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I.C.T.P., P.O. BOX 586, 34100 TRIESTE, ITALY, CABLE: CENTRATOM TRIESTE



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**Determinant line bundles revisited**

D.S. FREED  
Department of Mathematics  
University of Texas at Austin  
Austin, TX 78712  
U.S.A.

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These are preliminary lecture notes, intended only for distribution to participants

# Determinant Line Bundles Revisited

DANIEL S. FREED

Department of Mathematics  
University of Texas at Austin

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This note is an addendum to joint work with Xianzhe Dai [DF].<sup>1</sup> In that paper we investigate the geometric theory of  $\eta$ -invariants of Dirac operators on manifolds with boundary. We summarize the main results below. One key geometric observation is that the exponentiated  $\eta$ -invariant naturally takes values in the *determinant line* of the boundary. As such it is intimately related to the geometry of determinant line bundles for *families* of Dirac operators. The differential geometry of determinant line bundles was developed first by Quillen [Q] in a special case, and then by Bismut and Freed [BF1], [BF2] in general. (See [F1] for an exposition of these results.) In §5 of [DF] the results on  $\eta$ -invariants are used to reprove the holonomy formula for determinant line bundles, also known as Witten's global anomaly formula [W]. However, the argument there is unnecessarily complicated. The main purpose of this note, then, is to reprove *both* the curvature and holonomy formulas for determinant line bundles using the results of [DF].

To avoid repetitious recitation of requirements, we set some conventions here which apply throughout. We work with *compact* Riemannian manifolds. If the boundary is nonempty we assume that the metric is a product near the boundary. Our results hold for any Dirac operator on a spin<sup>c</sup> manifold coupled to a vector bundle with connection, but for simplicity we state the formulas only for the basic Dirac operator on a spin manifold. Thus all manifolds are assumed spin. We use the  $L^2$  metric on the spinor fields  $S$ . A *family of Riemannian manifolds* is a smooth fiber bundle  $\pi: X \rightarrow Z$  together with a metric on the relative (vertical) tangent bundle  $T(X/Z)$  and a distribution of "horizontal" complements to  $T(X/Z)$  in  $TX$ . We assume that  $T(X/Z)$  is endowed with a spin structure. Also, when working with families of manifolds with boundary, we assume that the Riemannian metrics on the fibers are products near the boundary. There is an

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<sup>1</sup>In [DF] the reader will find an extensive discussion of related work and a bibliography.

induced family  $\partial\pi: \partial X \rightarrow Z$  of closed manifolds. Finally, we will always use ‘ $X$ ’ to denote an odd dimensional manifold and ‘ $Y$ ’ to denote an even dimensional manifold.

As stated earlier this is a continuation of joint work with Xianzhe Dai.

### Eta Invariants on Manifolds with Boundary

First recall that on a *closed* odd dimensional manifold  $X$  the Dirac operator  $D_X$  is self-adjoint and has a discrete spectrum  $\text{spec}(D_X)$  extending to  $+\infty$  and  $-\infty$ . The  $\eta$ -invariant of Atiyah-Patodi-Singer [APS] is defined by meromorphic continuation of the function

$$\eta_X(s) = \sum_{\substack{\lambda \neq 0 \\ \lambda \in \text{spec}(D_X)}} \frac{\text{sign } \lambda}{|\lambda|^s},$$

which by general estimates converges for  $\text{Re}(s)$  sufficiently large. In fact, for Dirac operators the meromorphic continuation is analytic for  $\text{Re}(s) > -2$  [BF2, Theorem 2.6]. In any case  $\eta_X$  is regular at  $s = 0$ , and we set

$$(1) \quad \tau_X = \exp \pi i(\eta_X(0) + \dim \text{Ker } D_X) \in \mathbb{C}.$$

The general theory of  $\eta$ -invariants shows that  $\tau_X$  varies smoothly in families, whereas the  $\eta$ -invariant  $\eta_X(0)$  is discontinuous in general. Note that  $|\tau_X| = 1$ .

On a manifold with boundary we need to specify elliptic boundary conditions to obtain an operator with discrete spectrum. We use the boundary conditions introduced by Atiyah-Patodi-Singer, but adapted to odd dimensional manifolds  $X$ . This involves an additional piece of information concerning  $\text{Ker } D_{\partial X}$ . Recall that on an even dimensional manifold  $Y$  the spinor fields  $S_Y$  split as  $S_Y = S_Y^+ \oplus S_Y^-$ , and the Dirac operator  $D_Y: S_Y^\pm \rightarrow S_Y^\mp$  interchanges the positive and negative pieces. (In the sequel we use ‘ $D_Y$ ’ to denote the operator  $D_Y: S_Y^+ \rightarrow S_Y^-$ .) If  $Y = \partial X$  is the boundary of an odd dimensional manifold  $X$ , then  $\dim \text{Ker}^+ D_{\partial X} = \dim \text{Ker}^- D_{\partial X}$ . The additional piece of information we must choose as part of the boundary condition is an isometry

$$T: \text{Ker}^+ D_{\partial X} \longrightarrow \text{Ker}^- D_{\partial X}.$$

Then the basic analytic properties of  $D_X$  with these boundary conditions are the same as those of the Dirac operator on a closed manifold, and so the invariant (1) is defined. Its dependence on  $T$  is simple, and factoring this out we observe that

$$(2) \quad \tau_X \in \text{Det}_{\partial X}^{-1},$$

where  $\text{Det}_{\partial X}$  is the *determinant line* of the Dirac operator  $D_{\partial X}$  on the boundary:

$$(3) \quad \text{Det}_{\partial X} = (\text{Det } \text{Ker}^- D_{\partial X}) \otimes (\text{Det } \text{Ker}^+ D_{\partial X})^{-1}.$$

(Recall that  $\text{Det } V = \bigwedge^n V$  for an  $n$  dimensional vector space  $V$ . Also  $L^{-1} = L^*$  for a one dimensional vector space  $L$ .) Properly normalized we have  $|\tau_X| = 1$  in the *Quillen metric* on  $\text{Det}_{\partial X}^{-1}$ .

Now suppose  $X \rightarrow Z$  is a family of odd dimensional manifolds with boundary. Then  $\partial X \rightarrow Z$  is a family of closed even dimensional manifolds. The determinant lines (3) patch together to form a smooth determinant line bundle  $\text{Det}_{\partial X/Z} \rightarrow Z$ . Furthermore, it carries the *Quillen metric* and a canonical connection  $\nabla$ , as defined in [BF1]. The exponentiated  $\eta$ -invariant is now a smooth section

$$\tau_{X/Z} : Z \longrightarrow \text{Det}_{\partial X/Z}^{-1}.$$

There are two basic results about this invariant: a variation formula and a gluing law. The variation formula computes the derivative of  $\tau_{X/Z}$  in a family.

**Theorem 4** [DF, Theorem 1.9]. *With respect to the canonical connection  $\nabla$  on  $\text{Det}_{\partial X/Z}^{-1}$ ,*

$$\nabla \tau_{X/Z} = 2\pi i \left[ \int_{X/Z} \hat{A}(\Omega^{X/Z}) \right]_{(1)} \cdot \tau_{X/Z}.$$

Here  $\Omega^{X/Z}$  is the Riemannian curvature of  $X \rightarrow Z$  and  $\hat{A}$  is the usual  $\hat{A}$ -polynomial. (For other Dirac operators substitute the appropriate index polynomial in place of  $\hat{A}$ .) The ‘(1)’ denotes the 1-form piece of the differential form. For a family of *closed* manifolds this is a result of Atiyah-Patodi-Singer. The new point here is the relationship of  $\tau$  with the canonical connection  $\nabla$ . This plays a crucial role in the next section.

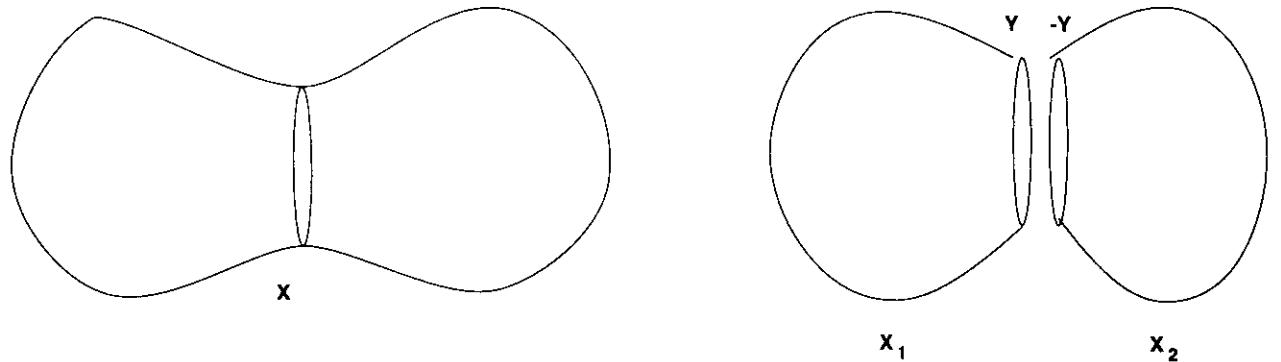


FIGURE 1. Cutting a closed manifold into two pieces.

The simplest case of the gluing law is for a *closed* manifold  $X$  split into two pieces  $X_1, X_2$  along a closed oriented codimension one submanifold  $Y \hookrightarrow X$ . (See Figure 1.) Then  $\tau_{X_1} \in \text{Det}_Y^{-1}$  and  $\tau_X \in \mathbb{C}$ .

**Theorem 5** [DF, Theorem 2.20]. *In this situation*

$$\tau_X = (\tau_{X_1}, \tau_{X_2})_{\text{Det}_Y^{-1}}.$$

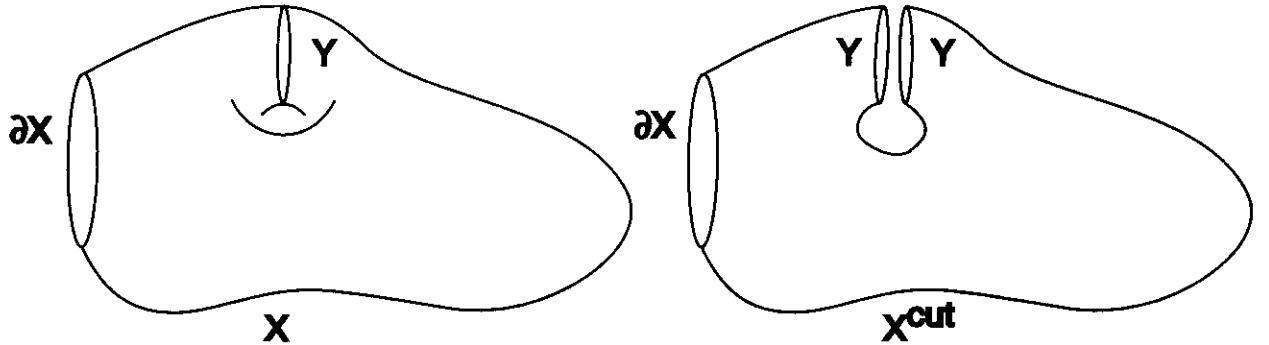


FIGURE 2. Cutting a manifold along a submanifold.

The more general gluing formula, which we need in the next section, applies when  $X$  has boundary. Then for  $Y \hookrightarrow X$  a closed oriented codimension one submanifold we cut along  $Y$  to obtain a new manifold  $X^{\text{cut}}$  with  $\partial X^{\text{cut}} = \partial X \cup Y \cup -Y$ . (See Figure 2.) Now

$$(6) \quad \begin{aligned} \tau_X &\in \text{Det}_{\partial X}^{-1} \\ \tau_{X^{\text{cut}}} &\in \text{Det}_{\partial X}^{-1} \otimes \text{Det}_Y^{-1} \otimes \text{Det}_{-Y}^{-1} \\ &\cong \text{Det}_{\partial X}^{-1} \otimes L_Y \otimes L_Y^{-1}, \end{aligned}$$

where  $L_Y = \text{Det}_Y^{-1}$ . There is now a sign which enters the gluing formula, and it is nicely taken care of by the following device. In general we view the determinant line  $\text{Det } V$  of a vector space  $V$  as a one dimensional *graded* vector space whose grading is given by  $\dim V$ . Applied to (3) we see that  $\text{Det}_Y$  (and so also  $\text{Det}_Y^{-1}$ ) is graded by the *index* of the Dirac operator  $D_Y$ . Notice that in our current situation  $Y$  does not necessarily bound a 3-manifold, and so its index may be nonzero. Let

$$(7) \quad \text{Tr}_s: L_Y \otimes L_Y^{-1} \longrightarrow \mathbb{C}$$

be the usual contraction times the grading; i.e., if index  $D_Y$  is even it is the usual contraction and if index  $D_Y$  is odd it is minus the usual contraction. That understood, we state the general gluing formula.

**Theorem 8** [DF, Theorem 2.20]. *In this situation*

$$(9) \quad \tau_X = \text{Tr}_s(\tau_{X^{\text{cut}}}).$$

One of the novel points of [DF] is the proof of the gluing law, which we do not discuss here.

### Determinant Line Bundles and Adiabatic Limits

The application we discuss is to the geometry of the determinant line bundle. Suppose  $\pi: Y \rightarrow Z$  is a family of *closed* even dimensional manifolds. Let  $L = \text{Det}_{Y/Z}^{-1}$  be the inverse determinant line bundle of the family. The results in the last section use the Quillen metric and the construction of the canonical connection  $\nabla$ . But they do not depend on the formulas for the curvature and holonomy of  $\nabla$ , which were proved in [BF1], [BF2]. Here we derive the curvature and holonomy formulas from Theorem 4 and Theorem 8.<sup>2</sup> The basic idea is to use the  $\tau$ -invariant (2) to define the parallel transport of a new connection  $\nabla'$  on  $L$ . Thus suppose  $\gamma: [0, 1] \rightarrow Z$  is a smooth path<sup>3</sup> in  $Z$ . Denote  $I = [0, 1]$ . Let  $Y_\gamma = \gamma^*(Z) \rightarrow I$  be the pullback of the family  $\pi: Y \rightarrow Z$  by the path  $\gamma$ . Then  $Y_\gamma$  is an odd dimensional manifold with  $\partial Y_\gamma = Z_{\gamma(1)} \sqcup -Z_{\gamma(0)}$ . The standard metric  $g_I$  on  $I = [0, 1]$  determines a metric on  $Y_\gamma$ , since we already have a metric  $g_{Y_\gamma/I}$  on the fibers and a distribution of horizontal planes. (The projection  $\pi: Y_\gamma \rightarrow I$  is then a Riemannian submersion.) The  $\tau$ -invariant of  $Y_\gamma$  is a linear map

$$(10) \quad \tau_{Y_\gamma}: L_{\gamma(0)} \longrightarrow L_{\gamma(1)},$$

exactly what we need to define parallel transport. However, (10) does *not* define parallel transport since it is not independent of the parametrization of the path  $\gamma$ . To get a quantity independent of parametrization we introduce the *adiabatic limit* as follows. For each  $\epsilon \neq 0$  consider the metric

$$(11) \quad g_\epsilon = \frac{g_I}{\epsilon^2} \oplus g_{Y_\gamma/I}$$

on  $Y_\gamma$  relative to the decomposition  $TY_\gamma \cong \pi^*TI \oplus T(Y_\gamma/I)$ . Let  $\tau_{Y_\gamma}(\epsilon)$  be the  $\tau$ -invariant for this metric.

**Lemma 12.** *The adiabatic limit*

$$(13) \quad \tau_\gamma = \text{a-lim } \tau_{Y_\gamma} = \lim_{\epsilon \rightarrow 0} \tau_{Y_\gamma}(\epsilon)$$

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<sup>2</sup>As was mentioned in the introduction, this was done in [DF, §5] in an unnecessarily complicated way. Also, there we *used* the curvature formula instead of *proving* it. This section should be considered a rewrite of [DF, §5].

<sup>3</sup>Since we need a cylindrical metric near the boundary of  $Y_\gamma$  defined below, we require that  $\gamma([0, \delta])$  and  $\gamma([1-\delta, 1])$  be constant for some  $\delta$ .

exists and is invariant under reparametrization of  $\gamma$ .

Notice that the adiabatic limit is introduced for a simple geometrical reason—to scale out the dependence of  $\tau$  on the parametrization of  $\gamma$ .

*Proof.* Here we follow [DF, §5].<sup>4</sup> As a preliminary we state without proof a simple result about the Riemannian geometry of adiabatic limits. Let  $\nabla^{Y_\gamma}(\epsilon)$  denote the Levi-Civita connection on  $Y_\gamma$  of the metric (11) and  $\Omega^{Y_\gamma}(\epsilon)$  its curvature. The result we need, which follows from a straightforward computation, is that  $\text{a-lim } \nabla^{Y_\gamma} = \lim_{\epsilon \rightarrow 0} \nabla^{Y_\gamma}(\epsilon)$  exists and is torsionfree. Furthermore, the curvature of this limiting connection is the limit of the curvatures of  $\nabla^{Y_\gamma}(\epsilon)$  and has the form

$$\text{a-lim } \Omega^{Y_\gamma} = \lim_{\epsilon \rightarrow 0} \Omega^{Y_\gamma}(\epsilon) = \begin{pmatrix} 0 & 0 \\ * & \Omega^{Y_\gamma/I} \end{pmatrix}$$

It follows that

$$(14) \quad \text{a-lim } \hat{A}(\Omega^{Y_\gamma}) = \lim_{\epsilon \rightarrow 0} \hat{A}(\Omega^{Y_\gamma}(\epsilon)) = \hat{A}(\Omega^{Y_\gamma/I}).$$

We apply this result to arbitrary families of manifolds, where it also holds.

To prove that the adiabatic limit exists, consider the family of Riemannian manifolds  $Y_\gamma \times \mathbb{R}^{\neq 0} \rightarrow \mathbb{R}^{\neq 0}$ , where the metric on the fiber at  $\epsilon$  is (11). According to the variation formula Theorem 4 we have

$$\frac{d}{d\epsilon} \tau_{Y_\gamma}(\epsilon) = 2\pi i \left[ \int_{(Y_\gamma \times \mathbb{R}^{\neq 0})/\mathbb{R}^{\neq 0}} \hat{A}(\Omega^{(Y_\gamma \times \mathbb{R}^{\neq 0})/\mathbb{R}^{\neq 0}}) \right]_{(1)}.$$

Now (14) implies that

$$\lim_{\epsilon \rightarrow 0} \hat{A}(\Omega^{(Y_\gamma \times \mathbb{R}^{\neq 0})/\mathbb{R}^{\neq 0}}) = \hat{A}(\Omega^{(Y_\gamma \times \mathbb{R}^{\neq 0})/(I \times \mathbb{R}^{\neq 0})}) = \rho^*(\hat{A}(\Omega^{Y_\gamma/I})),$$

where  $\rho: Y_\gamma \times \mathbb{R}^{\neq 0} \rightarrow Y_\gamma$  is the projection. In particular, this limit is independent of  $\epsilon$ , and so

$$\lim_{\epsilon \rightarrow 0} \frac{d}{d\epsilon} \tau_{Y_\gamma}(\epsilon) = \lim_{\epsilon \rightarrow 0} 2\pi i \left[ \int_{(Y_\gamma \times \mathbb{R}^{\neq 0})/\mathbb{R}^{\neq 0}} \rho^* \hat{A}(\Omega^{Y_\gamma/I}) \right]_{(1)} = 0.$$

Therefore,  $\text{a-lim } \tau_{Y_\gamma} = \lim_{\epsilon \rightarrow 0} \tau_{Y_\gamma}(\epsilon)$  exists.

A similar argument proves that  $\tau_\gamma$  is invariant under reparametrization. Let  $\mathcal{D}$  denote the space of diffeomorphisms  $\phi: [0, 1] \rightarrow [0, 1]$  with  $\phi(0) = 0$  and  $\phi(1) = 1$ . We pull back  $\pi: Y \rightarrow Z$  via the map

$$\begin{aligned} [0, 1] \times \mathbb{R}^{\neq 0} \times \mathcal{D} &\longrightarrow Z \\ \langle t, \epsilon, \phi \rangle &\longmapsto \gamma(\phi(t)) \end{aligned}$$

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<sup>4</sup> And we correct a mistake in the exposition there.

to construct the family of manifolds

$$\mathcal{Y} \longrightarrow \mathbb{R}^{\neq 0} \times \mathcal{D},$$

where the metric on the fiber over  $\langle \epsilon, \phi \rangle$  is (11). As in the previous argument we compute the differential of  $\tau_{Y_{\gamma \circ \phi}}(\epsilon, \phi)$  in the adiabatic limit:

$$(15) \quad \lim_{\epsilon \rightarrow 0} d\tau_{\langle \epsilon, \phi \rangle} = 2\pi i \sigma^* \left[ \int_{Y/Z} \hat{A}(\Omega^{Y/Z}) \right]_{(2)},$$

where

$$\begin{aligned} \sigma: [0, 1] \times \mathcal{D} &\longrightarrow Z \\ \langle t, \phi \rangle &\longmapsto \gamma(\phi(t)) \end{aligned}$$

We conclude that (15) vanishes since the image of  $\sigma$  is one dimensional—the pullback of a 2-form vanishes.

**Lemma 16.** *The maps  $\tau_\gamma$  are the parallel transport of a connection  $\nabla'$  on  $L \rightarrow Z$ .*

*Remark.* Since  $\tau_\gamma$  is a unitary transformation ( $|\tau_\gamma| = 1$ ), the connection  $\nabla'$  is also unitary.

*Proof.* By a general result [F2, Appendix B] it suffices to show that the fiducial parallel transport  $\tau_\gamma$  is invariant under reparametrization and composes under gluing. The first statement is contained in the previous lemma. For the second, if  $\gamma_1, \gamma_2$  are paths with  $\gamma_2(0) = \gamma_1(1)$ , then we can compose to get a path  $\gamma = \gamma_2 \circ \gamma_1$ . The gluing law Theorem 8 then implies  $\tau_\gamma = \tau_{\gamma_2} \circ \tau_{\gamma_1}$  as required. (Theorem 8 applies to a fixed metric and then we take the adiabatic limit.)

*Remark.* It is instructive to see in detail how the sign works in this application of the gluing law. Here we cut  $Y_\gamma$  along  $Y = Y_{\gamma_2(0)} = Y_{\gamma_1(1)}$  to obtain  $Y_\gamma^{\text{cut}} = Y_{\gamma_1} \sqcup Y_{\gamma_2}$ . So

$$\begin{aligned} \tau_{\gamma_1} &\in \text{Hom}(L_{\gamma_1(0)}, L_{\gamma_1(1)}) \cong L_Y \otimes L_{\gamma_1(0)}^{-1}, \\ \tau_{\gamma_2} &\in \text{Hom}(L_{\gamma_2(0)}, L_{\gamma_2(1)}) \cong L_{\gamma_2(1)} \otimes L_Y^{-1}, \end{aligned}$$

where we write  $L_Y = L_{\gamma_1(1)} = L_{\gamma_2(0)}$ . Thus

$$(17) \quad \tau_{Y_\gamma^{\text{cut}}} = \tau_{\gamma_2} \otimes \tau_{\gamma_1} \in L_{\gamma_2(1)} \otimes L_Y^{-1} \otimes L_Y \otimes L_{\gamma_1(0)}^{-1}.$$

The key point is that the factors are in a different order than in (6) and (7)—now the factor  $L_Y^{-1}$  precedes the factor  $L_Y$ . So the contraction is the usual trace. Put differently, to move (17) to the standard form (6) we introduce a factor of  $(-1)^{\text{index } D_Y}$  and this is cancelled by the factor  $(-1)^{\text{index } D_Y}$  in the supertrace (9). The upshot is that in this situation the right hand side of (9) is  $\tau_{\gamma_2} \circ \tau_{\gamma_1}$  as desired.

It is quite easy to prove from the variation formula Theorem 4 that this new connection agrees with the canonical connection  $\nabla$ .

**Proposition 18.**  $\nabla' = \nabla$ .

*Proof.* We must show that the parallel transports agree. Let  $\gamma: [0, 1] \rightarrow Z$  be a path and fix an element  $\ell_0 \in L_{\gamma(0)}$  of unit norm. Then if  $\gamma: [0, t] \rightarrow Z$ ,  $0 \leq t \leq 1$ , is the restriction of  $\gamma$ , and  $\tau_t: L_{\gamma(0)} \rightarrow L_{\gamma(t)}$  the parallel transport of  $\nabla'$ , by definition the path  $\ell_t = \tau_t(\ell_0)$  is parallel for  $\nabla'$ . It suffices to show that  $\frac{D\tau_t}{Dt} = 0$ , where  $\frac{D}{Dt} = \nabla$  along the path  $\gamma$ . For then  $\frac{D\tau_t(\ell_0)}{Dt} = 0$  as well, since  $\ell_0$  is a constant.

Define  $T = \{\langle t, s \rangle \in [0, 1] \times [0, 1] : s \leq t\}$  with projection

$$\begin{aligned}\rho: T &\longrightarrow [0, 1] = I \\ \langle t, s \rangle &\longmapsto t\end{aligned}$$

and a map

$$\begin{aligned}\Gamma: T &\longrightarrow Z \\ \langle t, s \rangle &\longmapsto \gamma(s).\end{aligned}$$

Then the pullback  $\pi: \Gamma^*Y \rightarrow T$  determines a family of manifolds  $\rho \circ \pi: \Gamma^*Y \rightarrow [0, 1]$  parametrized by  $I = [0, 1]$ . We use the flat metric on  $T$  and make  $\pi: \Gamma^*Y \rightarrow T$  a Riemannian submersion. The variation formula Theorem 4 implies

$$(19) \quad \frac{D\tau_t}{Dt} = 2\pi i \int_{\Gamma^*Y/I} \text{a-lim} \left[ \hat{A}(\Omega^{\Gamma^*Y/I}) \right]_{(1)}.$$

Even before taking the adiabatic limit, the fact that  $\Gamma$  factors through the projection  $\langle t, s \rangle \mapsto s$  implies that the right hand side of (19) vanishes.

In view of Proposition 18, to compute the curvature and holonomy of  $\nabla$  it suffices to compute the curvature and holonomy of  $\nabla'$ . Notice that since  $L = \text{Det}_{Y/Z}^{-1}$  is the *inverse* determinant line bundle our formulas here have opposite signs to those for  $\text{Det}_{Y/Z}$  computed in [BF1], [BF2]. The holonomy is computed from the parallel transport by a straightforward application of the gluing law. We must only be careful about the spin structure. Recall that  $S^1$  has two spin structures. The *nonbounding* spin structure is the trivial double cover of the circle; the *bounding* spin structure is the nontrivial double cover.

**Theorem 20** [BF2, Theorem 3.18]. *Suppose  $\gamma: [0, 1] \rightarrow Z$  is a closed path.<sup>5</sup> There is an induced manifold  $\hat{Y}_\gamma \rightarrow S^1$  obtained by gluing the ends of  $Y_\gamma$ . Then the holonomy of  $L$  around  $\gamma$  is*

$$(21) \quad \text{hol}_L(\gamma) = \begin{cases} (-1)^{\text{index } D_Y} \text{a-lim } \tau_{\hat{Y}_\gamma}, & \text{nonbounding spin structure on } S^1; \\ \text{a-lim } \tau_{\hat{Y}_\gamma}, & \text{bounding spin structure on } S^1. \end{cases}$$

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<sup>5</sup>Recall that we require that  $\gamma([0, \delta])$  and  $\gamma([1 - \delta, 1])$  be constant for some  $\delta$ .

Here the spin structure on  $S^1$  combines with the spin structure on  $T(\hat{Y}_\gamma/S^1)$  to give a spin structure on  $\hat{Y}_\gamma$ .

*Proof.* This follows directly from the definition (13) of parallel transport and the gluing law applied to  $X = \hat{Y}_\gamma$  and  $X^{\text{cut}} = Y_\gamma$ . Take first the nonbounding spin structure on  $S^1$ , lifted to a spin structure on  $\hat{Y}_\gamma$ . The induced spin structure on the cut manifold  $Y_\gamma$  is the standard one, with the ends each identified with  $Y_z$ , where  $z = \gamma(0) = \gamma(1)$ . Now for each  $\epsilon$  the  $\tau$ -invariant of  $Y_\gamma$  is an element

$$\tau_{Y_\gamma(\epsilon)} \in L_z \otimes L_z^{-1}.$$

Then Theorem 8 implies

$$\tau_{\hat{Y}_\gamma(\epsilon)} = (-1)^{\text{index } D_Y} \tau_{Y_\gamma(\epsilon)},$$

where on the right hand side we identify  $L_z \otimes L_z^{-1}$  with  $\mathbb{C}$  using the *usual contraction*. Now the first equation in (21) follows from the definition of holonomy in terms of parallel transport. To obtain the second equation, consider the identity map of  $Y_z$  lifted to the *nontrivial* deck transformation on the spin bundle of  $Y_z$ . It induces multiplication by  $(-1)^{\text{index } D_Y}$  on the inverse determinant line  $L_z$ . Apply this transformation to  $Y_\gamma$  before gluing in order to switch spin structures on  $\hat{Y}_\gamma$ . Then the second equation in (21) follows from the first.

**Theorem 22** [BF2, Theorem 1.21]. *The curvature  $\Omega^L$  of the inverse determinant line bundle  $L \rightarrow Z$  is*

$$(23) \quad \Omega^L = -2\pi i \left[ \int_{Y/Z} \hat{A}(\Omega^{Y/Z}) \right]_{(2)}.$$

*Proof.* For any line bundle we can determine the curvature once we know the holonomy as follows. Suppose  $\Gamma: D \rightarrow Z$  is a map of a disk into  $Z$  with boundary map  $\gamma$ . Let  $Y_\Gamma = \Gamma^* Y \rightarrow D$  be the pullback manifold; then  $\partial Y_\Gamma = \hat{Y}_\gamma$ . In the following calculation we use the bounding spin structure on  $S^1$  and the induced spin structure on  $\hat{Y}_\gamma$ .

$$(24) \quad \begin{aligned} \int_D \Omega^L &= -\log \text{hol}_L(\gamma), \\ &= \text{a-lim}(-\log \tau_{\hat{Y}_\gamma}), \\ &= \text{a-lim} \left\{ -2\pi i \int_{Y_\Gamma} \hat{A}(\Omega^{Y_\Gamma}) \right\}, \\ &= \int_D (-2\pi i) \int_{Y_\Gamma/D} \hat{A}(\Omega^{Y_\Gamma/D}), \\ &= \int_D \Gamma^* \left\{ -2\pi i \left[ \int_{Y/Z} \hat{A}(\Omega^{Y/Z}) \right]_{(2)} \right\}. \end{aligned}$$

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In the fourth line we apply (14). In the third line we apply the index theorem of Atiyah-Patodi-Singer [APS] which asserts that

$$\int_{Y_\Gamma} \hat{A}(\Omega^{Y_\Gamma}) - \frac{\eta_{Y_\Gamma}(0) + \dim \text{Ker } D_{Y_\Gamma}}{2}$$

is a certain index, so in particular is an integer. When  $\Gamma$  shrinks the disk to a point both sides of (24) vanish, so we have chosen the correct logarithm on the right hand side of (24). Since (24) holds for all  $\Gamma: D \rightarrow Z$ , equation (23) follows.

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