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SMR.858 - 2

Lecture I

SUMMER SCHOOL IN HIGH ENERGY PHYSICS AND COSMOLOGY

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MONOPOLE SOLUTIONS IN SUPERSYMMETRIC THEORIES

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Trieste Lectures, Summer '95 J. Harvey

"Monopoles, Supersymmetry and Duality"

Plan of Lectures:

1. EM Duality, Dirac monopole, 't Hooft - Polyakov monopole

2. Bogomolnyi bound, properties of Prasad - Sommerfield solution, collective coordinates, Witten effect, $SL(2,Z)$ duality

3. Monopoles coupled to fermions, zero modes, charge fractionalization, Callan - Rubakov effect? Start Supersymmetry

4. Monopoles in $N=2,4$ supersymmetric gauge theory, Central charges, Bog bound, rep's of $N=2$, $N=4$, Supersymmetric collective coordinates

5. Moduli spaces, analysis of M_2, Sen's argument for S-duality.

Goal is to provide foundation for lectures next week on Seiberg - Witten, dynamics of susy gauge theory where monopoles and duality play a central role.

Lecture 1

Maxwell eqn's in free space:

$$\vec{\nabla}\cdot\vec{E} = \rho_e = 0 \qquad \vec{\nabla}\cdot\vec{B} = 0$$

$$\vec{\nabla}\times\vec{B} - \frac{\partial\vec{E}}{\partial t} = \vec{J}_e = 0 \qquad \vec{\nabla}\times\vec{E} + \frac{\partial\vec{B}}{\partial t} = 0$$

are invariant under

- duality: $\vec{E}\to\vec{B}$, $\vec{B}\to-\vec{E}$

or more generally
$$\vec{E}\to\cos\theta\,\vec{E} + \sin\theta\,\vec{B}$$
$$\vec{B}\to-\sin\theta\,\vec{E} + \cos\theta\,\vec{B}$$

if we write
$$F_{0i} = E_i$$
$$F_{ij} = -\epsilon_{ijk}B_k$$

then Maxwell: $\partial_\mu F^{\mu\nu} = j_e^\nu = 0 \qquad j_e^\nu = (\rho_e, \vec{J}_e)$

$$\partial_\mu {}^*F^{\mu\nu} = 0$$

w/ ${}^*F_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu}{}^{\rho\sigma}F_{\rho\sigma}$

and duality is $F^{\mu\nu}\to{}^*F^{\mu\nu}$

but this symmetry is broken by electric sources if there are no magnetic sources

$$\partial_\mu F^{\mu\nu} = j_e^\nu \neq 0 \;;\; \partial_\mu {}^*F^{\mu\nu} = j_m^\nu = 0$$

We usually use this to introduce a vector potential

$$\partial_\mu {}^*F^{\mu\nu} = 0 \Rightarrow F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$$

~~[illegible]~~

Of course association of $F^{\mu\nu}$ to A^μ is ambiguous because of gauge inv.

$$A_\mu \to A_\mu - \partial_\mu \chi \qquad F_{\mu\nu} \to F_{\mu\nu}$$

recall in coupling to QM

$$\vec{p} \Rightarrow -i\vec{\nabla} \Rightarrow -i(\vec{\nabla} - ie\vec{A}) \quad \text{for a particle of charge } e$$

Sch. eq'n $\quad i\dfrac{\partial\psi}{\partial t} = \dfrac{-1}{2m}(\vec{\nabla} - ie\vec{A})^2\psi + V\psi$

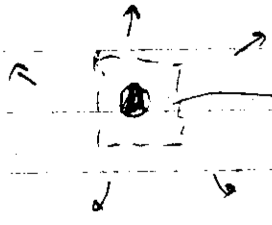
inv. under $\quad \psi \to e^{-ie\chi}\psi$

$$\vec{A} \to \vec{A} - \vec{\nabla}\chi = \vec{A} - \frac{i}{e} e^{ie\chi}\vec{\nabla} e^{-ie\chi}$$

Dirac - 1931 $\quad$ "Can we restore duality by adding magnetic sources in a way compatible with QM?"

Wu-Yang version of Dirac argument:

Suppose we have $\quad \vec{B} = \dfrac{g\hat{r}}{4\pi r^2}$



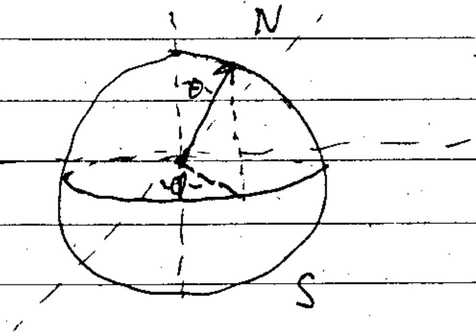
black box of size r_0 we do not look in.

Look for description valid outside "black box"

Mathematically - description in $\mathbb{R}^3 - \{0\}$

Can construct A_μ as follows:

Work in sph. coor. at each $r > r_0$ have S^2 (θ, ϕ)



On North half $(0 \le \theta \le \pi/2)$ $\vec{A}_N = \frac{g}{4\pi r} \frac{(1-\cos\theta)}{\sin\theta} \hat{e}_\phi$

singular at $\theta = \pi$, but not on N, $\vec{B}_N = \vec{\nabla} \times \vec{A}_N = \frac{g \hat{r}}{4\pi r^2}$

Similarly on S $(\pi/2 \le \theta \le \pi)$ $\vec{A}_S = -\frac{g}{4\pi r} \frac{(1+\cos\theta)}{\sin\theta} \hat{e}_\phi$

sing. at $\theta = 0$

Is this OK? $\vec{A}$ is not physical, but ambiguity in $\vec{A}$ is gauge transf, so $\vec{A}$ must differ in overlap E (equator $\theta = \pi/2$) by gauge transf.

$\theta = \pi/2$ $\vec{A}_N = \frac{g}{4\pi r} \hat{e}_\phi$ $\vec{A}_S = -\frac{g}{4\pi r} \hat{e}_\phi$

so $\vec{A}_N - \vec{A}_S = -\vec{\nabla}\chi$ w/ $\chi = -\frac{g}{2\pi}\phi$

Note χ is not continous - changes by g going from $\varphi = 0$, $\varphi = 2\pi$.

To see why,

$$g = \int_N \vec{B}_N \cdot d\vec{S} + \int_S \vec{B}_S \cdot d\vec{S} = \int_N \vec{\nabla} \times \vec{A}_N \cdot d\vec{S} + \int_S \vec{\nabla} \times \vec{A}_S \cdot d\vec{S}$$

$$= \oint_E (\vec{A}_N - \vec{A}_S) \cdot d\vec{l} = -\oint_E \vec{\nabla}\chi \cdot d\vec{l} = \chi(0) - \chi(2\pi)$$

Now χ need not be continous, but wave fn ψ had better be.

$$\psi \to e^{-ie\chi}\,\psi \quad \text{under g.t.}$$

so ψ continous if $e^{-ie\chi}$ continous. i.e.

$$e^{-ie[\chi(0) - \chi(2\pi)]} = e^{-ieg} = 1$$

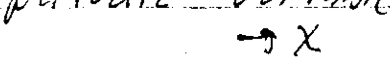
$$\Rightarrow \boxed{eg = 2\pi n, \quad n \in \mathbb{Z}} \qquad \text{Dirac quantization.}$$

Comments and Exercises

C1. 1 monopole in universe $\Rightarrow$ electric charge quantized as observed in nature

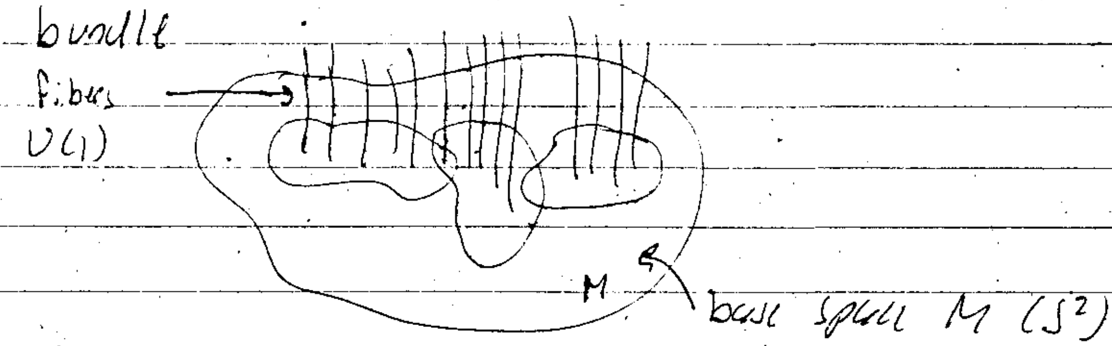
C2. $U(1)$ group of g.t. has elements $e^{-ie\chi}$. If charge quantized in units of e, then $\chi = 0$, $\chi = 2\pi/e$, are same g.t. i.e. χ is a periodic variable

compact $U(1)$, charge quantized



non-compact $\mathbb{R}$, charge not quantized.

C3. Mathematically is an example of $U(1)$ principal fibre bundle



in each region U_i looks locally like $(M \times U(1))$
↑ phase of ψ

in overlap must relate by $U(1)$ gauge transf.
(transition fcn)

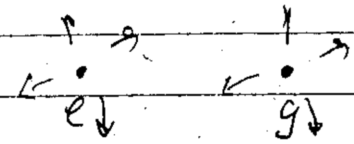
n = monopole charge = 1st Chern class of $U(1)$ bundle.

C4. In real world have $\underbrace{SU(3)_c \times SU(2)_L \times U(1)_Y}_{U(1)_{EM}}$

and quarks w/ fractional charge.
Have to generalize Dirac condition to unbroken non-abelian theories to study properly - roughly ~~[illegible]~~ Dirac monopole ok w/ quarks if it carries color magnetic field - may discuss later.

~~[illegible]~~

Exercise 1. Consider point electric charge e and point magnetic charge g. Compute field angular momentum



$$\vec{L} = \int d^3r \; \vec{r} \times (\vec{E} \times \vec{B})$$

a) Show $\vec{L}$ well defined and ind. of separation

b) Show $|\vec{L}| = \frac{n}{2} (\hbar) \Leftrightarrow$ Dirac condition.

Ex. 2. a) By generalizing Ex. 1 or otherwise prove Dirac - Zwanziger - Schwinger condition for pair w/ electric, magnetic charges (e_1, g_1), (e_2, g_2)

$$e_1 g_2 - e_2 g_1 = 2\pi n$$

b) Explore allowed soln's assuming existence of electron $(e, 0)$. Show there are soln's w/ a CP non-invariant spectrum.

So far monopole is not defined everywhere, so can't calculate properties, mass, spin, etc.

We know $U(1)$ compact if embedded in compact non-abelian Lie group e.g. $SU(2)$.

So will study YM-Higgs theory w/ $SU(2) \to U(1)$

Gauge field A_μ^a both in adjoint rep $\underline{3}$ of $SU(2)$
Higgs " Φ^a

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + e\,\epsilon^{abc} A_\mu^b A_\nu^c$$

$$(D_\mu \Phi)^a = \partial_\mu \Phi^a + e\,\epsilon^{abc} A_\mu^b A_\nu^c$$

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu} - \frac{1}{2} D^\mu \Phi^a D_\mu \Phi^a - V(\Phi)$$

$$\text{w/ } V(\Phi) = \frac{\lambda}{4}\left(\Phi^a \Phi^a - v^2\right)^2$$

We have Bianchi $D_\mu {}^*F^{\mu\nu a} = 0$

Eqn's of Motion
$$D_\mu F^{a\mu\nu} = e\,\epsilon^{abc} \Phi^b D^\nu \Phi^c$$
$$(D^\mu D_\mu \Phi)^a = -\lambda \Phi^a (\Phi^b \Phi^b - v^2)$$

We will be looking at soln's to classical eqn's - then will discuss quantum effects around this classical soln. Energy of soln from energy density

$$\Theta_{00} = \frac{1}{2}\left\{ \vec{E}^a \cdot \vec{E}^a + \vec{B}^a \cdot \vec{B}^a + \Pi^a \Pi^a + \vec{D}\Phi^a \cdot \vec{D}\Phi^a \right\} + V(\Phi)$$

$$E_i^a = F_{0i}^a$$
$$B_i^a = -\frac{1}{2}\epsilon_{ijk} F_{jk}^a$$
$$\Pi^a = (D^0 \Phi)^a$$

$\Theta_{00} \geq 0$ and 0 only if $F^{a\mu\nu} = D^{\mu}\Phi^{a} = V(\Phi) = 0$

Satisfied by Φ^{a} const. $\langle\Phi^{a}\rangle \neq 0$ such that

$\langle\Phi^{a}\rangle\langle\Phi^{a}\rangle = v^{2}$ e.g. $\langle\Phi^{a}\rangle = v\,\delta^{a3}$ $\Big\}$ break $SU(2) \to U(1)$ w/ only gen. T_3 leaving $\langle\Phi^{a}\rangle$ invariant

Note: will often write Φ^{a} for $\langle\Phi^{a}\rangle$ if no confusion.

Higgs vacuum $\quad \mathcal{M} = \{\Phi : V(\Phi) = 0\}$

here $\Rightarrow \sum_{a=1}^{3} \Phi^{a}\Phi^{a} = v^{2}$ i.e. $\mathcal{M} = S^{2}$

Now ~~as~~ ~~r→∞~~ finite E field config. need not lie in $\mathcal{M}$ everywhere, but must have $\Phi \in \mathcal{M}$ as $r \to \infty$.

$\therefore$ $\Phi^{a}(r)$ as $r \to \infty$ gives map

$$S^{2}_{\infty} \to \mathcal{M} = S^{2}$$

Characterized by integer winding #

<u>Ex. 3</u> a) If $\Phi^{a} \to v\hat{r}^{a}$ as $r \to \infty$ show

$$N = \frac{1}{4\pi v^{3}} \int_{S^{2}_{\infty}} dS^{i}\, \tfrac{1}{2}\epsilon^{ijk}\epsilon^{abc}\,\Phi^{a}\,\partial^{j}\Phi^{b}\,\partial^{k}\Phi^{c} = 1$$

b) Construct a map from $S^{2}_{\infty} \to S^{2}$ having arbitrary integer N.

We have $SU(2) \xrightarrow[\langle\Phi\rangle]{} U(1)$ and topological classification of field configurations. What is relation to monopoles?

heuristic (sloppy) argument:

Consider Higgs field Φ^a w/ $N \neq 0$. If $A_\mu^a = 0$

$$\text{Energy} = \int d^3x \; \tfrac{1}{2}\vec{\nabla}\Phi^a \cdot \vec{\nabla}\Phi^a + \tfrac{1}{2}\dot{\Phi}^a\dot{\Phi}^a + V(\Phi) \geq \int d^3x \; \tfrac{1}{2}\vec{\nabla}\Phi^a\cdot\vec{\nabla}\Phi^a + V(\Phi)$$

$$(\vec{\nabla}\Phi^a)^2 = \underbrace{\left(\frac{\partial\Phi}{\partial r}\right)^2}_{\text{radial}} + \underbrace{(\hat{r}\times\vec{\nabla}\Phi^a)^2}_{\text{angular}}$$

$N \neq 0 \Rightarrow \Phi$ has non-trivial angular dependence on S^2_∞

$\Rightarrow (\vec{\nabla}\Phi^a)^2 \simeq \left(\frac{1}{r}\right)^2$

$\Rightarrow$ Energy $\sim \int \frac{r^2 dr}{r^2} \sim$ linearly divergent.

$\therefore$ w/o gauge fields no top. non-trivial finite energy sol'ns

But w/ $A_\mu^a \neq 0$ can have

- Φ^a w/ $N \neq 0$
- $D^\mu\Phi^a = \underbrace{\underbrace{\partial^\mu\Phi^a}_{\sim\frac{1}{r}} + \underbrace{e\,\epsilon^{abc}A^{\mu b}\Phi^c}_{\sim\frac{1}{r}}}_{\text{cancel.}} \to 0 \quad \sim 1/r^2$

then

- Energy finite
- $A \sim 1/r \Rightarrow \vec{B} \sim 1/r^2 \Rightarrow$ magnetic monopole.

In a little more detail:

We want $D^\mu \Phi^a = \partial^\mu \Phi^a + e\,\epsilon^{abc} A_\mu^b \Phi^c = 0$ - meaning approp. fall off

gen'l soln is $A_\mu^a = -\frac{1}{ev^2}\,\epsilon^{abc}\,\Phi^b \partial_\mu \Phi^c + \frac{1}{v}\,\Phi^a \underset{\uparrow}{A_\mu}$
arbitrary

and compute $F^{a\mu\nu} = \frac{1}{v}\,\Phi^a F^{\mu\nu}$ w/

$$F^{\mu\nu} = \frac{-1}{ev^3}\,\epsilon^{abc}\,\Phi^a \partial^\mu \Phi^b \partial^\nu \Phi^c + \partial^\mu A^\nu - \partial^\nu A^\mu$$

and $\partial_\mu F^{\mu\nu} = \partial_\mu {}^*F^{\mu\nu} = 0$ (again meaning $\to 0$ suitably fast)

So, outside monopole gauge field is in direction of Φ^a - i.e. in unbroken $U(1)$. $F^{\mu\nu} = \frac{1}{v}\,\Phi^a F^{a\mu\nu}$

Now compute

$$g = \int_{S_2^\infty} \vec{B}\cdot d\vec{S} = \frac{1}{2ev^3}\int_{S_2^\infty} \epsilon_{ijk}\,\epsilon^{abc}\,\Phi^a \partial^j \Phi^b \partial^k \Phi^c\, dS^i$$

($\partial^\mu A^\nu - \partial^\nu A^\mu$ does not contribute by Bianchi)

$$g = \frac{4\pi}{\cancel{2}e} N \quad \Rightarrow \quad \boxed{eg = 4\cancel{2}\pi N}$$

again Dirac, but now related to topology of Higgs fields and larger by a factor of 2.

~~Dirac~~ X X ~~///~~

In general if $G \to H$, $\mathcal{M} = G/H$

Top. of Higgs field: maps $S^2_\infty \to \mathcal{M}$: $\pi_2(G/H)$

Dirac pt. of view - top. non-trivial patching along equator S^1 depends on :

$\pi_1(H)$ - classifies monopoles at large distances

It is a mathematical fact that

$\pi_2(G/H) = \pi_1(H)$ (assuming G simply connected)

End of Lecture 1.

~~Lecture 2~~