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Introduction to Gauge Theory

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Intro to Gauge Theory

1. Global symmetry

$$\mathcal{L} = |\partial_\mu \varphi|^2 - \frac{\lambda}{2}(|\varphi|^2 - a^2)^2$$

$\varphi(x,t)$ = complex scalar field in 1+1 dim.

$\varphi(x,t) \to \varphi'(x,t) = e^{i\theta}\varphi(x,t)$ is a sym.
(θ = constant)

Global $U(1)$ invariance $\Rightarrow$

Conservation of electric charge:

$$\partial^\mu j_\mu = 0$$

$$j_\mu = i(\varphi^* \partial_\mu \varphi - \varphi \partial_\mu \varphi^*)$$

$$\partial_\mu = (\partial_0, \partial_1) = (\partial_t, \partial_x)$$
$$\partial^\mu = (\partial_0, -\partial_1) = (\partial_t, -\partial_x)$$

2. Broken Symm.

Broken symm is a symm which does not respect the ground state.

e.g. $\mathcal{L} = |\partial_\mu \varphi|^2 - \frac{\lambda}{2}(|\varphi|^2 - a^2)^2$

The symm. is global $U(1)$: $\varphi \to e^{i\theta}\varphi$.

But $\varphi_{vac} = a e^{i\eta}$ $\quad (|\varphi_{vac}| = a)$

Then $\varphi_{vac} \to e^{i\theta}\varphi_{vac} = \underbrace{a e^{i(\eta+\theta)}}_{\text{still a vacuum}} \neq a e^{i\eta}$

Goldstone's thm: There is a massless excitation for each broken symm.

$\varphi(x,t)$ = complex $\Rightarrow$ 2 degrees of freedom

write $\varphi(x,t) = \left(\frac{\rho(x,t)}{\sqrt{2}} + a\right) e^{i\xi(x,t)/a\sqrt{2}} \Longrightarrow$

Goldstone Mode

$$\mathcal{L} = \underbrace{\tfrac{1}{2}(\partial_\mu \xi)^2}_{\text{massles}} + \tfrac{1}{2}(\partial_\mu \rho)^2 + \underbrace{\tfrac{1}{2}m^2\rho^2}_{m = \sqrt{2\lambda}\, a} + \text{interactions}$$

3. Local Symm. (Gauge Symm)

Require a symm. for $\theta(x,t)$:

$$\varphi(x,t) \rightarrow \varphi'(x,t) = e^{i\theta(x,t)}\varphi(x,t)$$

$$\partial_\mu \varphi \rightarrow e^{i\theta}[\varphi + i\partial_\mu \theta]$$

Introduce gauge field A_μ :

$$D_\mu \varphi \equiv (\partial_\mu - iA_\mu)\varphi$$ ← Covariant Derivative

$$\boxed{\begin{aligned} \varphi &\rightarrow \varphi' = e^{i\theta}\varphi \\ A_\mu &\rightarrow A'_\mu = A_\mu - \partial_\mu \theta \end{aligned}}$$ gauge transformation

Then $D_\mu \varphi \rightarrow e^{i\theta} D_\mu \varphi$

$\therefore$ $|D_\mu \varphi|^2$ is gauge invariant.

Gauge field dynamics :

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

(Field strength)

$$-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \equiv -\frac{1}{4} F_{\mu\nu}^2 \leftarrow \text{dynamic term}$$

$$\boxed{\mathcal{L} = -\frac{1}{4} F_{\mu\nu}^2 + |D_\mu \varphi|^2 - \frac{\lambda}{2}(|\varphi|^2 - a^2)^2}$$

Abelian-Higgs Model (in 1+1 dim)

4. Higgs Mechanism.

$$\varphi = \left(\frac{h}{\sqrt{2}} + a\right) e^{i\xi}$$

Choose the gauge such that $\xi(x,t) \equiv 0$.

i.e. $\varphi \to \varphi' = e^{-i\xi}\varphi = \left(\frac{h}{\sqrt{2}} + a\right)$

In this gauge (drop the prime on φ')

$$D_\mu \varphi = (\partial_\mu - iA_\mu)\left(\frac{h}{\sqrt{2}} + a\right)$$

$$= \frac{1}{\sqrt{2}} \partial_\mu h - ia A_\mu + \cdots$$

$$\therefore \quad |D_\mu \varphi|^2 = \tfrac{1}{2}(\partial_\mu h)^2 + a^2 A_\mu^2 + \cdots$$

photon mass.

field content:

massive vector field A_μ $(m_\gamma = a)$

massive Higgs field h $(m_h = \sqrt{2\lambda}\, a)$

1) This is a way of giving mass to the photon in a gauge invariant manner. Note that simply adding $m^2 A_\mu^2$ is not gauge invariant under $A_\mu \to A_\mu - i\partial_\mu \Theta$.

2) The Goldstone mode (the phase of φ) is said to be "eaten" to form the longitudinal component of the massive field A_μ.

5. Abelian Higgs on the Lattice (1+1).

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}^2 + |D_\mu \varphi|^2 - \frac{\lambda}{2}(|\varphi|^2 - 1)^2$$

Worke in $A_0 = 0$ gauge ($A_1 \equiv W$) $\Rightarrow$

$$\mathcal{L} = -\frac{1}{2}\dot{W}^2 + |\dot{\varphi}|^2 - |\partial_1 \varphi - iW|^2 - \frac{\lambda}{2}(|\varphi|^2 - 1)^2$$

There is still a t-indep. gauge symm.

$$\varphi(x,t) \to e^{i\theta(x)}\varphi(x)$$

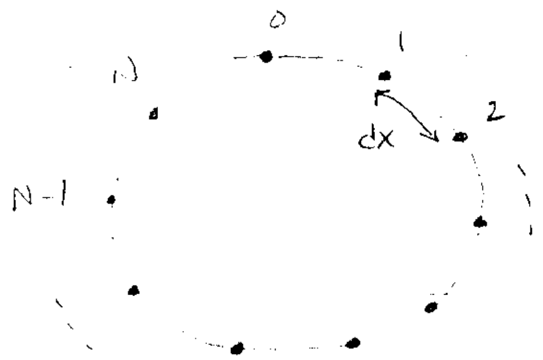
$$W(x,t) \to W(x,t) - \partial_x \theta(x)$$

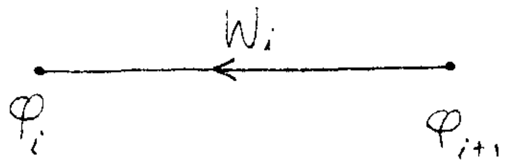
The goal: place the theory on the lattice and Preserve the gauge symm.

$$L = \sum_{i=0}^{N} \left\{ \frac{1}{2} \dot{W}_i^2 + |\dot{\varphi}_i|^2 - \frac{1}{dx^2} \left| \varphi_{i+1} - e^{iW_i dx} \varphi_i \right|^2 - \frac{\lambda}{2} \left(|\varphi|^2 - 1 \right)^2 \right\} dx$$

periodic BC:

$\varphi_0 = \varphi_{N+1}$
$W_0 = W_{N+1}$





φ_i : lives on sites (N of them)
W_i : lives on Links (N of them)

note: for fixed BC there are N-1 W_is.

Lattice covariant derivative (lives on links)

$$(D_x\varphi)_i \equiv \frac{1}{dx}\left[\varphi_{i+1} - e^{iW_i dx}\varphi_i\right]$$

In Cont. recall: $D_\mu\varphi \to e^{i\theta} D_\mu\varphi$

under $\varphi \to e^{i\theta}\varphi$

$A_\mu \to A_\mu + \partial_\mu\theta$.

The latice gauge transformation is

$$\varphi_i \to e^{i\theta_i}\varphi_i$$

$$W_i \to W_i + \frac{1}{dx}\left[\theta_{i+1} - \theta_i\right]$$

Then

$$(D_x\varphi)_i \to \frac{1}{dx}\left[e^{i\theta_{i+1}}\varphi_{i+1} - \exp\left[iW_i dx + \theta_{i+1} - \theta_i\right]e^{i\theta_i}\varphi_i\right]$$

$$= e^{i\theta_{i+1}}\frac{1}{dx}\left[\varphi_{i+1} - e^{iW_i dx}\varphi_i\right]$$

$\therefore$ $(D_x\varphi)_i \to e^{i\theta_{i+1}}(D_x\varphi)_i$

$\therefore$ gauge invariant

It is useful to expand the covar. der. term

$$\sum_i \frac{1}{dx^2} |\varphi_{i+1} - e^{iW_i dx}\varphi_i|^2$$

$$= \sum_i \frac{1}{dx^2} [\varphi_{i+1}^* \varphi_{i+1} + \varphi_i^* \varphi_i - \varphi_{i+1} e^{-iW_i dx}\varphi_i^* - \varphi_{i+1}^* e^{iW_i dx}\varphi_i]$$

$$= \sum_i \frac{1}{dx^2} [2\varphi_i^* \varphi_i - e^{iW_i dx}\varphi_{i+1}^* \varphi_i - e^{-iW_i dx}\varphi_{i+1}\varphi_i^*]$$

Thus,

$$L = \sum_i \{ \tfrac{1}{2}\dot{W}_i^2 + \dot{\varphi}_i^* \dot{\varphi}_i - \frac{1}{dx^2}[2\varphi_i^*\varphi_i - e^{iW_i dx}\varphi_{i+1}^*\varphi_i - e^{-iW_i dx}\varphi_{i+1}\varphi_i^*] - \frac{\lambda}{2}(\varphi_i^*\varphi_i - 1)^2$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{W}_i} = \frac{\partial L}{\partial W_i} \quad \Rightarrow$$

$$\ddot{W}_i = \frac{i}{dx}[e^{iW_i dx}\varphi_{i+1}^*\varphi_i - e^{-iW_i dx}\varphi_{i+1}\varphi_i^*]$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\varphi}_i^*} = \frac{\partial L}{\partial \varphi_i^*} \Rightarrow$$

$$\ddot{\varphi}_i = -\frac{1}{dx^2}\{2\varphi_i - e^{iW_{i-1}dx}\varphi_{i-1} - e^{-iW_i dx}\varphi_{i+1}\}$$
$$-\lambda(\varphi_i^*\varphi_i - 1)\varphi_i$$

$$\boxed{\begin{aligned} \ddot{W}_i &= \frac{i}{dx}\left[e^{iW_i dx}\varphi_{i+1}^*\varphi_i - e^{-iW_i dx}\varphi_{i+1}\varphi_i^*\right] \\ \ddot{\varphi}_i &= \frac{1}{dx^2}\left[e^{-iW_i dx}\varphi_{i+1} + e^{iW_{i-1}dx}\varphi_{i-1} - 2\varphi_i\right] \\ &\quad -\lambda(\varphi_i^*\varphi_i - 1)\varphi_i \end{aligned}}$$