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Lecture V

SUMMER SCHOOL IN HIGH ENERGY PHYSICS AND COSMOLOGY

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MONOPOLE SOLUTIONS IN SUPERSYMMETRIC THEORIES

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Please note: These are preliminary notes intended for internal distribution only.

Lecture 5

Recap: Last time wrote down $N=2$ SUSY Lagrangian and showed that

$$\delta_\alpha \psi = \delta_{\alpha_+}\psi + \delta_{\alpha_-}\psi = (\sigma^{\mu\nu}F_{\mu\nu} - \not{D}S)\left[\tfrac{1}{2}(1+\gamma_5)\alpha + \tfrac{1}{2}(1-\gamma_5)\alpha\right]$$

$$= 0 + \gamma^i B_i \alpha_-$$

α_+ - unbroken susy's - 4 real param
α_- - broken susy's - " " "

Ex. 13. Verify that $\delta_{\alpha_-}\chi = \gamma^i B_i \alpha_-$ is a z.m. of Dirac eq'n.

We can think of coefficients of fermion zero modes as "fermionic collective coordinates"

Will return to this idea shortly.

First I want to describe how susy also gives new insight into Bog. bound following
Witten + Olive, Phys. Lett. 78B (1978) 97.

Recall $N=2$ susy algebra has possible central charges. In terms of 2 Majorana spinors Q_i $i=1,2$ have

$$\{Q_{\alpha i}, \bar{Q}_{\beta j}\} = \delta_{ij}\gamma^\mu_{\alpha\beta}P_\mu + \delta_{\alpha\beta}U_{ij} + (\gamma_5)_{\alpha\beta}V_{ij}$$

w/ $V_{ij} = -V_{ji}$; $U_{ij} = -U_{ji}$ $\quad i,j = 1 \ldots N$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\uparrow$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ # susys

For $N=2$ just V_{12}, U_{12}

They can be evaluated in a particular theory by constructing Q's as field ops and using canonical commutation relations. One finds

$$U_{ij} = \epsilon_{ij}\, v\, Q_e \qquad Q_e = \tfrac{1}{v}\int d^3x\, \partial_i (S^a E^a)$$
$$V_{ij} = \epsilon_{ij}\, v\, Q_m \qquad Q_m = \tfrac{1}{v}\int d^3x\, \partial_i (S^a B^a)$$

So $\{Q_{\alpha i}, \bar{Q}_{\beta j}\} = \delta_{ij}\gamma^\mu_{\alpha\beta} P_\mu + v\epsilon_{ij}(\delta_{\alpha\beta} Q_e + (\gamma_5)_{\alpha\beta} Q_m)$

This implies the Bog. bound $M^2 \geq v^2 (Q_e^2 + Q_m^2)$

- see notes by Van Proeyen

roughly go to rest frame $P_\mu = (M, \vec{0})$ and consider $Q_m = 0$ after mult. by γ^0

$$\underbrace{\{Q_{\alpha i}, Q_{\beta j}\}}_{\text{pos. def.}} = \underbrace{\delta_{ij}\delta_{\alpha\beta} M}_{M} + \underbrace{v\epsilon_{ij}\gamma^0_{\alpha\beta} Q_e}_{\pm v Q_e}$$

$\therefore$ $M \geq v Q_e$, $M \geq -v Q_e$ $\Rightarrow$ $M \geq v|Q_e|$

When is bound saturated? When $\{Q_{\alpha i}, Q_{\beta j}\}$ has a zero eigenvalue, i.e. when some $Q_{\beta j}$ annihilates the state

$\therefore$ some unbroken susy $\Longleftrightarrow$ Bog. Bound

There is also a close connection between BPS saturated states and the size of $N=2$ supermult.

Without too much detail...

to construct $N=2$ irreps $\{Q_{\alpha i}, \bar{Q}_{\beta j}\} = \delta_{ij} \gamma^{\mu}_{\alpha\beta} P_\mu + \cdots$

Massive reps: $P_\mu = (M, \vec{0})$ treat Q's as creation, ann. ops
- have to represent them all non-trivially and rep. has dim $2^{2N} = 2^4 = 16$ here

Massless reps: $P_\mu = (p, 0, 0, p)$ - ½ of Q's have $\{Q, Q\} = 0$ and rep. by 0 - rep. of others gives dim $2^N = 4$ irrep.

Now (A_μ, ψ, S, P) has 8 states $(4B, 4F)$ and is 2 irreps combined to satisfy CPT

In Higgs mech some states eat others but total # stays the same. So what gives?

Must have special relation between M, V_{12}, U_{12} (i.e. M, Q_e, Q_m) so that ½ of Q's represented by 0 again $\Rightarrow$ Bog. bound.

Back to idea of fermion z.m. as collective coordinates.

Given $A_i(\vec{x}, z^\alpha(t))$, $\Phi(\vec{x}, z^\alpha(t))$ we saw

$$S_4 \to S_{eff} = \int dt\, g_{\alpha\beta}\, \dot{z}^\alpha \dot{z}^\beta$$

$\uparrow$ metric on $\mathcal{M}_K$

If we restrict to energies $\ll M_W$ then field theory about monopole bkgnd $\to$ QM since there are just a finite # of d.o.f.

If we add fermion z.m. $\psi_{0\alpha}$ we could write

$$\psi = \sum_\alpha \lambda_\alpha(t)\, \psi_{0\alpha}(\vec{x}) \quad + \text{non z.m.}$$

and get S_{eff} for $\lambda_\alpha(t)$ as well. It is a bit more subtle but this can be carried out for monopoles in $N=2,4$ susy gauge theory

$N=2$	J. Gauntlett	9305068
$N=4$	J. Blum	9401133

but answer is no surprise - you get a supersymmetric QM on $\mathcal{M}_K$.

Unbroken spacetime susy $\to$ world-line susy acting on $(z^\alpha(t), \lambda^\alpha(t))$.

For $N=2$ 4 real unbroken susy's $\Rightarrow$ $N=4$ world-line susy.

One finds

$$S_{eff} = \frac{1}{2}\int dt\, g_{\alpha\beta}\left(\dot{z}^\alpha \dot{z}^\beta + 4i\lambda^{\dagger\alpha} D_t \lambda^\beta\right) \quad + \text{const}$$

$$\text{w/} \quad D_t\lambda^\beta = \dot{\lambda}^\beta + \Gamma^\beta{}_{\alpha\gamma}\,\dot{z}^\alpha \lambda^\gamma$$

Now $N=4$ susy $\Rightarrow$ restrictions on target space $\mathcal{M}_K$.

In particular, $N=4 \Rightarrow \mathcal{M}_K$ is hyperKahler a fact known to mathematicians without using supersymmetry.

This means there are 3 complex structures on $\mathcal{M}_K$

$J^{(m)}{}_\beta{}^\alpha \qquad m = 1,2,3$

$$J^{(m)}{}_\beta{}^\alpha\, J^{(n)}{}_\alpha{}^\gamma = -\delta_\beta{}^\gamma \delta^{mn} + \epsilon^{mnp} J^{(p)}{}_\beta{}^\gamma$$

and the 4 susy's are

$$\delta z^\alpha = i\beta_4\lambda^\alpha + i\beta_m \lambda^\beta J^{(m)}{}_\beta{}^\alpha \qquad \beta_4, \beta_m \text{ 4 Grassmann param}$$

$$\delta\lambda^\alpha = -\dot{x}^\alpha \beta_4 - \beta_m \dot{x}^\beta J^{(m)}{}_\beta{}^\alpha$$

Now λ's are 1 compt. objects. To see structure of states it is useful to use complex structure to introduce complex coordinates on $\mathcal{M}_K$. We then have $\lambda^\alpha, \lambda^{\bar\beta}$ w/

$$\{\lambda^\alpha, \lambda^{\bar\beta}\} = \delta^{\alpha\bar\beta}$$

and the states are

$f_0(z)\,|\Omega\rangle$ $\qquad \psi^a|\Omega\rangle = 0$

$f_{\bar a}(z)\,\lambda^{\bar a}\,|\Omega\rangle$

$f_{\bar a_1 \cdots \bar a_p}\,\lambda^{\bar a_1}\lambda^{\bar a_2}\cdots\lambda^{\bar a_p}\,|\Omega\rangle$

Here is 1-1 correspondence between

State $|f\rangle = f_{\bar a_1 \cdots \bar a_p}\lambda^{\bar a_1}\cdots\lambda^{\bar a_p}|\Omega\rangle \longleftrightarrow$ $(0,p)$ diff. form $f_{\bar a_1\cdots\bar a_p}\, dz^{\bar a_1}\wedge\cdots dz^{\bar a_p}$

<u>Ex. 14.</u> Construct supercharge Q, Q^* w/ $H = \{Q, Q^*\}$ and show that with the above correspondence

$$Q^* \leftrightarrow \bar\partial$$

$$Q \leftrightarrow \bar\partial^+$$

$$H = \{Q, Q^*\} = \bar\partial^+\bar\partial + \bar\partial\,\bar\partial^+ \underset{\text{Kahler}}{=} \tfrac{1}{2}(dd^+ + d^+d)$$

= Laplacian on forms.

Now I have emphasized $N=2$ because of role in later lectures, but we have seen ~~an~~ that pure $N=2$ is not a good candidate for a self-dual theory because monopoles have spin $0, 1/2$ but not 1.

$\therefore$ go to $N=4$ SYM.

I don't have time to discuss $N=4$ SYM in detail. good ref. for monopoles in $N=4$ is
H. Osborn, Phys. Lett. 83B (1979) 321.

- in $N=2$ have vector mult. (A_μ, Ψ, A, B)
  but also hyper mult. (2 Weyl fermions, 4 real scalars)

$N=2$ w/ 1 vector + 1 hyper both in adjoint rep $\Rightarrow N=4$.

- Also, $N=4$ is dim. red. of $N=1$ gauge theory in $d=10$, connected w/ string theory.

Have the following features

- V w/ flat directions, BPS monopoles
- Bog. bound follows from susy.
- Have twice as many fermions in $\mathcal{L}$ $\therefore$ have twice as many zero mode operators

$$a^n_{0\,1/2} \equiv a^n_{1/2} \qquad n=1,2$$
$$a^n_{0\,-1/2} \equiv a^n_{-1/2}$$

State	#	Spin
$\lvert\Omega\rangle$	1	0
$a^{n+}_{1/2}\lvert\Omega\rangle$, $a^{n+}_{-1/2}\lvert\Omega\rangle$	2, 2	1/2, −1/2
$a^{n+}_{1/2}\,a^{m+}_{-1/2}\lvert\Omega\rangle$	4	0
$a^{1+}_{1/2}\,a^{2+}_{1/2}\lvert\Omega\rangle$	1	1
$a^{1+}_{-1/2}\,a^{2+}_{-1/2}\lvert\Omega\rangle$	1	−1
3 a^+'s $\lvert\Omega\rangle$	2, 2	1/2 −1/2
4 a^+'s $\lvert\Omega\rangle$	1	0

16 states 2 spin 1, 6 spin 0, 8 spin $\pm 1/2$
$\Rightarrow$ same as gauge supermultiplet. (1 vector + 1 hyper)

$\therefore$ Monopole and $W^{\pm}$ supermult. are the same (including spin 1) so have a good candidate for a self-dual theory.

Also, $N=4$ is a finite theory w/ $\beta_g = 0$, as a result there are no quantum corr. to Bog. bound, potential etc. and semi-classical analysis should be exact for BPS states

We have taken care of

1. quantum corrections ✓
2. spin 1 ✓

? → 3. how to test

A. Sen: To test $SL(2,\mathbb{Z})$ which takes

$$\begin{pmatrix} n_e \\ n_m \end{pmatrix} \to \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} n_e \\ n_m \end{pmatrix} \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$$

$$\tau \to \frac{a\tau - b}{c\tau - d}$$

relates weak, strong coupling but we do not expect BPS saturated states to disappear when we vary e (see SW for exceptions for $N=2$).

At small e W^+ $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ exists and is BPS saturated. Make very plausible assumption that $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ exists at any coupling.

If $SL(2,\mathbb{Z})$ then $\begin{pmatrix} n_e \\ n_m \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix}$ exists at any τ, including weak coupling.

So $SL(2,\mathbb{Z}) \Rightarrow$ existence of state $\binom{a}{c}$ $\quad (a,c)=1$ from $ad-bc=1$

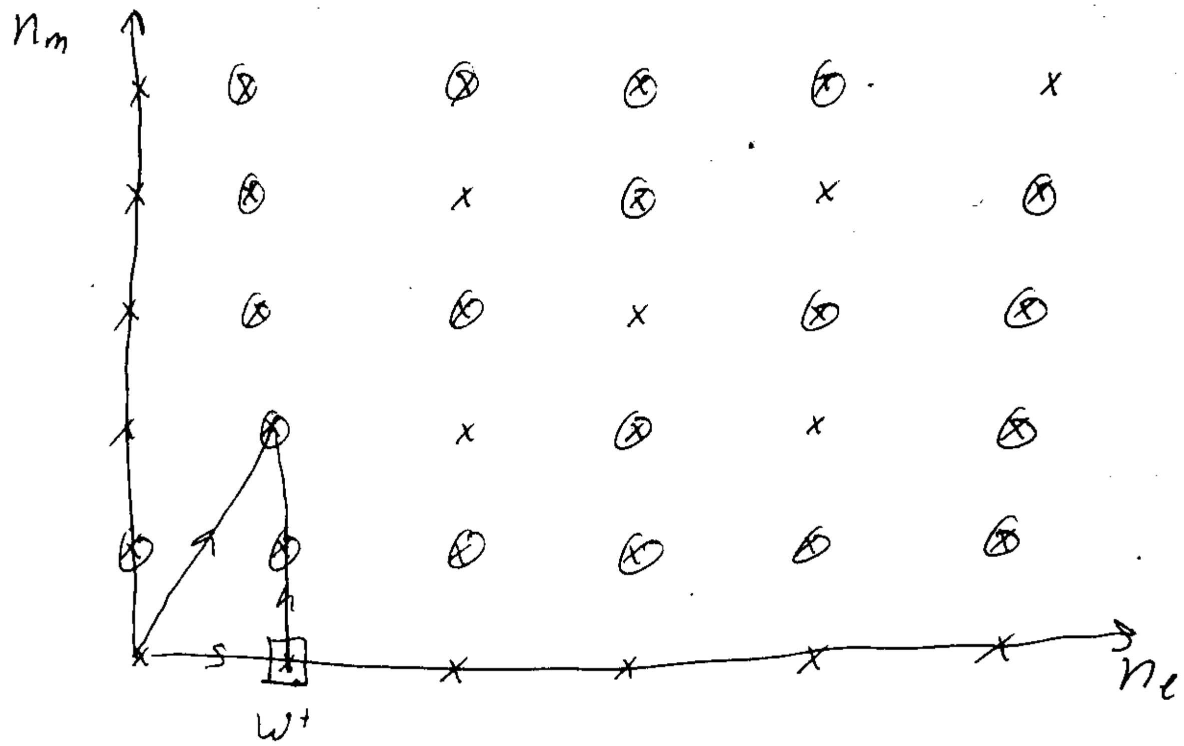
and w/ multiplicity $16 \cdot 1$

$N=4$ mult.

$c=1$ $\quad \binom{a}{1}$ $a \in \mathbb{Z}$ must exist $\Rightarrow$ dyon excitations of single monopole ✓

$c=2$ $\quad \binom{a}{2}$ a odd must exist.

In general have



$\otimes$ must exist and by triangle inequality are true bound states.

Now $\mathcal{M}_2$ is known explicitly, the statement about $(\frac{9}{2})$ states can be checked using SSQM on $\mathcal{M}_2$

By carrying out coll. coor. exp. one finds

$$S = \int dt \; \tfrac{1}{2} g_{\alpha\beta}\left(\dot{z}^\alpha \dot{z}^\beta + i\bar{\psi}^\alpha \gamma^0 D_t \psi^\beta\right) + \tfrac{1}{6} \dot{R}_{\alpha\beta\gamma\delta} (\bar{\psi}^\alpha \psi^\gamma)(\bar{\psi}^\beta \psi^\delta)$$

where now ψ^α is a <u>2-cmpt.</u> spinor rather than 1-cmpt.

This changes the structure of SSQM by adding back in holo. forms missing in $N=2$ so that

Hilbert space $\leftrightarrow$ diff. forms on $\mathcal{M}_K$

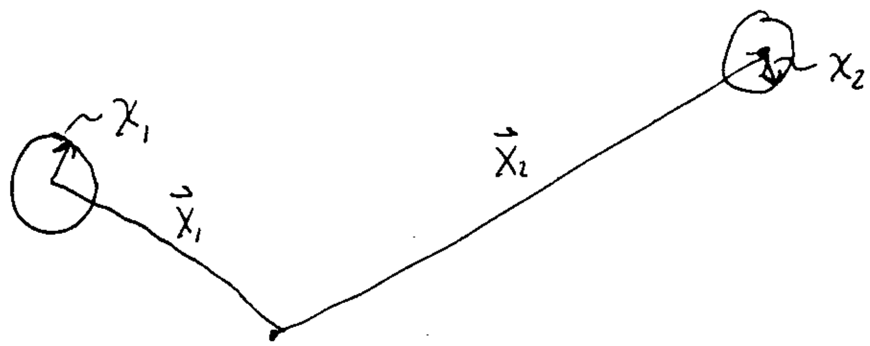
BPS sat. state $\leftrightarrow$ L^2 harmonic form on $\mathcal{M}_K$

Thus mathematically duality $\Rightarrow$ unique L^2 harmonic form on $\tilde{\mathcal{M}}_2$
after removing $\mathbb{R}^3 \times S^1$
carrying <u>odd</u> electric charge.

Now let's discuss the structure of $\mathcal{M}_2$:

$$\mathcal{M}_2 = \mathbb{R}^3 \times \left(\frac{S^1 \times \mathcal{M}_{AH}}{\mathbb{Z}_2} \right)$$

when 2 monopoles are well separated we have



in this limit $\chi = \frac{1}{2}(\chi_1 + \chi_2)$ is S^1 coordinate

$\vec{\mathbb{X}} = \frac{1}{2}(\vec{X}_1 + \vec{X}_2)$ is $\mathbb{R}^3$ "

and $\psi = \frac{1}{2}(\chi_1 - \chi_2)$ are coor. on $\mathcal{M}_{AH}$

$\vec{X} = \frac{1}{2}(\vec{X}_1 - \vec{X}_1)$

No such decomp when monopoles are close
Metric on $\mathcal{M}_{AH}$ found by Atiyah - Hitchin using
a) hyperkahler $\to$ self-dual in $d=4$
b) $SO(3)$ isometry

$\Rightarrow \quad ds^2 = f(r)^2 dr^2 + a(r)^2 \sigma_1^2 + b(r)^2 \sigma_2^2 + c(r)^2 \sigma_3^2$

σ_i - left inv. 1-forms on $SO(3) = S^3/\mathbb{Z}_2$.

have exp. formulae for f, a, b, c in terms of elliptic functions

Note: • round metric on S^3 has $SO(3)_L \times SO(3)_R$ isometry - this has only $SO(3)_L$ since constructed from σ_i

• In terms of Euler angles

$$\sigma_1 = -\sin\psi\, d\theta + \cos\psi \sin\theta\, d\phi$$

$$\sigma_2 = \cos\psi\, d\theta + \sin\psi \sin\theta\, d\phi$$

$$\sigma_3 = d\psi + \cos\theta\, d\phi$$

$$0 \le \theta \le \pi,\ 0 \le \phi \le 2\pi,\ 0 \le \psi \le 2\pi$$

$\mathbb{Z}_2$ quotient acts as $\psi \to \psi + \pi,\ \chi \to \chi + \pi$

- This is $\chi_1 \to \chi_1 + 2\pi$
  $\chi_2 \to \chi_2$.

  for well sep. monopoles

- $\mathbb{Z}_2$ means forms on M_{AH} even under $\mathbb{Z}_2$ → even electric charge, odd forms → odd electric charge.

So, can we show there is an unique L^2 harmonic form on M_{AH} which is odd under $\mathbb{Z}_2$?
$\hat{\omega}$

i.e. $\int_M \omega \wedge * \omega < \infty$, $\Delta\omega = (dd^\dagger + d^\dagger d)\omega = 0$.

Now if ω is harmonic, so is its Hodge dual $*\omega$, ∴ to be unique ω must be self-dual or anti-self-dual.

With this information one can write down an ansatz.

$$\omega = F(r)\left(d\sigma_1 - \frac{f a}{b c}\, dr \wedge \sigma_1 \right).$$

this is used by construction using $d\sigma_i = \frac{1}{2}\epsilon_{ijk}\sigma_j \wedge \sigma_k$

and $d\omega = 0 \Rightarrow d^{+}\omega = 0$ so $\Delta\omega = 0$

$d\omega = 0$ gives eqn $\frac{dF}{dr} = -\frac{fa}{bc} F$ and using asymptotic

~~asymptotic~~ forms for f, a, b, c one can verify that

ω is L^2.

This establishes the existence of the 2^{nd} row in the figure w/ correct multiplicity.

What about for general $\binom{q}{c}$? There is a construction by G. Segal of compact cohomology of M_n which shows $SL(2,\mathbb{Z})$ structure but does not settle question.

Also, a recent paper by Porrati claiming a gen'l proof but I have not yet had time to study the paper.

In conclusion, there is new evidence for $SL(2,Z)$ duality of $N=4$ SYM. However there are still many open questions

- extension to dynamic processes i.e. can we show duality relates corr. fcn's, S-matrix elements and not just spectrum?

- Duality although not exact in $N=2,1$ SUSY gauge theories plays a crucial role in the analysis of dynamics as you will hear next week

- Roots of duality still mysterious but developments over the last year suggest its origin in part may be in string theory.