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CONFERENCE ON

TOPOLOGICAL AND GEOMETRICAL PROBLEMS

RELATED TO QUANTUM FIELD THEORY

(13 - 24 March 1995)

Chern-Simons Perturbation Theory
about
an Instanton Moduli Space

LECTURE 3

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<u>Talk 3 Outline</u>

I) The Theorem

II) Recap

III) Universal Bundle

IV) Universal Propagator

V) Feynman Rule Algebra

VI) The Definition

VII) The Proof

VIII) Direct Applications

no time?

<u>Interpretation:</u>

[Symbol Calculus] ⇒ Index Thy. Ints

[Calculus of Geometry near $\mathcal{M}$] ⇒ Higher Loop Ints

(2)

I)

<u>Thm:</u> Let $\mathcal{M}$ be a smooth component of the moduli space of flat connections on $G^{cpct} \to P \to M^3_{\text{closed, oriented}}$

I.e. $\mathcal{S} \to \mathcal{M}$ is a smooth bundle w/ fibers $\mathcal{B}/\mathcal{G}$ and

$T_{A_0}\mathcal{S} = \operatorname{Ker}(D_m)_{A_0} : \Omega^1(M;\mathfrak{g}) \to \Omega^2(M;\mathfrak{g})$

Then the quantity

$Z_K^{pert}(M,\mathcal{M}) \in PS(K)$ defined

below is a differential invariant.

(2a)

Recap

Talk 1: Derive Formulation by General Method

Stationary phase algorithm (Feynman) (p.4) using evaluation map (p.5) and structures on deformation complex (p.11, 12) to define change of supervariables (p.12) (which were introduced à la Mathai-Quillen (p.11)) leads to definition (p. 10, 13-14) which is an invariant a priori for finite dimensional version.

Trivial Example (p. 8, 9) illustrates:

Cancellation among Feynman diagrams with different number of vertices (points in M).

Talk 2: Acyclic Case (repackaged) (2b)

For $\mathcal{M} = \{[A_0]\}$ $\left(D_m\right|_{A_0} : \Omega^*(M;\mathfrak{g}) \to \Omega^{*+1}(M;\mathfrak{g})$ acyclic),

$$Z_k^{\text{pert}} = \left[e^{-\sum (ik)^{1-\ell} \beta_\ell \, CS_{grav}(g)} \right] \left[\sum_{\ell=1}^{\infty} (ik)^{1-\ell} I_\ell([A_0], g) \right]$$

$$= C_k \left[\sum_{\ell=1}^{\infty} (ik)^{1-\ell} C_{v,I} \int_{M[v]} \mathrm{Tr}\left(\check{L}^{I}_{tot,v} \right) \right] \qquad \begin{array}{l} I = 3(\ell-1) \\ v = 2(\ell-1) \end{array}$$

$$= C_k \int_{e^M} \mathrm{Tr}_{\tilde{k}} \left(e^{\frac{1}{2ik} \tilde{L}_{tot}} \right)$$

$\in \Omega^0(\mathrm{Met}) \otimes PS(k)$

is an invariant (i.e. is closed on Met).

Here: $e^M = \bigcup_{v=0}^{\infty} M[v]/S_v$

$$\Omega^*(e^M) = \bigoplus_{v=0}^{\infty} \underbrace{\Omega^*(M[v]; \Lambda^*(\mathfrak{g}_1 \otimes \cdots \otimes \mathfrak{g}_v))}_{A^*_v}{}^{S_v}$$

$$\tilde{L} = IK(\tilde{\partial}^{-1}) \in \Omega^2_{cl}(M[2]; \mathfrak{g} \otimes \mathfrak{g})$$

$$\tilde{L}_{tot,v} = \sum_{i,j=1}^{v} (\pi_{ij})^*(\tilde{L}_s) \in \Omega^2(M[v]; A^2_v)$$

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$$\{\tilde{L}_{tot,v}\}_{v=0}^{\infty} = \tilde{L}_{tot} \in \Omega^2(\mathcal{C}^m \times Met)$$

<u>Acyclic PF</u>: $\tilde{L}_{tot}$ closed $\Rightarrow$

$$d_{Met} Z_K^{n\ell} = C_K \int_{\partial \mathcal{C}^m} Tr_R\left(e^{\frac{1}{2ik}\tilde{L}_{tot}}\right) + d_{Met} C_K \int_{\mathcal{C}^m} Tr(\ldots)$$

<u>Use</u>:

$$\partial M[V] =_{\substack{open \\ dense}} \bigcup_{S \subset \{1,\ldots,V\}} [\partial Bl(M, \Delta_S)] \times M^{\{1,\ldots,V\} \setminus S}$$

$$\tilde{L}\Big|_{\partial M[2]} = \underbrace{\pi^*(\tilde{\rho})}_{\text{smooth part}} + \underbrace{\mathbb{1}_g(\hat{\omega} + v\cdot\tilde{\lambda})}_{\text{singular part}}$$

Lie alg. coho., symmetries & form degrees to see that

C_K is correct "counterterm."

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III) <u>Universal Bundle</u>

Assume $\mathcal{H} = \{1\}$ for simplicity, so

$$\begin{array}{ccc} \mathcal{S} & & \hat{\mathcal{S}} = \mathcal{S} \times Met \\ \downarrow & & \downarrow \\ \mathcal{M} & & \hat{\mathcal{M}} = \mathcal{M} \times Met \end{array}$$

are principle $\mathcal{G} = Aut(P)$ bundles.

(Otherwise they're $\mathcal{G}/\mathcal{H}$ "homogeneous bundles" and we work on $\mathcal{S}$..)

Universal bundle (with varying metric)

$$\begin{array}{ccc} G \rightarrow P_U & = & \hat{\mathcal{S}} \times_{\mathcal{G}} P \\ \downarrow & & \downarrow \\ \hat{\mathcal{M}} \times M & = & \hat{\mathcal{S}} \times_{\mathcal{G}} M \end{array}$$

Recall (p. 6) $\nabla^{\hat{\mathcal{S}} \to \hat{\mathcal{M}}}$ - trivial in Met direct.
- $\perp$ to $\mathcal{G}$ orbits in $\mathcal{M}$ direction:

Universal connection on P_U is assoc. conn to $\nabla^{\hat{\mathcal{S}} \to \hat{\mathcal{M}}}$ in $\hat{\mathcal{M}}$ direct. & A_0 in M direc.

Horizontal Tangent Space:

$$T_{hor}(\hat{\mathcal{L}} \to \widehat{\mathcal{M}})\Big|_{(A_0,g)} = \hat{\Omega}^1_h\Big|_{(A_0,g)} \times TM_e T_g$$

(harmonic forms)

Let

$\{\psi_\alpha\}$ be o.n. basis for $\hat{\Omega}^1_h\Big|_{(A_0,g)} \Leftrightarrow$ basis $\{T_\alpha\}$ for $T\mathcal{M}_{[A_0]}$

$\{\psi^\alpha\}$ be dual basis for $\hat{\Omega}^1_h \Leftrightarrow$ basis $\{E^\alpha\}$ for $T^*\mathcal{M}_{[A_0]}$

Connection 1-Form: $\Theta \in \Omega^1(\hat{\mathcal{L}}; \mathrm{Lie}(\mathcal{G}))$

$$T\hat{\mathcal{L}}\Big|_{(A_0,g)} \to T\mathcal{L}\Big|_{A_0} = \mathrm{Ker}(D_M|_{A_0}) \subset \Omega^1(M;\mathfrak{g}) \xrightarrow{(D_M^0)^{-1}} \Omega^0(M;\mathfrak{g})$$

i.e.: $\Theta = D_M^{-1} \circ \Pi_{T\hat{\mathcal{L}} \to T\mathcal{L} \subset T\mathcal{F}}$

Curvature 2-Form:

$$F_{\hat{\mathcal{L}} \to \widehat{\mathcal{M}}} = E^\alpha \left(\{D_{\widehat{\mathcal{M}}}, D_M^{-1}\}\psi_\alpha\right) \in \Omega^2_{hor}\left(\hat{\mathcal{L}}; \Omega^0(M;\mathfrak{g})\right)^{\mathcal{G}}$$

Universal Curvature.

$$D_{\widehat{\mathcal{M}} \times M} : \Omega^*(\widehat{\mathcal{M}} \times M; \mathfrak{g}_U) = \Omega^*(\widehat{\mathcal{M}}; \hat{\mathcal{L}} \times_{\mathcal{G}} \Omega^*(M;\mathfrak{g}))$$

$$\left(D_{\widehat{\mathcal{M}} \times M}\right)^2 = F_U = \left(D_{\widehat{\mathcal{M}}}\right)^2 + \{D_{\widehat{\mathcal{M}}}, D_M\} + \left(D_M\right)^2_{= 0}$$

$$= \left(\left[\mathbb{1} - \{D_{\widehat{\mathcal{M}}}, D_M^{-1}\}\right]\psi_\alpha\right) E^\alpha$$

$$\equiv \hat{\psi}_\alpha \, E^\alpha$$

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IV Universal Propagator

$$\hat{D} = P_d \circ D_{\hat{m} \times M} \circ P_\delta : \Omega^*(\hat{m} \times M ; g_\nu) \circlearrowleft$$

$$\Omega^*(\hat{m} ; \Omega^*(M; g_\nu))$$

Lemma:

$$\{D_{\hat{m} \times M}, \hat{D}^{-1}\} = \mathbb{1} - \hat{P}_h + \hat{D}^{-1} \circ F_\nu \circ \hat{D}^{-1}$$

$$\hat{P}_h = [\mathbb{1} - \{D_{\hat{m}}, \hat{D}^{-1}\}] P_h [\mathbb{1} - \{D_{\hat{m}}, \hat{D}^{-1}\}]$$

Pf: $\{Y, X^{-1}\} = -X^{-1}\{Y, X\}X^{-1} + P_{Ker(X)} Y X^{-1} + X^{-1} Y (\mathbb{1} - P_{Im(X)})$

$Y \to D_{\hat{m} \times M}$, $X \to \hat{D}, \ldots$ ∎

For $\gamma \in \Omega^*_h$, let

$$\hat{\gamma} = [\mathbb{1} - \{D_{\hat{m}}, \hat{D}^{-1}\}]\gamma \quad \text{(consistent w/ } \hat{\psi}_\alpha)$$

Theorem:

$$\hat{L} = IK(\hat{D}^{-1}) \in \Omega^*(\hat{m} \times M[2]; g \otimes g)$$

satisfies:

(i) $\hat{L}|_{\partial M[2]} = (\partial b)^*(\hat{\rho}) + \mathbb{1}_g(\hat{\omega} + \nu \cdot \hat{\Omega}\ldots$

as before

(ii) $D_{\hat{m} \times M[2]} \hat{L} = -\hat{P}_h(x, y) + \int_{z \in M} \hat{L}(x, z) \hat{F}_\nu(z) \hat{L}(\ldots$

$$\hat{P}_h(x, y) = \sum \hat{\psi}_\alpha(x) \hat{\psi}^\alpha(y) + (x \leftrightarrow y)$$

Pf of (i): new terms: $\{D_{\hat{m}}^+, D_{\hat{m}}\} = E^\alpha (g^{\mu\nu} I(\frac{\partial}{\partial a^\nu}) \psi_\alpha\ldots$

decreases degree in a, da by at most: Leading singularity from terms decreasing by $(x, y) = (\exp_z a, \exp_z -a)$

Pf of (ii): Previous lemma.

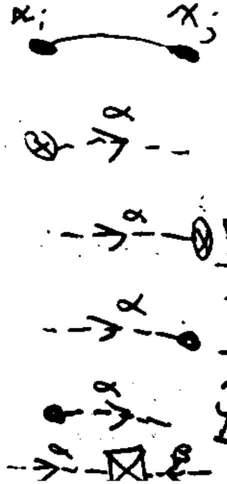
*) Feynman Rule Algebra

General trick:

Introduce (graded) algebra $\mathcal{A}^*$ which captures the Feynman rules and their variations.

$$\mathcal{A}^* = \mathbb{R}\{\text{completely labelled graphs}\}$$
$$= \Big[\bigoplus_{V=0}^{\infty} \mathbb{1}_V\Big] \otimes FG\{\underline{\hat{L}}(x_i, x_j), \underline{I}_\alpha, \underline{E}^\alpha, \underline{\hat{\psi}}_\alpha, \underline{\hat{\psi}}^\alpha, \underline{G}^{\alpha\beta}\}$$

$$\mathcal{A}^n < \bigoplus_{\substack{k,l,V \\ k-l-3V=n}} \mathcal{A}^{(k,l,V)} \xrightarrow{\ I\ } \Omega^*(\hat{m} \times C^m; \Lambda^*(TM))^n$$
$$= \bigoplus_{k,l,V} \Omega^k(\hat{m} \times M[V]; \Lambda^l(TM) \otimes A_V^*)$$



$$\underline{\hat{L}}(x_i, x_j) \longrightarrow \hat{L}_s(x_i, x_j) \in \Omega^2(\hat{m} \times M[2]; A_2^*)$$

$$\underline{I}_\alpha \longrightarrow I_\alpha \in TM = \Lambda^1(TM)$$

$$\underline{E}^\alpha \longrightarrow E^\alpha \in T^*M = \Lambda^1(T^*M)$$

$$\underline{\hat{\psi}}^\alpha(x_i) \longrightarrow \hat{\psi}^\alpha(x_i) \in \Omega^2(\hat{m} \times M_i; \mathfrak{g})$$

$$\underline{\hat{\psi}}_\alpha(x_i) \longrightarrow \hat{\psi}_\alpha(x_i) \in \Omega^1(\hat{m} \times M_i; \mathfrak{g})$$

$$G^{\alpha\beta} \longrightarrow G^{\alpha\beta} = G(D^\alpha \Gamma \psi^\beta$$

Differentials

$$\mathcal{A}^* \circlearrowleft \underline{D} = \underline{D}_0 + \underline{D}_1 \quad , \quad \underline{D}_1 \text{ increases } V \text{ by } 1$$

$$I \downarrow$$

$$\Omega^*(\hat{m} \times e^M; \Lambda^*(TM)) \circlearrowleft D_{\hat{m} \times e^M}$$

univ. cat. deriv + div. w.r.t. $Z_k^{sc} \sim \frac{1}{\sqrt{|\tau_{an}|}} \in \Omega^{|m|}(M)$

$$D I(\underline{\omega}) = I(\underline{D}_0 \underline{\omega}) + \int_M I(\underline{D}_1 \underline{\omega})$$

Example:

$$D\hat{L}(x,y) = \Big[\textstyle\sum \hat{\psi}_\alpha(x) \hat{\psi}^\alpha(y) + (x \leftrightarrow y)\Big] + \int_{z \in M} \hat{L}(x,z) \hat{F}(z) \hat{L}(z,y)$$

$$\updownarrow$$

$$\underline{D}\Big(\underset{x}{\bullet}\overset{\hat{L}(x,y)}{\text{———}}\underset{y}{\bullet}\Big) = \Big[\underline{D}_0\Big(\overset{\underline{L}(x,y)}{\bullet\text{———}\bullet}\Big)\Big] + \Big[\underline{D}_1\Big(\overset{L(x,y)}{\bullet\text{———}\bullet}\Big)\Big]$$

$$= \Big[\hat{\psi}_\alpha(x) \overset{\alpha}{\dashrightarrow} \hat{\psi}^\alpha(y) + (x \leftrightarrow y)\Big] + \Big[\hat{L}(x,z)\hat{\psi}(z)\hat{L}(z,y),\ z \overset{\alpha}{\dashrightarrow} E^\alpha\Big]$$

Remark: $\underline{D}^2 \neq 0$, $(D_{\hat{m} \times e^M})^2 \neq 0$.

II) The Definition

$$\Phi = \left[\sum_{i,j=1}^{V} \hat{L}_s(x_i, x_j)\right] + \left[I_\alpha \sum_{i=1}^{V} \hat{\Psi}^{\alpha}(x_i)\right] + I_\alpha I_\beta \zeta^{\alpha\beta}$$

$$\Phi = I(\Phi)$$

$$\hat{Z}_k^{he} = C_k \int_{e^m} \mathrm{Tr}_k \left(e^{\frac{1}{2ik}\Phi}\right) \in \bigoplus_k \Omega^k(\hat{\mathfrak{m}}; \Lambda^k T\hat{\mathfrak{m}})$$

$$\hat{Z}_k^{sc} \propto \frac{1}{\sqrt{|\tau_{on}|}} \in \Omega^{|m|}(\mathfrak{m}) \subset \Omega^{|m|}(\hat{\mathfrak{m}})$$

$$\hat{Z}_k^{pert} = \hat{Z}_k^{he} \lrcorner \hat{Z}_k^{sc} \in \Omega^{|m|}(\hat{\mathfrak{m}})$$

$$Z_k^{pert} = \int_{\mathfrak{m}} \hat{Z}_k^{pert} \in \Omega^0(M_\theta) \otimes PS(k)$$

VI) The Proof

$e^{\frac{1}{2ik}\Phi}$ is closed

(direct calc. or
a priori since derived by talk 1!)

$\Downarrow$

$\hat{Z}_k^{he}$ is closed

($[\hat{L}]_{\partial m[a]}$ as before

$\Rightarrow C_{1k}$ counterterm ok as before)

$\Downarrow$

$\hat{Z}_k^{pert}$ closed

$\Downarrow$

Z_k^{pert} closed, i.e. an invariant!

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VIII)
Direct Applications

- Link invts. when $\exists$ homology
- Verlinde Formula:

$$\int_{\mathcal{M}(\Sigma \times S^1)} \hat{Z}_k^{\text{pert}} \longrightarrow \int_{\mathcal{M}(\Sigma^d)} ch(\mathcal{L}^k)\, td(T\mathcal{M}) = \dim(\mathcal{H})$$

- String Theory:

$\mathcal{M} \longleftrightarrow$ moduli space of C.F.T.'s

$\cup$

Yang-Mills + Einstein moduli

See gravity & gauge theory in strings not just at one critical point

???

- Needed for gluing formula