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# College on Computational Physics

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## *Introduction to Solitons*

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# Solitons

$$\mathcal{L} = \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - U(\varphi) \qquad 1+1 \text{ dim}$$

$$\partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)} - \frac{\partial\mathcal{L}}{\partial\varphi} = 0 \quad \Rightarrow$$

$$\boxed{\partial_\mu^2\varphi + U'(\varphi) = 0} \qquad \text{Klein-Gordon equation}$$

$$E = \int dx \left[ \frac{1}{2}(\partial_0\varphi)^2 + \frac{1}{2}(\partial_1\varphi)^2 + U(\varphi) \right]$$

$$= T + V$$

where $V = \int dx \left[ \frac{1}{2}(\partial_1\varphi)^2 + U(\varphi) \right]$

examples:

$U = \frac{\lambda}{4}\varphi^4 + \frac{1}{2}m^2\varphi^2$ — $\varphi^4$ theory

$U = \frac{\lambda}{4}(\varphi^2 - v^2)^2$ — broken $\varphi^4$ theory

$U = \frac{\alpha}{\beta^2}(1 - \cos\beta\varphi)$ — sine-Gordon

Look for static solutions (then boost): $-\frac{d^2\varphi}{dx^2} + U'(\varphi) =$

Hamilton's Principle

$$\delta V = 0 \Rightarrow \delta \int dx \left[ \frac{1}{2} \left(\frac{\partial\varphi}{\partial x}\right)^2 + U(\varphi) \right] = 0$$

cf. single particle mechanics

$$\delta \int dt \underbrace{\left[\frac{1}{2}\left(\frac{dX}{dt}\right)^2 - V(X)\right]}_{L = T - V} = 0$$

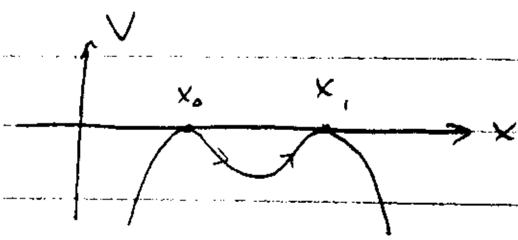
$$x \sim t$$
$$\varphi(x) \sim X(t)$$
$$U(\varphi) \sim -V(X)$$

Can use our mechanical intuition in an inverted potential.

Finite Energy Solutions have $U(\varphi) \to 0$ as $x \to \pm\infty$.

$X(t) \to X_0$ as $t \to -\infty$

$X(t) \to X_1$ as $t \to +\infty$.



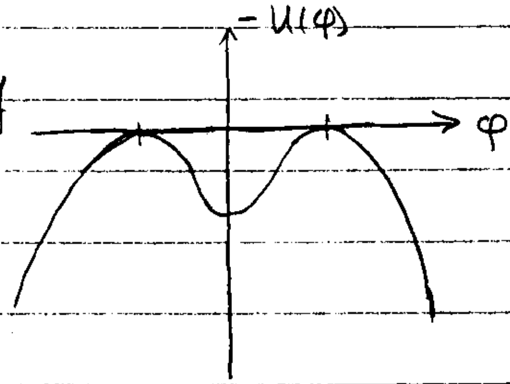

Zero "energy" i.e.

$$\int dt \; \frac{1}{2}\dot{X}^2 + V(X) = 0 \quad \text{(otherwise over/under shoots)}$$

1. Unique ground state $\Rightarrow$ no non-trivial solutions

2. Multiple ground states $\Rightarrow$ non-trivial solutions

note: $\varphi(x)$ increases monotonically




3. Explicit Solution

$$\frac{1}{2}\left(\frac{\partial\varphi}{\partial x}\right)^2 - U(\varphi) = 0 \quad \leftarrow 1^{st} \text{ Integral}$$

$$\pm\int_{\varphi_0}^{\varphi} \frac{d\varphi}{\sqrt{2U(\varphi)}} = \int_{x_0}^{x} dx = x - x_0$$

$$x - x_0 = \pm\int_{\varphi_0}^{\varphi} d\varphi\,(2U(\varphi))^{-1/2}$$

$+$ : kink

$-$ : anti-kink

$$\Rightarrow \quad \varphi(x) = f(x - x_0)$$

$\uparrow$

translation invariance

4. The Energy of solition can be simplified

$$E = \int dx \left[ \tfrac{1}{2} (\partial_x \varphi)^2 + U(\varphi) \right]$$

$$= \int dx \left( \frac{d\varphi}{dx} \right)^2$$

$$= \int \underbrace{dx \frac{d\varphi}{dx}}_{d\varphi} \underbrace{\left( \frac{d\varphi}{dx} \right)}_{[2U(\varphi)]^{1/2}}$$

$$\therefore \quad E = \int d\varphi \, [2U(\varphi)]^{1/2}$$

More Explicit Solutions

1. $U(\varphi) = \frac{\lambda}{4} (\varphi^2 - v^2)^2 \quad \Rightarrow$

$$\varphi(x) = v \tanh\left(\sqrt{\tfrac{\lambda}{2}}\, v \varphi\right) \quad \text{and} \quad E = \tfrac{2}{3} \left(\tfrac{\lambda}{2}\right)^{1/2} v^3$$

2. $U = \frac{\alpha}{\beta^2} (1 - \cos\beta\varphi) \qquad \alpha, \beta > 0 \quad \Rightarrow$

$$\varphi(x) = \frac{4}{\beta} \tan^{-1}\left[ \exp(\sqrt{\alpha}\, x) \right] \quad \text{and} \quad E = \frac{8\alpha^{1/2}}{\beta^2}$$

Small oscillations + stability

$$\Box\varphi + U'(\varphi) = 0$$
$$\underbrace{\Box}_{\partial_\mu^2} = \partial_t^2 - \partial_x^2$$

$$\varphi(x,t) = f(x) + \delta(x,t) \quad \Rightarrow$$

$$\Box\delta + \underbrace{U''(f)}_{t\text{-indep.}}\,\delta = 0$$

$$\delta(x,t) = \mathrm{Re}\sum_n a_n e^{-i\omega_n t}\psi_n(x) \quad \Rightarrow$$

$$-\frac{d^2\psi_n}{dx^2} + U''(f)\,\psi_n = \omega_n^2\psi_n$$

Schrödinger equ. in a potential $U''(f(x))$.
Stable if $E_n = \omega_n^2 \geq 0$.

note: there is a zero mode (because of transl. invar).

$$-\frac{d^2}{dx^2}f(x) + U'(f(x)) = 0$$

but, $f(x-x_0)$ also a solution $\forall\, x_0$.

$$-\frac{d^2}{dx^2} f(x-x_0) + U'(f(x-x_0)) = 0$$

$$\quad ① \qquad\qquad\qquad ②$$

$$① = -\frac{d^2}{dx^2}\left[ f(x) - x_0 f'(x) + \cdots \right]$$

$$= -\partial_x^2 f(x) + x_0 \partial_x^2 f'(x)$$

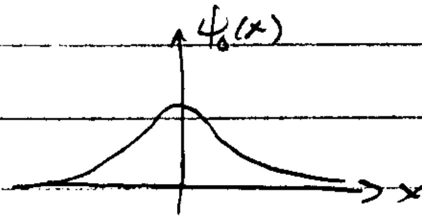
$$② = U'(f(x-x_0)) = U'(f(x)) - x_0 \frac{d}{dx} U'(f(x)) + \cdots$$

$$= U'(f(x)) - x_0 U''(f(x)) f'(x)$$

$$\therefore \quad -\frac{d^2}{dx^2} f'(x) + U''(f'(x)) f'(x) = 0$$

$\therefore \quad \psi_0 = f'(x)$ is a zero mode.

<u>But</u>, $f(x)$ is monotonic $\Rightarrow$




Since $\psi_0(x)$ has no nodes, it is the ground state. Hence $E_n \geq 0 \Rightarrow$ stability. //

We can find find t-dependent solutions by boosting:

$$\varphi'(x',t') = \varphi(x,t) = f(x)$$

$$t' = \gamma(t + vx) \qquad \text{or} \qquad t = \gamma(t' - vx')$$
$$x' = \gamma(x + vt) \qquad\qquad x = \gamma(x' - vt')$$

$$\varphi'(x',t') = f(\gamma(x' - vt'))$$

↑ Lorentz contracted

$$\varphi'(x,t) = f(\gamma(x - vt)) \quad \leftarrow \text{moving soliton}$$

Higher dim. (t-indep)

Derrick's Thm:

Consider $D$ space dim. and $N$ scalar fields $\varphi$, where $\mathcal{L} = \frac{1}{2}(\partial_\mu \varphi)^2 - U(\varphi)$. Let $U \geq 0$ and $=0$ for the ground states. Then for $D \geq 2$, the only static solutions (finite energy) are the ground states.

pf.

$$V_1 = \int d^D x \ \frac{1}{2}(\underline{\nabla}\varphi)^2 \geq 0$$

$$V_2 = \int d^D x \ U(\varphi) \geq 0$$

Simultaneously zero only for ground states. Let $\varphi(\underline{x})$ be a static solution. Write $\varphi(\underline{x};\lambda) = \varphi(\lambda \underline{x})$

$$V(\lambda) = \lambda^{2-D} V_1 + \lambda^{-D} V_2$$

$\left.\frac{dV}{d\lambda}\right|_{\lambda=1} = 0$ by Hamilton's principle $\Rightarrow$

$$(D-2)V_1 + D V_2 = 0.$$

1. $D > 2 \Rightarrow V_1 + V_2 = 0 \Rightarrow V_1 = 0$ & $V_2 = 0 \Rightarrow$ ground state
2. $D = 2 \Rightarrow V_2 = 0$. $\delta V_2 = 0$ since $V_2$ has a min. at zero. May apply Hamilton's principle to $V_1$ alone