(n,d) = \dim \bigoplus_{i=1}^{n} H^0(X, K_X^i) = n^2(g-1)+1 \ ,$$

where $U(n,d)$ is the moduli space of stable vector bundles of rank n and degree d defined on an algebraic curve X of genus g, and K_X denotes the canonical bundle of X. Since the cotangent space $T_E^*U(n,d)$ of the moduli space $U(n,d)$ at a vector bundle E is given by $H^0(X, End(E)\otimes K_X)$, the cotangent bundle $T^*U(n,d)$ can be identified with the set of pairs (E,ϕ) with $\phi \in H^0(X, End(E)\otimes K_X)$. If we consider ϕ as a linear operator acting on E with coefficients in K_X, then the i-th coefficient s_i of the characteristic polynomial

$$\det(y-\phi) = y^n - s_1 y^{n-1} + s_2 y^{n-2} - \cdots + (-1)^n s_n$$

of ϕ is an element of $H^0(X, K_X^i)$. Let us denote by

$$ch(\phi) = (s_1, s_2, \cdots, s_n) = s \in \bigoplus_{i=1}^{n} H^0(X, K_X^i)$$

the characteristic coefficients of ϕ. It defines the *Hitchin map*

$$H : T^*U(n,d) \ni (E,\phi) \longmapsto ch(\phi) \in \bigoplus_{i=1}^{n} H^0(X, K_X^i) \ . \tag{1}$$

Since the target space of the map is a vector space of dimension $n^2(g-1)+1$, the Hitchin map can be thought of as a set of $n^2(g-1)+1$ holomorphic functions defined on the cotangent bundle $T^*U(n,d)$. An amazing fact Hitchin discovered is that these functions are *Poisson commutative* with respect to the canonical holomorphic symplectic structure of $T^*U(n,d)$. These commuting functions give rise to a set of commuting Hamiltonian vector fields. He proved that every integral submanifold (or orbit) of these commuting vector fields is a Zariski open subset of some Jacobian variety, and they are *linear* with respect to the canonical linear structure of the Jacobian variety.

It is a well-known fact that the *KP equations* define linear flows on every Jacobian variety. Therefore, it is quite reasonable to ask if the Hitchin systems and the KP equations are related. Even though the question is natural and the answer is most likely yes, nothing definite has been known till now. There have been several difficulties to answer this question. First of all, no explicit form of the differential equations representing the Hitchin system is known, because of the lack of natural coordinate systems on the moduli spaces of vector bundles. This situation makes it impossible to compare the Hitchin systems and the KP equations directly. The second difficulty is due to the absence of a relation between the moduli space $U(n,d)$ and the infinite-dimensional Grassmannian on which the KP flows are defined. Although a functorial relation between arbitrary vector bundles on curves and the Grassmannian has been known since [10], it does not give any map of the moduli spaces into the Grassmannian.

In the meantime, there have been some new developments ([1], [7]) in the relation between algebraic geometry of curves and infinite-dimensional integrable systems. In this new context, the algebro-geometric data consist of morphisms of algebraic curves and vector bundles on these curves, and the infinite-dimensional integrable systems appearing in this context are defined by the action of loop algebras on the Grassmannian of vector-valued functions. Since we can deal with arbitrary morphisms of curves, we can handle all Prym varieties in this framework. Actually, one of the main results of the work [7] is to give a characterization of arbitrary Prym varieties in terms of the multi-component KP equations generalizing the similar result for Jacobian varieties of [2] and [9]. The key idea of [7] is to determine the data in the context of integrable systems that correspond to the algebro-geometric information of a morphism between two curves. Since the authors have established a categorical equivalence between morphisms of algebraic curves and points of the Grassmannian, we can now define a mapping of the cotangent bundle of the moduli space of vector bundles into the Grassmannian, because the Hitchin map (1) gives a family of morphisms

$$f_s : X_s \longrightarrow X$$

of *spectral curves* parametrized by $s \in \bigoplus_{i=1}^{n} H^0(X, K_X^i)$. The spectral curve X_s is an n-sheeted covering of X defined as a subvariety of the total space $|K_X|$ of the canonical bundle by the characteristic equation

$$y^n - s_1(z)y^{n-1} + s_2(z)y^{n-2} - \cdots (-1)^n s_n(z) = 0 ,$$

where y is a linear coordinate of the fiber of K_X and z is a local coordinate of the curve X.

The Cartan subalgebra $\mathcal{H}$ of the loop algebra acts on the Grassmannian which produces the n-component KP equations. One can compare the vector fields on the Grassmannian given by the action of $\mathcal{H}$ and Hitchin's Hamiltonian vector fields through the mapping of $T^*U(n,d)$ into the Grassmannian.

Once we set the framework of our theory, it is automatic to extend the setting to include a generalization of the Hitchin system due to Markman [8], as well as the Hamiltonian systems related to the moduli spaces of vector bundles with a fixed determinant.

We begin this talk with reviewing some standard concepts of the holomorphic symplectic geometry in Part 1. In Part 2 we summarize the theory of spectral curves that we need to prove our main theorems. The theory was developed by Hitchin [5], Beauville, Narasimhan and Ramanan [4], Markman [8] and others. In Part 3 we define the Grassmannian and the multi-component KP flows, and prove the embedding theorem of a Zariski open subset of the moduli space of stable Higgs pairs into the Grassmannian. After giving a natural embedding, the compatibility of the Hitchin systems and the multi-component KP systems becomes an automatic consequence.

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