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RECENT DEVELOPMENTS IN SUPERSYMMETRIC N=2 YANG-MILLS THEORY

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Please note: These are preliminary notes intended for internal distribution only.

(I) MONOPOLES.

We begin by reviewing the properties of monopoles. We then formulate the Montonen-Olive conjecture and Witten's argument about how θ modifies the monopole and dyon electric charge. We take always $c=1$, and almost always $\hbar=1$ except when it is important to make the distinction between classical and quantum quantization. We take Maxwell's equations to be.

$$\left.\begin{array}{ll} \vec{\nabla}\cdot\vec{E} = \rho & \vec{\nabla}\times\vec{B} - \partial\vec{E}/\partial t = \vec{j} \\ \vec{\nabla}\cdot\vec{B} = 0 & \vec{\nabla}\times\vec{E} + \partial\vec{B}/\partial t = 0 \end{array}\right\} \qquad (1)$$

In these units there is a factor of $(4\pi)^{-1}$ in Coulomb's law. For a static point-like charge q at the origin, $\rho = q\,\delta^3(\vec{r})$, $\vec{E} = -\vec{\nabla}\phi$, and:

$$\Delta\phi = -q\,\delta^3(\vec{r}) \qquad (2)$$

by spherical symmetry $\int \vec{E}\,d\vec{r} = 4\pi.(q)/4\pi = \int q\,\delta^3(\vec{r})$, hence $\vec{E} = q\vec{r}/4\pi r^3$, $\phi = q/4\pi r$ as stated.

In relativistic form, introduce the field strength

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \qquad A^\mu = (\phi, \vec{A}) \qquad (3)$$

$$F_{0i} = \partial_0 A_i - \partial_i A_0 = -\partial_0 A^i - \partial_i A^0 = E^i$$
$$F_{ij} = \partial_i A_j - \partial_j A_i = -(\partial_i A^j - \partial^j A_i) = -\epsilon_{ijk} B^k$$

Hence

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_x & B_y & 0 \end{pmatrix} \qquad {}^*F^{\mu\nu} = \begin{pmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z & -E_y \\ B_y & -E_z & 0 & E_x \\ B_z & E_y & -E_x & 0 \end{pmatrix} \quad (4$$

Where

$$\epsilon^{0123} = +1 \qquad {}^*F^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}$$

Duality means $\vec{E}\to\vec{B}$ $\vec{B}\to-\vec{E}$. The equation of motion are:

$$\left.\begin{array}{l} \partial_\nu F^{\mu\nu} = -j^\mu \\ \partial_\nu {}^*F^{\mu\nu} = 0 \end{array}\right\} \qquad (5) \qquad j^\mu = (\rho, \vec{j})$$

The equations look asymmetric under E, B exchange. Only for $j^\mu = 0$ the equations are symmetric. Dirac introduced the magnetic current $k^\mu = (\sigma, \vec{k})$, and modified the Maxwell equations:

$$\boxed{\begin{aligned} \partial_\nu F^{\mu\nu} &= -j^\mu \\ \partial_\nu {}^*F^{\mu\nu} &= -k^\mu \end{aligned}} \qquad (6)$$

Symmetric under $F \to {}^*F$, ${}^*F \to -F$, $j \to k$, $k \to -j$.
The current densities for point particles can be written as:

$$j^\mu(x) = \sum_i q_i \int dx_i^\mu \, \delta^4(x - x_i), \qquad (7)$$
$$k^\mu(x) = \sum_i g_i \int dx_i^\mu \, \delta^4(x - x_i),$$

and the Lorentz equation is:

$$m \frac{d^2 x^\mu}{d\tau^2} = (q F^{\mu\nu} + g\, {}^*F^{\mu\nu}) \frac{dx_\nu}{d\tau} \qquad (8)$$

If $g \neq 0$, we cannot find $(\phi, \vec{A})$ everywhere. In QM this leads to crucial consequence: quantization of electric charge:

$$\frac{qg}{4\pi\hbar} = \frac{n}{2} \qquad n \in \mathbb{Z} \qquad (9)$$

1st Derivation

Consider a NR charge q in the presence of a magnetic pole at $\vec{r} = \vec{0}$:

$$\left.\begin{aligned} \vec{B} &= \frac{g}{4\pi r^3} \vec{r} \\ m\ddot{\vec{r}} &= q \, \dot{\vec{r}} \times \vec{B} \end{aligned}\right\} \qquad (10)$$

$\vec{L}$ changes

$$\frac{d}{dt} m \vec{r} \times \dot{\vec{r}} = m \vec{r} \times \ddot{\vec{r}} = \frac{qg}{4\pi r^3} \vec{r} \times (\dot{\vec{r}} \times \vec{r}) = \frac{d}{dt}\left(\frac{qg}{4\pi} \frac{\vec{r}}{r} \right)$$

Hence

$$\boxed{\vec{J} = \vec{r} \times m\dot{\vec{r}} - \frac{qg}{4\pi} \frac{\vec{r}}{r}} \qquad (11)$$

is conserved. The 2nd term comes from the EM field. Assuming that orbital angular momentum has integer eigenvalues, we conclude

That

$$\boxed{\frac{qg}{4\pi} = \frac{1}{2} n \hbar} \qquad (12)$$

For any pair of purely charged (either e or m) particles we learn that:

$$\frac{q_i g_j}{4\pi\hbar} = \frac{1}{2} n_{ij} \qquad (13)$$

Any e-charge is $\mathbb{Z} \cdot 2\pi\hbar/g_j$. The the highest common factor n_{0j}. Then all electric charges are multiples of $q_0 = n_{0j}\, 2\pi\hbar/g_j$. Same for the magnetic charge. The existence of a single monopole would explain charge quantization. The contribution $-qg\hat{r}/4\pi$ comes from the EM field.

$$\vec{J}_{em} = \int d^3\vec{r}\ \vec{r}\times(\vec{E}\times\vec{B}) = \frac{g}{4\pi}\int d^3\vec{r}\ \vec{r}\times\left(\vec{E}\times\frac{\vec{r}}{r^3}\right) \Rightarrow$$

$$J^i_{em} = \frac{g}{4\pi}\int d^3x\ E^j\left(\delta_{ij} - \frac{x_i x_j}{r^2}\right)\frac{1}{r} = \frac{g}{4\pi}\int d^3x\ E^j\partial_j(\hat{x}^i)$$

$$= -\frac{g}{4\pi}\int d^3x\ \vec{\nabla}\cdot\vec{E}\ \hat{x}^i = -\frac{gq}{4\pi}\hat{r}^i \qquad (14)$$

The bdry terms cancel by spherical symmetry if we use multipole expansions

<u>2nd Derivation</u>.

Use the canonical formalism:

$$L = \frac{1}{2}m\dot{\vec{r}}^2 + q\dot{\vec{r}}\vec{A} - q\phi = \frac{1}{2}m\dot{\vec{r}}^2 - qA_\mu\dot{x}^\mu \qquad (15)$$

$$\vec{p} = m\dot{\vec{r}} + q\vec{A}$$

$$H = \vec{p}\cdot\dot{\vec{r}} - L = \frac{1}{2m}(\vec{p} - q\vec{A})^2 + q\phi \qquad (16)$$

The Poisson brackets

$$\{\alpha,\beta\} = \sum_i \frac{\partial\alpha}{\partial r_i}\frac{\partial\beta}{\partial p_i} - \frac{\partial\alpha}{\partial p_i}\frac{\partial\beta}{\partial r_i} \qquad \left\{\begin{array}{l}\{r_i, r_j\} = 0\\ \{p_i, p_j\} = 0\\ \{r_i, p_j\} = \delta_{ij}\end{array}\right. \qquad (17)$$

$\dot{r}^i$ is gauge invariant. We obtain:

$$\boxed{\begin{array}{l}\{m\dot{r}^i, m\dot{r}^j\} = q\,\epsilon_{ijk}B^k\\ \{r^i, r^j\} = 0\\ \{r^i, m\dot{r}^j\} = \delta_{ij}\end{array}} \qquad (18)$$

Similarly, for orbital angular momentum:

$$\left.\begin{aligned} \{L^i, r^j\} &= \epsilon_{ijk} r^k \\ \{L^i, L^j\} &= \epsilon_{ijk} m\dot{r}^k + q(\delta_{ij}\vec{r}\cdot\vec{B} - B^i r^j) \\ L^i &= m(\vec{r}\times\dot{\vec{r}})^i \end{aligned}\right\} \quad (19)$$

Since

$$\frac{qg}{4\pi}(\delta_{ij} - \hat{r}^i\hat{r}^j) = \frac{qg}{4\pi}\{\hat{r}^i, m\dot{r}^j\}$$, once again

$$\vec{J} = \vec{L} - \frac{gq}{4\pi}\frac{\vec{r}}{r}$$ satisfies:

$$\{J^i, J^j\} = \epsilon_{ijk} J^k \quad (20)$$

Upon quantization:

$$\{\alpha, \beta\} \to \frac{1}{i\hbar}[\alpha, \beta]$$

$\vec{p} \to -i\hbar\vec{\nabla}$, the velocity is then:

$$m\dot{r}^i = \vec{p} - q\vec{A} = \frac{\hbar}{i}\vec{\nabla} - q\vec{A} = \frac{\hbar}{i}\left(\vec{\nabla} - i\frac{q}{\hbar}\vec{A}\right)$$

In covariant form we must be a little careful because $\vec{\nabla} = (\partial_x, \partial_y, \partial_z)$ with lower indices:

$$m\dot{r}^i = i\hbar\partial^i - qA^i = i\hbar\left(\partial^i + \frac{iq}{\hbar}A^i\right) \equiv i\hbar\mathcal{D}^i \quad (21)$$

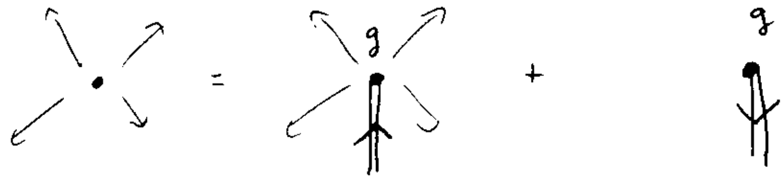
The Schrodinger equation is gauge covariant:

$$\left.\begin{aligned} &-\frac{\hbar^2}{2m}\mathcal{D}^2\psi + q\phi\psi = i\hbar\frac{\partial\psi}{\partial t} \\ &A_\mu \to A_\mu + \partial_\mu\chi \\ &\psi \to \exp -iq\chi/\hbar\ \psi = e^{-ie\chi}\psi \\ &e = q/\hbar \end{aligned}\right\} \quad (22)$$

Note that $e = q/\hbar$, q is the classical electric charge, and e is the EM coupling constant.

Dirac String.

Since $\vec{\nabla}\cdot\vec{B} \neq 0$ with monopoles, there is an ambiguity in the construction of A_μ. Dirac introduced a semi-infinite solenoid from $(0,0,-\infty)$ to the monopole position. With this string singularity we can write representatives of A for a monopole field.



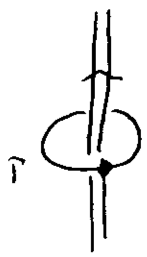
$$\vec{B}_{sol} = \frac{g}{4\pi r^2}\hat{r} + g\,\theta(-z)\,\delta(x)\,\delta(y)\,\hat{z} \qquad (23)$$

$$\vec{\nabla}\cdot\vec{B}_{sol} = g\,\delta(\vec{r}) - g\,\delta(r) = 0.$$

Hence $\vec{B}_{sol} = \vec{\nabla}\times\vec{A}$, and

$$\frac{g}{4\pi r^2}\hat{r} = \vec{\nabla}\times\vec{A} - g\,\theta(-z)\,\delta(x)\,\delta(y)\,\hat{z} \qquad (24)$$

The position of the string should be immaterial. It can be moved using singular gauge transformation. To make the string unobservable charge (rather flux) quantization is needed. Using the Aharonov-Bohm effect on the contour Γ:



$$q\oint_{\Gamma}\vec{A}\cdot d\vec{s} = q\,\Phi$$

which is unobservable if:

$$\frac{q\Phi}{2\pi\hbar} = n \in \mathbb{Z} \qquad \text{i.e.} \quad \boxed{\frac{qg}{4\pi\hbar} = \frac{1}{2}n}$$

Using $q/\hbar = e \Rightarrow eg/4\pi = n/2$.

Remark

If we have dyons instead (q_1, g_1), (q_2, g_2),

$$\frac{q_1 g_2 - q_2 g_1}{4\pi\hbar} = \frac{1}{2}n_{12} \qquad (25)$$

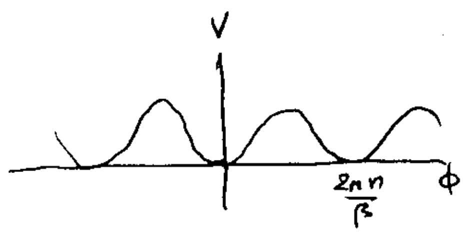
with a full $SO(2)$ symmetry.

SOLITONS

Simplest are found in 1+1-dimensions

$$\mathcal{L} = \frac{1}{2}(\dot{\phi}^2 - \phi'^2) - V(\phi)$$

$$V(\phi) = \frac{\alpha}{\beta}(1 - \cos\beta\phi)$$



For t-independent solution the energy is

$$M = \int_{-\infty}^{+\infty}\left(\frac{1}{2}\phi'^2 + V(\phi)\right)dx$$

Eq'n of motion:

$$\ddot{\phi} - \phi'' = -\alpha \sin\beta\phi$$

t-independence yields

$$\phi'' = \alpha \sin\beta\phi$$

equivalently, motion in a $-V(\phi)$ potential. Finite energy implies that $\phi \to$ vacuum as $|x| \to \pm\infty$. We have the conservation law. The simplest soliton or kink has $\phi \to 0$ $x \to -\infty$ $\phi \to 2\pi/\beta$ $x \to +\infty$ interpolating smoothly.

$$M = 2\int_0^\infty \left(\tfrac{1}{2}\phi'^2 + V(\phi)\right) dx \qquad \tfrac{1}{2}\phi'^2 - V(\phi) = 0 \qquad d\phi = (2V)^{1/2} dx$$

$$= 2\int_0^{2\pi/\beta} (2V(\phi))^{1/2} d\phi$$

For Sine-Gordon the meson mass is $m = \sqrt{\alpha\beta}$, and:

$$\boxed{M = 8m/\beta^2} \qquad (26)$$

We have a topological current associated to soliton number.

$$j^\mu_{top} = \frac{\beta}{2\pi} \varepsilon^{\mu\nu} \partial_\nu \phi$$

$$Q_{top} = \int_{-\infty}^{+\infty} dx\, j^0_{top} = \frac{\beta}{2\pi}\left(\phi(\infty) - \phi(-\infty)\right) = N \qquad (27)$$

N is an integer. Note:

1). j^μ_{top} is conserved trivially $\partial_\mu j^\mu_{top} = 0$

2). Q does not contain canonical momenta

3). j^0 is a space divergence, $\phi(\pm\infty) \in M_0 = \{$vacuum manifold$\}$.

In $D > 2$ without gauge fields there are no static solitons:

$$H \geq \int \left(\tfrac{1}{2}(\vec{\nabla}\phi)^2 + V(\phi)\right) d^Dx$$

$$(\vec{\nabla}\phi)^2 = (\partial_r\phi)^2 + (\hat{r} \times \vec{\nabla}\phi)^2$$

if ϕ_∞ is not constant, the 2nd term $\sim r^{-2}$ as $r \to \infty$. For $D \geq 2$ this term diverges. Hence ϕ is topologically trivial at ∞. With gauge fields, $D_\mu\phi = \partial_\mu\phi + ig W_\mu^a T^a \phi$, even if $\partial_\mu\phi \sim 1/r$ $W_\mu \sim 1/r$, $D_\mu\phi \sim 1/r^2$; and we may hope to have solitons in $D = 3$.

Georgi - Glashow Model

$$D_\mu \phi = \partial_\mu \phi + i e W_\mu^a T^a \phi \qquad G = SO(3) \qquad [T^a, T^b] = i f^{abc} T^c$$
$$[D_\mu, D_\nu]\phi = i e G_{\mu\nu}^a T^a \phi$$
$$G_{\mu\nu} = \partial_\mu W_\nu - \partial_\nu W_\mu + ie [W_\mu, W_\nu] \; ; \qquad W_\mu = T^a W_\mu^a, \quad T^{a\dagger} = T^a$$

In the SO(3) model, the simplest Lagrangian is:

$$\left.\begin{aligned} \mathcal{L} &= -\frac{1}{4} G_{\mu\nu}^a G^{a\mu\nu} + \frac{1}{2} \mathcal{D}^\mu \vec{\Phi} \cdot \mathcal{D}_\mu \vec{\Phi} - V(\phi) \\ \mathcal{D}_\mu \phi^a &= \partial_\mu \phi^a - e f^{abc} W_\mu^b \phi^c \end{aligned}\right\} \qquad (28)$$

In the adjoint rep. $(T^a)_{bc} = -i f^a{}_{bc}$. The equations of motion are:

$$\left.\begin{aligned} (\mathcal{D}_\nu G^{\mu\nu})_a &= -e f_{abc} \phi_b (\mathcal{D}^\mu \phi)_c \\ \mathcal{D}^\mu \mathcal{D}_\mu \phi_a &= -\lambda \phi_a (\phi^2 - a^2) \\ \mathcal{D}_\nu {}^* G^{\mu\nu a} &= 0 \end{aligned}\right\} \qquad (29) \qquad V(\phi) = \frac{\lambda}{4}(\phi^2 - a^2)^2$$

The energy density:

$$\Theta_{00} = \frac{1}{2} \left(\mathcal{E}_a^{i\,2} + \mathcal{B}_a^{i\,2} + \Pi_a^2 + (\mathcal{D}^i \phi)^2 \right) + V(\phi) \qquad (30)$$
$$\mathcal{E}_a^i = - G^{0i}{}_a \qquad G_a{}^{ij} = - \epsilon_{ijk} \mathcal{B}_a^k \qquad \Pi_a = (\mathcal{D}^0 \phi)_a$$

$\Theta_{00} \geq 0$, and it vanishes only if

$$G_a^{\mu\nu} = 0 \qquad \mathcal{D}_\mu \phi = 0, \qquad V(\phi) = 0 \qquad (31)$$

The Higgs vacuum is defined as:

$$\left.\begin{aligned} V(\phi) &= 0 \\ \mathcal{D}_\mu \phi &= 0 \end{aligned}\right\} \text{ Higgs vacuum} \qquad (32)$$

The unbroken generator is $\vec{\Phi} \cdot \vec{T}/a \propto Q$ (electric charge). The spectrum near the Higgs vacuum:

$$\phi^3 = a^3 + \phi'^3$$

$$\mu_H = a(2\lambda)^{1/2} \hbar \qquad q = 0$$

The spectrum of the theory is:

	Mass	Spin	Charge
Higgs	$a(2\lambda)^{1/2}\hbar$	0	0
γ	0	$\hbar$	0
$W^\pm$	$ae\hbar = aq$	$\hbar$	$\pm q = \pm e\hbar$

Identify $\vec{\Phi}\cdot\vec{W}^\mu/a$ with ~~a~~ A^μ, and we find:

$$\boxed{Q = \hbar e\, \vec{\Phi}\cdot\vec{T}/a} \qquad (33)$$

If the group is compact electric charge is quantized $(\partial^\mu + iQA^\mu/\hbar)$

'tHooft - Polyakov ansatz

They looked for a spherically symmetric P-invariant ~~solution~~ static solution.

$$\phi^a = \frac{r^a}{er^2} H(aer) \qquad W_a^i = -\epsilon_{aij}\frac{r^j}{er^2}(1-K(aer)) \qquad (34)$$

$$W_a^0 = 0$$

$$\phi^a_\infty(\hat{r}) = \lim_{r\to\infty} \phi_a(r\hat{r}) = a\hat{r}^a \Rightarrow \quad H(\xi)\sim\xi \quad \xi\to\infty$$

$$(\xi = aer)$$

$$E = -\int \mathcal{L}\, d^3\vec{r}$$

at $\vec{r}\to 0$, $H\sim\xi$, $(K-1)\sim\xi$. As $\xi\to\infty$ $H\sim\xi$, $K\to 0$ fast.
Changing variables to $\xi = aer$:

$$\boxed{M = \frac{4\pi a}{e} f(\lambda/e^2)} \qquad (35)$$

The Prasad - Sommerfeld (PS) limit corresponds to $\lambda\to 0$ but keeping the B.C. ($\phi^2 = a^2$ asymptotically). At large distance

$$W_a^i = -\epsilon_{aij}\frac{r^j}{er^2} + \cdots \Rightarrow \quad B^i = -\frac{1}{e}\frac{r^i}{r^3}, \quad G_a^{ij} \sim \frac{1}{er^4}\epsilon_{ijk} r^a r^k$$

Hence

$$\boxed{g = -4\pi/e}$$

$q = e\hbar \Rightarrow gq = -4\pi\hbar$, The smallest charge in the theory is $q_0 = \frac{1}{2}q$.

For the vector boson $q = \pm e\hbar$, thus $qg = -4\pi\hbar$. For q_0, $q_0 g = -4\pi\hbar(1/2)$.
The 'tHP monopoles has the minimal charge

$$K \sim O(e^{-Mr/\hbar}) \qquad M = ea\hbar$$

$$H - \xi \sim O(e^{-\mu r/\hbar}) \qquad \mu = (2\lambda)^{1/2} a\hbar \qquad (36)$$

outside ~~and~~ some $R_0 - R_0(\hbar/M, \hbar/\mu)$ we are effectively in the Higgs vacuum.

For $SO(3)$ the Higgs vacuum satisfies:

$$\left.\begin{aligned} \mathcal{D}_\mu \phi^a &= \partial_\mu \phi^a - e\,\varepsilon^{abc} W_\mu^b \phi^c = 0 \\ \phi^2 &= a^2 \end{aligned}\right\} \quad (37)$$

The first eq'n is.

$$\partial_\mu \vec{\phi} - e \vec{W}_\mu \times \vec{\phi} = 0$$

which can be solved for $\vec{\phi}$. The general solution is a particular solution plus the general solution of the homogeneous equation.

$$\vec{W}_\mu = \frac{1}{ea^2}\, \vec{\phi} \times \partial_\mu \vec{\phi} + \frac{1}{a} \vec{\phi} A_\mu \qquad (38)$$

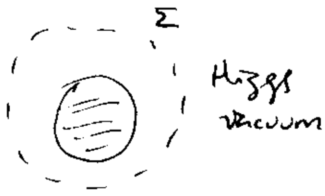
In the Higgs vacuum $\partial_\mu \vec{\phi} \cdot \vec{\phi} = 0$, then

$$\frac{1}{ea^2} (\vec{\phi} \times \partial_\mu \vec{\phi}) \times \vec{\phi} = \frac{1}{a^2} (\partial_\mu \vec{\phi}\, a^2 - \vec{\phi}\; \phi \cdot \partial_\mu \phi) = \partial_\mu \vec{\phi}$$

The 1st term on ~~the~~ RHS of (38) is the particular solution, and $\vec{\phi} A_\mu$ is the general solution to the homogeneous equation. Computing the curvature, after some algebra we obtain.

$$\left.\begin{aligned} \vec{G}_{\mu\nu} &= \frac{1}{a} \vec{\phi} F_{\mu\nu} \\ F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu + \frac{1}{a^2 e} \vec{\phi} \cdot (\partial_\mu \vec{\phi} \times \partial_\nu \vec{\phi}) \end{aligned}\right\} \quad (39)$$

Next we compute the magnetic charge in a region in the Higgs vacuum ~~con~~ containing a region with twist.



Using the eq'ns of motion in the Higgs vacuum,

$$\partial_\mu F^{\mu\nu} = 0 \qquad \partial_\mu {}^*F^{\mu\nu} = 0$$

because:

$$\mathcal{D}_\nu G_a^{\mu\nu} = - e\, \varepsilon_{abc}\, \phi_b (\mathcal{D}^\mu \phi)_c$$
$$\mathcal{D}^\mu \mathcal{D}_\mu \phi_a = - \lambda \phi_a (\phi^2 - a^2)$$

$$g_\Sigma = \int_\Sigma \vec{B} \cdot d\vec{S} \qquad F^{ij} = - \epsilon_{ijk} B^k \Rightarrow$$

$$\boxed{g_\Sigma = - \frac{1}{ea^3} \int_\Sigma \frac{1}{2} \varepsilon_{ijk}\, \phi \cdot (\partial^j \phi \times \partial^k \phi)\; dS^i} \qquad (40)$$

The winding number of $\Phi : \Sigma \longrightarrow \mathcal{M}_0$ (vacuum manifold) is

$$N = \frac{1}{4\pi a^3} \int dS^i \, \frac{1}{2} \epsilon_{ijk} \, \phi \cdot (\partial^j \phi \times \partial^k \phi) \qquad (41)$$

Hence.

$$\boxed{g_\Sigma = -\frac{4\pi N}{e}} \qquad (42)$$

In coordinate-free notation:

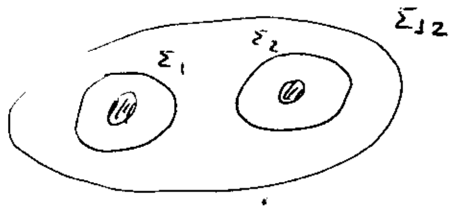
$$F = \frac{1}{2ea^3} \vec{\Phi} \cdot (d\vec{\Phi} \times d\vec{\Phi}) + dA . \qquad (43)$$

then

$$\boxed{g_\Sigma = \int_\Sigma \frac{1}{2ea^3} \vec{\Phi} \cdot (d\vec{\Phi} \times d\vec{\Phi})} \qquad (44)$$

For several regions

$$g_{\Sigma_{12}} = g_{\Sigma_1} + g_{\Sigma_2}$$



Since the smallest e-charge expected is $q_0 = e\hbar/2$, we obtain Dirac's condition

$$\frac{ge}{4\pi} = -N \iff g \frac{2q_0/\hbar}{4\pi} N \iff \boxed{\frac{g q_0}{4\pi\hbar} = -\frac{N}{2}} \qquad (45)$$

If we choose the unitary gauge, we recover the Dirac string.

BPS Bound

The electric and magnetic charges are defined as follows:

$$F^{\mu\nu} = \vec{\Phi} \cdot \vec{G}^{\mu\nu}/a$$

$$g = \int_{S(\infty)} \vec{B} \cdot d\vec{S} = \frac{1}{a} \int B_a^k \phi_a \cdot dS^k = \frac{1}{a} \int B_a^k (D^k \phi)_a \cdot d^3\vec{r} \qquad (46)$$

$$q = \int \vec{E} \cdot d\vec{S} = \frac{1}{a} \int E_a^k \phi_k \, dS^k = \frac{1}{a} \int \partial_k (E_a^k \phi_a) \, d^3\vec{r} = \frac{1}{a} \int E_a^k D^k \phi_a \, d^3 r \qquad (47)$$

$D_k E^k_a = \epsilon_{abc} \phi_b D_0 \phi_c \Rightarrow (D_k E^k_a) \phi_a = 0$ by antisymm. of ϵ, In the CM the monopole mass is:

$$M = \int d^3\vec{r} \left\{ \frac{1}{2} \left[(\mathcal{E}_a^k)^2 + (\mathcal{B}_a^k)^2 + (\mathcal{D}^0\vec{\phi})^2 + (\mathcal{D}^i\vec{\phi})^2 \right] + V(\phi) \right\} \geq$$

$$\geq \int d^3\vec{r} \, \frac{1}{2} \left(\mathcal{E}_a^{k\,2} + \mathcal{B}_a^{k\,2} + (\mathcal{D}^i\phi)^2 \right) =$$

$$= \frac{1}{2} \int d^3 r \left\{ (\mathcal{E}_a^k - \mathcal{D}^k\phi_a \sin\theta)^2 + (\mathcal{B}_a^k - \mathcal{D}^k\phi_a \cos\theta)^2 \right\} +$$

$$+ \sin\theta \int d^3\vec{r}\, \mathcal{E}_a^k \mathcal{D}^k\phi_a + \cos\theta \int d^3\vec{r}\, \mathcal{B}_a^k \mathcal{D}^k\phi_a =$$

$$= \frac{1}{2} \int d^3\vec{r} \left[(\mathcal{E}_a^k - \mathcal{D}^k\phi_a \sin\theta)^2 + (\mathcal{B}_a^k - \mathcal{D}^k\phi_a \cos\theta)^2 \right] +$$

$$+ a(q\sin\theta + g\cos\theta) \quad \rightarrow \quad M \geq a(q\sin\theta + g\cos\theta)$$

The RHS is maximum at $q\cos\theta - g\sin\theta = 0$ $\tan\theta = q/g$, hence:

$$\boxed{M \geq a(g^2+q^2)^{1/2}} \qquad (48)$$

For 'tHP, $q=0$, $M \geq a|g|$. Since $|g| = 4\pi/e$, $M_W = ae\hbar = aq$,

$$M \geq a\frac{4\pi}{e} = \frac{4\pi}{e^2\hbar} M_W = \frac{4\pi\hbar}{q^2} M_W = \frac{\nu}{\alpha} M_W \qquad (49)$$

$\nu = 1$ or $1/4$ depending on whether the electron charge is q or $\frac{1}{2}q$.

BPS Monopoles

It is obtained for $\mathcal{E}_a^i = 0$, $\lambda \to 0$ (keeping $\phi^2 = a^2$ B.C.). The bound is saturated when:

$$\mathcal{D}^0\phi = 0 \qquad \mathcal{B}_a^i = \pm(\mathcal{D}^i\phi)_a \qquad (g \gtrless 0) \qquad (50)$$

Now $H - \xi = 1 + O(e^{-\xi})$ as $\xi \to \infty$ because $m_{Higgs} = 0$. In the BPS limit we also have dyon solutions

$$\mathcal{D}^0\vec{\phi} = 0 \qquad \mathcal{E}_a^k = (\mathcal{D}^k\phi)_a \sin\theta \qquad \mathcal{B}_a^k = (\mathcal{D}^k\phi)\cos\theta$$
$$q = g\tan\theta = -\frac{4\pi}{e}\tan\theta \qquad (51)$$

Semiclassical arguments (& Tomboulis, Woo) lead to $q = n\hbar e$ $n = 0, \pm1, \pm2, \ldots$

A simple candidate for the magnetic current is:

$$F^{\mu\nu} = \hat{\phi}\cdot G^{\mu\nu} + \frac{1}{e}\,\hat{\phi}\cdot(\mathcal{D}^\mu\hat{\phi}\times\mathcal{D}^\nu\hat{\phi}) \qquad \hat{\phi} = \frac{\phi}{|\phi|}$$

by the Bianchi identity $\partial_\nu {}^*F^{\mu\nu}=0$. ~~Magnon~~ Magnetic charge resides at zeroes of the Higgs field. Another (simpler) proposal is:

$$F^{\mu\nu} = \vec{\phi}\;\vec{G}^{\mu\nu}/a \qquad (52)$$

$k^\mu = \partial_\nu(\vec{\phi}\cdot{}^*\vec{G}^{\mu\nu})/a = \mathcal{D}_\nu\vec{\phi}\cdot{}^*\vec{G}^{\mu\nu}/a$. Automatically $\partial_\mu k^\mu=0$, k^0 has no canonical momentum, hence $\int k^0 d^3x$ is a topological quantum number.

Monopoles From Afar

Assume G breaks to H. We want to construct the monopole charge in terms of H-fields alone. G, H are compact. To determine macroscopic properties of the system we study the Higgs vacuum region.

$$\begin{aligned}
&\mathcal{D}_\mu\phi = \partial_\mu\phi + ieW_\mu\phi \qquad W_\mu = W_\mu^a T^a\\
&G_{\mu\nu} = \partial_\mu W_\nu - \partial_\nu W_\mu + ie[W_\mu, W_\nu] \qquad (53)\\
&\mathcal{L} = -\frac{1}{4}(G^a_{\mu\nu})^2 + (\mathcal{D}^\mu\phi)^+\mathcal{D}_\mu\phi - V(\phi) \;;\; V(D(g)\phi) = V(\phi)\\
&\Theta^{\mu\nu} = -G^{\mu\lambda}_a G_a{}^\nu{}_\lambda + \frac{1}{2}(\mathcal{D}^\mu\phi)^+(\mathcal{D}^\nu\phi) + \frac{1}{2}(\mathcal{D}^\nu\phi)^+\mathcal{D}^\mu\phi - g^{\mu\nu}\mathcal{L}\\
&j^\mu_a = ie\,\phi^+T^a\mathcal{D}^\mu\phi - ie(\mathcal{D}^\mu\phi)^+T^a\phi
\end{aligned}$$

The equations of motion are

$$\mathcal{D}^\mu\mathcal{D}_\mu\phi_a = -\frac{\partial V}{\partial\phi_a} \qquad \mathcal{D}_\nu G^{\mu\nu}_a = -j^\mu_a \qquad (54)$$

For finite energy solutions, at large distances we are in the Higgs vacuum

$$\left.\begin{aligned} V(\phi) &= 0\\ \mathcal{D}_\mu\phi &= 0\end{aligned}\right\}$$

$M_0 = \{\phi;\ V(\phi)=0\}$. Assume ~~M~~ G is transitive on M_0. Then the little group H of different ϕ's $\in M_0$ are isomorphic.

$$\mathcal{D}_\mu\phi = 0 \Rightarrow [\mathcal{D}_\mu, \mathcal{D}_\nu]\phi = D(G_{\mu\nu})\,\phi = 0 \qquad (D(g)\text{ rep of }G\text{ of }\phi)$$

Thus ~~B~~ G is in H in the Higgs vacuum.

Only H-gauge fields permeate the Higgs vacuum. In a region $\mathcal{H}$ surrounding monopoles $M_1, \ldots, M_N$ $\phi(r) \in \mathcal{M}_0$ $\vec{r} \in \mathcal{H}$. Take Σ in $\mathcal{H}$ with $M_1 \subset \mathring{\Sigma}$, $\Phi: \Sigma \to \mathcal{M}_0$. In there is no monopole crossing Σ, ϕ changes continuously with t. Hence ϕ is classified topologically by $\pi_2(\mathcal{M}_0) = \pi_2(G/H)$ by the assumption that G is transitive on $\mathcal{M}_0$. Use the exact sequence:

$$\pi_2(G) \to \pi_2(G/H) \to \pi_1(H) \to \pi_1(G) \to \pi_1(G/H) \to \pi_0(H) \ldots$$

Then

$$\pi_2(G/H) \simeq \pi_1(H)_G = \text{Ker}\{\pi_1(H) \to \pi_1(G)\}$$

These classes are gauge invariant by continuity. If $\pi_1 G \neq \phi$, $\pi_1(H)_G$ is a normal subgroup of $\pi_1(H)$. If Dirac strings are not allowed, we can only see $\pi_1(H)_G$. Let $\tilde{G}$ be the universal cover of G, the little group will be $\tilde{H}$, $\tilde{G}/\tilde{H}$, $\pi_1(\tilde{H}) = \pi_1(H)_G$. If Dirac strings are allowed, all elements of $\pi_1(H)$ appear. ~~If~~ In the Georgi-Glashow model $G = SO(3)$, could be replaced by $\tilde{G} = SU(2)$. The homotopically distinct paths in H are

$$h(t) = \exp\left(i t\, \vec{\Phi}\cdot\vec{T}\, 4\pi N/a\right) \quad 0 \le t \le 1 \qquad (55)$$

$(T_a)_{ij} = -i\epsilon_{aij}$ $\quad SO(3) \qquad T_a = \frac{1}{2}\sigma_a \quad SU(2)$

$G = SU(2)$, $H = U(1)$, $\vec{\Phi}\cdot\vec{T} \in \frac{1}{2}\mathbb{Z}$, $\qquad N \in \mathbb{Z}$

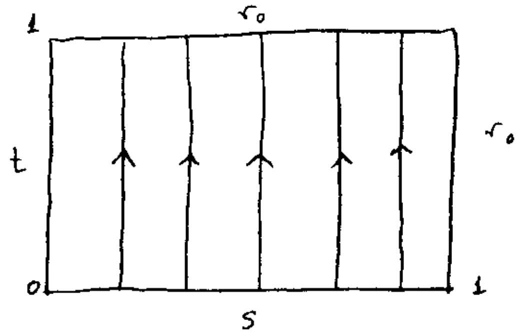
$G = SO(3)$, $H = SO(2)$, $\vec{\Phi}\cdot\vec{T} \in \mathbb{Z}$ $\qquad N \in \frac{1}{2}\mathbb{Z}$, but only for $N \in \mathbb{Z}$ we obtain a trivial path in $SO(3)$. After all $\pi_1 SO(3) = \mathbb{Z}_2$. Later we show that $g = 4\pi N/e$. In each case $N \in \mathbb{Z}$. For $SU(2)$ $q_0 = e\hbar/2$ $\quad g q_0/4\pi\hbar = N/2$. For $SO(3)$ $q_0 = e\hbar$, $g q_0/4\pi\hbar = N$, and to obtain all solutions allowed by the Dirac conditions strings should be introduced

For GWS, $G \approx SU(2) \times U(1)$ $\quad H = U(1)$, but $U(1)$ is a linear combination of $SU(2)$ & $U(1)$. Although $\pi_1(H) = \mathbb{Z}$, $\pi_1(H)_G = 0$, so any non-trivial monopole must have a string.

Topology in terms of H-fields

On Σ we solve now $\mathcal{D}_\mu \phi = 0$. $\Sigma \simeq S^2 = \{ r(s,t),\ 0 \le s,t \le 1 \}$ with the bdry identified

$(0,t) = (1,t) = (s,0) = (s,1) = r_0$



For fixed s $\vec{r}(s,t)$ describes a closed path $\gamma_s(t)$. We have a loop of loops. $\phi : \Sigma \to \mathcal{M}_0$, $\phi_0 = \phi(\vec{r}_0)$ The covariant derivative along t is:

$$\mathcal{D}_t \equiv \frac{\partial r^i}{\partial t} \mathcal{D}_i$$

$$\mathcal{D}_t \phi = 0 \qquad \phi(s,0) = \phi_0 \qquad 0 \le s \le 1 \tag{56}$$

$$\mathcal{D}_\mu \phi = 0 \qquad \mathcal{D}_i \phi = \partial_i \phi + ie W_i \phi = \partial_i \phi - ie W^i \phi = 0$$

For s fixed

$$\mathcal{D}_t \phi = 0 \Leftrightarrow \partial_t \phi = ie \vec{W} \cdot \frac{\partial \vec{r}}{\partial t} \cdot \phi \tag{57}$$

Consider a group element $g(s,t)$ such that

$$\mathcal{D}_t g(s,t) = 0 \qquad g(s,0) = 1$$

$$g(s,t) = \mathcal{P} \exp ie \int_0^t \vec{W} \cdot \frac{\partial \vec{r}}{\partial t} dt \tag{58}$$

On the bdry g should have the same value acting on ϕ_0, hence $g(s,1) \in H_0$. At $s = 0, 1$ $\partial \vec{r}/\partial t = 0$, thus $g(0,t) = g(1,t) = 1$. At intermediate paths

$$\boxed{\begin{aligned} h(s) &\equiv g(s,1) = \mathcal{P} \exp\left(ie \int_0^1 \vec{W} \cdot \frac{\partial \vec{r}}{\partial t} dt \right) \\ h(0) &= h(1) = 1 \end{aligned}} \tag{59}$$

This describes the map $\pi_2(G/H) \cong \pi_1(H)_G$. The path in H is clearly trivial in $\pi_1 G$, the homotopy is just $g(s,t)$. Next we write $h(s)$ in terms of the field strength

$\mathcal{D}_t g(s,t) = 0 \qquad g(s,0) = 1$, which can be thought of as:

$$\mathcal{D}_t \cdot g \cdot = g\, \partial_t \Rightarrow g^{-1} \cdot \mathcal{D}_t \cdot = \partial_t g^{-1} \cdot \tag{60}$$

Then:

$$\partial_t \left(g^{-1} \mathcal{D}_s g \right) = g^{-1} \mathcal{D}_t \mathcal{D}_s g = g^{-1} [\mathcal{D}_t, \mathcal{D}_s] g =$$

$$= ie\, g^{-1} G_{ij}\, g \frac{\partial r^i}{\partial t} \frac{\partial r^j}{\partial s} \quad . \quad \text{Since}$$

$g^{-1} \mathcal{D}_s g = 0$ $\quad t = 0 \quad$ $g^{-1} \mathcal{D}_s g = h^{-1} \frac{dh}{ds}$ $\quad t = 1$, integrate t:

$$\boxed{h^{-1} \frac{dh}{ds} = ie \int_0^1 dt\; g^{-1} G_{ij}\, g \frac{\partial r^i}{\partial t} \frac{\partial r^j}{\partial s}} \qquad (61)$$

(1). In the abelian case $h(s) = \exp ie \int \vec{B} \cdot d\vec{S}$, $h(1) = 1 \Rightarrow$
$\Rightarrow eg = 2\pi n$ or $qg/4\pi\hbar = n/2$ q.e.d.

(2). An interesting application is charge quantization with color. ~~Imag~~ Imagine that locally $H \simeq U(1) \times K$, but the global structure is not fixed. Take ϕ in the adjoint: $[M, \phi] = 0$ defines the little group of ϕ. $\phi = \phi_a T^a$. $\phi \in L(H)$, and commutes with the other generators of H. ϕ generates an invariant ~~subgroup~~ $U(1)$ subgroup of H that we identify with electromagnetism; $M \in L(H) \Leftrightarrow [M, \phi] = 0$ $\quad L(H) \simeq u(1) \oplus L(K)$. K is "color".

$$L(K) = \{ M \in L(H) \mid \operatorname{Tr} \phi M = 0 \}$$

$$W^\mu = A^\mu \frac{\phi}{a} + X^\mu \qquad \operatorname{Tr} \phi X^\mu = 0 \qquad (62)$$

$$\partial^\mu + i \frac{Q}{\hbar} A^\mu \quad : \quad \partial^\mu + ie W^\mu \qquad Q = \frac{e\hbar}{a} \phi$$

Since $h^{-1} dh/ds$ is a generator of H, we may write

$$h^{-1} \frac{dh}{ds} = ie\, \alpha(s) \frac{\phi_0}{a} + i \beta_a K^a \qquad (63)$$

$$\operatorname{Tr} T_a T_b = \eta\, \delta_{ab}$$

$$\alpha(s) = -\frac{i}{ae\eta} \operatorname{Tr} \phi_0\, h^{-1} \frac{dh}{ds} = \frac{1}{a\eta} \operatorname{Tr} \phi_0 \int_0^1 g^{-1}(r) G_{ij}\, g \frac{\partial r^i}{\partial t} \frac{\partial r^j}{\partial s} dt$$

$$= \frac{1}{a\eta} \int_0^1 \operatorname{Tr} \phi(r)\, G_{ij} \frac{\partial r^i}{\partial t} \frac{\partial r^j}{\partial s} dt$$

(ϕ is covariantly constant)
$g \phi_0 g^{-1} = \phi(r)$)

Identify the e.m. field

$$F^{\mu\nu} = \frac{1}{a\eta} \operatorname{Tr} \phi \cdot G^{\mu\nu} \Rightarrow \alpha(s) = \int_0^1 F_{ij} \frac{\partial r^i}{\partial t} \frac{\partial r^j}{\partial s} dt = \frac{d\Phi(s)}{ds} \quad (64)$$

Thus:

$$h(s) = k(s)\, e^{i\frac{e}{a}\Phi(s)\,\phi_0} = \{ Q = \frac{e\hbar}{a}\phi_0 \} = $$
$$= k(s)\, e^{iQ\Phi(s)/\hbar}$$

$h(1) = 1$, $\Phi(1) = g$, total magnetic charge inside Σ. Hence

$$\boxed{e^{igQ/\hbar} = k \in Z(K)} \qquad (65)$$

Take $K = SU(N)$. $k = \lambda \ni \lambda^N = 1$. $\lambda = e^{2\pi i n/N}$

$U(1) \cap K \subset Z_N$. If $|s\rangle$ is a color singlet, $k|s\rangle = |s\rangle$,

and $e^{igQ/\hbar} = 1$ i.e. $e^{igq_s/\hbar} = 1$

If q_0 is the unit of charge for color singlets $q_s = n q_0$ $n \in \mathbb{Z}$,

the possible g values are $g = m g_0$ with $\boxed{g_0 q_0 = 2\pi\hbar}$ (66)

If the states have N-ality:

$$k|c\rangle = e^{2\pi i\, t(c)/N}|c\rangle = e^{igQ/\hbar}|c\rangle$$

For a minimal monopole g_0, we obtain

$$g_0 q/\hbar = 2\pi\left(m + t(c)/N\right) \Rightarrow \boxed{q_c = q_0\left(m + t(c)/N\right)} \qquad (67)$$

Giving a form of charge fractionalization.

Non-Abelian Magnetic Charge

Goddard - Nuyts - Olive attempted to classify all H-monopole configurations. Take a static configuration, choose $\vec{r}\cdot\vec{W}_a = 0$ outside some region. It is plausible to argue that

$$\left.\begin{aligned} G_{ij} &= \frac{1}{4\pi r^2}\epsilon_{ijk}\hat{r}^k G(r) + O(r^{-2-\delta}) \\ \partial_r G(r) &= 0 \end{aligned}\right\} \qquad (68)$$

Since $G(r)$ is in the adjoint rep.

$$G(r) = g(r) G_0\, g(r)^{-1} \qquad (69)$$

$$h^{-1}\frac{dh}{ds} = ie\int_0^1 g^{-1} G_{ij}\, g \frac{\partial r^i}{\partial t}\frac{\partial r^j}{\partial s} = ie\, G_0 \int_0^1 \frac{1}{4\pi}\epsilon_{ijk}\hat{r}^k \frac{\partial r^i}{\partial t}\frac{\partial r^j}{\partial s}\, dt$$

$$= ie\, G_0 \frac{d}{ds}\Omega(s) \qquad (70)$$

Ω is the solid angle subtended by the loop γ_s.

$$\left.\begin{aligned} h(s) &= e^{\frac{ie}{4\pi} G_{(0)} \Omega(s)} \\ e^{ie G_{(0)}} &= 1 = h(1) \end{aligned}\right\} \qquad (71)$$

By conjugation we can set $G_{(0)}$ to the Cartan subalgebra, $G_{(0)} = \vec{\beta}\cdot\vec{H}$, hence (71) is equivalent to

$$\boxed{e\vec{\beta}\cdot\vec{H} = 2\pi \mathbb{Z}} \qquad (72)$$

Let Λ_W be the weight lattice of the group H. Then

$$\boxed{e\vec{\beta} \in \Lambda_W^*}$$

These are the magnetic weights, and they define a dual group $H^\vee$ which plays a central rôle in the Montonen-Olive conjecture.

Consider $H = SO(3)$ or $SU(2)$ for simplicity.

$$G_{(0)} = \beta T^3 \qquad e^{ie\beta T^3} = 1 .$$

1) $H = \cancel{S(2)}\ SO(3)$ T^3 is integral $e\beta = 4\pi \frac{n}{2}$

2) $H = SU(2)$ T^3 is $\frac{1}{2}$ integral $e\beta = 4\pi n$

Hence:

$$\begin{aligned} &H = SU(2) & e\beta &= 4\pi\,(\text{weight of } SO(3)) \\ &H = SO(3) & e\beta &= 4\pi\,(\text{weight of } SU(2)) \end{aligned} \qquad (73)$$

Given G, let $\tilde{G}$ be the universal cover $\pi_1\tilde{G} = 0$. $\Lambda_R(\tilde{G})$ the root lattice $\Lambda_W(\tilde{G}) = \Lambda_R(\tilde{G})^*$. $Z(\tilde{G})$ center of $\tilde{G}$, then $\Lambda_R(\tilde{G}) = \Lambda_W(\tilde{G})/Z(\tilde{G})$. $G \subset \tilde{G}$, $Z(G)$ center of $G \subset Z(\tilde{G})$

$\mu \in \Lambda_W(G)$.

$$e\vec{\beta}\cdot\vec{\mu}\ \cancel{\in} = 2\pi \mathbb{Z} \qquad (74)$$

For $\tilde{G}$ $e\vec{\beta} \in 2\pi \Lambda_R(\tilde{G})$ $\qquad |\Lambda^*(G)/\Lambda_R(G)| = Z(G)$

Examples.

$$\begin{aligned} SU(N)^\vee &= SU(N)/\mathbb{Z}_N \\ (SU(NM)/\mathbb{Z}_N)^\vee &= (SU(NM)/\mathbb{Z}_M)^\vee \end{aligned} \qquad (75)$$

The Montonen-Olive conjecture states that a gauge theory is characterized by $H \times H^\vee$. That we have two equivalent descriptions. One in terms of H-gauge fields and ordinary particles, the other with $H^\vee$ gauge fields and monopoles, with Noether and topological currents interchanged (also with $g \sim 1/g$ or something similar). Probably the conjecture makes sense in $N=2$, or better in $N=4$ SUSY.

θ And The Monopole Charge (E.W. Henf. 79)

We have seen that for two dyons of charges (e_1, g_1), (e_2, g_2) the relation:

$$e_1 g_2 - e_2 g_1 = 2\pi n \qquad (76)$$

For charges $eg = 2\pi n$. The quantization condition says something about the difference between the electric charges of two monopoles. Take two poles of minimal charge $2\pi/e$ and charges e_1, e_2; then ~~the~~ (76) implies $(e_1 - e_2)g = 2\pi(e_1 - e_2)/e = 2\pi n \Rightarrow e_1 - e_2 = ne$, but q, q' could be anything. If the theory is CP conserving, then q, q' must be either of the form ne, or $(n+1/2)e$. All monopoles must satisfy one of the two possibilities. The proof is as follows: Under CP $(q, 2\pi/e) \to (-q, 2\pi/e)$, applying (76) leads to $e_1 g_2 - e_2 g_1 = 4\pi q/e = 2\pi n \Rightarrow q = ne$ or $(n+1/2)e$. If we took $(ne, 2\pi/e)$, $((m+1/2)e, 2\pi/e)$, then according to (76), $\frac{2\pi}{e}(ne - (m+1/2)e) = 2\pi(n-m+1/2) \notin 2\pi\mathbb{Z}$. Unfortunately in the real world we have CP violation. In particular the θ-parameter. Take the GG model for simplicity:

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}^2 + \frac{1}{2}(D_\mu \vec{\phi})^2 - \lambda(\phi^2 - a^2)^2 \qquad (77)$$

and add the θ term:

$$\frac{\theta e^2}{32\pi^2} F_{\mu\nu} \cdot \tilde{F}_{\mu\nu} \qquad (78)$$

$$(\varepsilon^{\mu\nu\alpha\beta} F_{\mu\nu} F_{\alpha\beta})$$

Define the operator N which implements rotation around the $\hat{\phi}$ direction. These transformations correspond to electric charge.

$\delta\vec{v} = \frac{1}{a}\vec{\phi}\times\vec{v}$ $\quad \delta\vec{A}_\mu = \frac{1}{ea} D_\mu\vec{\phi}$. The operator $e^{2\pi i N} = 1$ because $e^{2\pi i N}$ is a 2π-rotation about $\hat{\phi}$. Only at spatial infinity where $|\phi| = a$ does $\exp 2\pi i N$ generate a 2π rotation. Elsewhere the angle is $2\pi|\phi|/a$. However, by Gauss' law if the gauge transformation is 1 at ∞ it leaves the state invariant, giving $e^{2\pi i N} = 1$. Thus we use Noether's formula to compute N, including the contribution from the θ-term.

$$N = \int d^3x \left(\frac{\delta \mathcal{L}}{\delta \partial_0 A_i^a} \delta A_i^a + \frac{\delta \mathcal{L}}{\delta \partial_0 \phi^a} \delta\phi^a \right) \qquad (79)$$

since $\delta\phi^a = 0$, only the gauge part contributes

$$\frac{\delta}{\delta\partial_0 A_i^a}\left(-\frac{1}{4}F_{\mu\nu}^a F^{\mu\nu a}\right) = -F^{0ia} = (E^i)^a = F_{0i}^a$$

$$\frac{\delta}{\delta\partial_0 A_i^a}\left(\tilde{F}^{\mu\nu}F_{\mu\nu}\right) = \frac{\delta}{\delta(\ )}\left(\tilde{F}^{0i}F_{0i} + \tilde{F}^{jk}F_{jk}\right)$$

$$\tilde{F}^{\mu\nu}F_{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\alpha\beta}F_{\mu\nu}^a F_{\alpha\beta}^a = 2\,\varepsilon^{0ijk}F_{0i}^a F_{jk}^a$$

(There is a factor of 2 not clear in Witten's computation, unless it is responsible for the difference between SO(3) and SU(2))

$$\frac{\delta}{\delta\partial_0 A_i^a}\tilde{F}\cdot F = 2\,\varepsilon^{ijk}F_{jk}^a$$

$$N = \frac{1}{ae}\int d^3x\, D_i\vec{\phi}\cdot\vec{F}_{0i} + \frac{e\theta}{8\pi^2 a}\int d^3x\, D_i\vec{\phi}\cdot\frac{1}{2}\epsilon^{ijk}\vec{F}_{jk}$$

$$= \left(\frac{1}{e}Q - \frac{e\theta}{8\pi^2}M\right)$$

Given the magnetic ~~charge~~ charge, for the 't Hooft-Polyakov monopole $M = 4\pi/e$, and $q = ne + e\theta/2\pi$

$$\boxed{q = ne + \frac{e\theta}{2\pi}} \qquad (80)$$

There is a general argument which leads to the previous result. The θ-term is:

$$\mathcal{L}' = \frac{\theta e^2}{32\pi^2} \varepsilon^{\mu\nu\lambda\sigma} F^a_{\mu\nu} F^a_{\lambda\sigma} \qquad (81)$$

We ignore the heavy fields and consider only long distance effects. Thus we concentrate directly on EM fields. Replace $F^a_{\mu\nu}$ by $F_{\mu\nu}$,

$$F_{0i} = E_i \qquad F_{ij} = -\epsilon_{ijk} B_k$$

Then

$$\mathcal{L}' = \frac{\theta e^2}{32\pi^2} 4\, \epsilon^{0ijk} F_{0i} F_{jk} = -\frac{\theta e^2}{4\pi^2} \vec{E}\cdot\vec{B}$$

Consider:

$$\vec{E} = -\vec{\nabla}\phi$$
$$\vec{B} = \vec{\nabla}\times\vec{A} + \frac{g\vec{r}}{4\pi r^3} \qquad (\text{monopole background}) \qquad (82)$$

Then:

$$L' = \int d^3r\, \mathcal{L}' = \frac{\theta e^2}{4\pi^2}\int d^3\vec{r}\; \vec{\nabla}\phi\left(\vec{\nabla}\times\vec{A} + \frac{g\vec{r}}{4\pi r^3}\right)$$
$$= -\frac{\theta e^2}{4\pi^2} g \int d^3\vec{r}\; \phi(\vec{r})\, \delta^{(3)}(\vec{r})$$

This is the coupling of ϕ to an electric charge of magnitude $-\theta e^2 g/4\pi^2$ at the origin. Since $(eg/4\pi)$ is always $\frac{1}{2}\mathbb{Z}$, then ~~a~~ charge is reduced by θ. If we add an electric pole, after integration by parts nothing happens. Hence the dyon charge is:

$$Q = e\left(m + \frac{eg\,\theta}{4\pi^2}\right)$$

Since $eg = \frac{1}{2}n$ when $\theta \to \theta + 2\pi$, the original set of charges is recreated, with each term moving $\left(\frac{eg}{2\pi}\right)$ forward. The argument outlined applies no matter which group G guarantees the existence of monopoles, as long as there are abelian fields far away.

We will show later that the GG model extended to $N=2$ SUSY has some outstanding properties: It contains central charges, and BPS saturation can be understood quantum mechanically.

II. SUPERSYMMETRY.

Conventions. $\eta_{ab} = \mathrm{diag}(1,-1,-1,-1)$. The spinors of $SL(2,\mathbb{C})$ are dotted and undotted

$$\psi'_\alpha = M_\alpha{}^\beta \psi_\beta \qquad \bar\psi'_{\dot\alpha} = M^*{}_{\dot\alpha}{}^{\dot\beta}\bar\psi_{\dot\beta} \qquad (83)$$

Spinor indices are raised or lowered with ϵ-metric

$$\epsilon^{\alpha\beta} = \epsilon^{\dot\alpha\dot\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = (i\sigma_y) \qquad (84)$$

Since $M \in SL_2(\mathbb{C})$,

$$\epsilon^{\alpha\beta} = \epsilon^{\gamma\delta} M_\gamma{}^\alpha M_\delta{}^\beta$$

This can be written as:

$$M\sigma_y M^T = \sigma_y \Rightarrow \sigma_y M = (\sigma_y M^T)^{-1} = (M^T)^{-1}\sigma_2$$

Which implies:

$$\psi'^\alpha = \psi^\beta (M^{-1})_\beta{}^\alpha \qquad \bar\psi'^{\dot\alpha} = \bar\psi^{\dot\beta}(M^*)^{-1}{}_{\dot\beta}{}^{\dot\alpha} \qquad (85)$$

Define:

$$(\sigma^a)_{\alpha\dot\alpha} \equiv (1, \vec\sigma)$$

$$\sigma^a P_a = \begin{pmatrix} P^0 - P^3 & -P^1 + iP^2 \\ -P^1 - iP^2 & P^0 + P^3 \end{pmatrix} \qquad (86)$$

$$\det \sigma^a P_a = P_a P^a$$

Lorentz transformations are generated by:

$$(\sigma^{nm})_\alpha{}^\beta = \frac{1}{4}\left(\sigma^n{}_{\alpha\dot\beta}\,\bar\sigma^{m\dot\beta\beta} - (n \leftrightarrow m)\right)$$

$$(\bar\sigma^{nm})^{\dot\alpha}{}_{\dot\beta} = \frac{1}{4}\left(\bar\sigma^{n\dot\alpha\beta}\,\sigma^m{}_{\beta\dot\beta} - (n \leftrightarrow m)\right) \qquad (87)$$

Where

$$(\bar\sigma^a)^{\dot\alpha\alpha} = \epsilon^{\dot\alpha\dot\beta}\epsilon^{\alpha\beta}(\sigma^a)_{\alpha\dot\alpha} \qquad (88)$$

numerically:

$$(\bar\sigma^a) = (i\sigma_y)(\sigma^a)^T(i\sigma_y)^T = \sigma_y(\sigma^a)^T\sigma_y = (1, -\vec\sigma) \qquad (89)$$

In this basis, Dirac matrices and spinors are:

$$\gamma^a = \begin{pmatrix} 0 & \sigma^a \\ \bar\sigma^a & 0 \end{pmatrix} \qquad \psi_D = \begin{pmatrix} \psi_\alpha \\ \bar\chi^{\dot\alpha} \end{pmatrix} \qquad \psi_M = \begin{pmatrix} \psi_\alpha \\ \bar\psi^{\dot\alpha} \end{pmatrix} \qquad (90)$$

Majorana

$\gamma_5 = -i\gamma^0\gamma^1\gamma^2\gamma^3$, hence $\gamma_5 = +1 \Rightarrow$ −ve helicity

$\gamma_5 = -1 \Rightarrow$ +ve helicity

For a massless fermion moving in the z-direction $P^\mu = E(1,0,0,1)$, hence $(\gamma^0-\gamma^3)\psi=0$. Since the helicity now is $J_3 = \frac{i}{2}\gamma^1\gamma^2$, $\Rightarrow$
$\Rightarrow$ $J_3\psi = \frac{i}{2}\gamma^{0\,2}\gamma^1\gamma^2\psi = \frac{i}{2}\gamma^0\gamma^1\gamma^2\gamma^3\psi = -\frac{1}{2}\gamma_5\psi$. Hence the neutrino field has $\gamma_5=+1$.

Scalar Product.

$$\begin{aligned} \psi\chi &\equiv \psi^\alpha\chi_\alpha = -\psi_\alpha\chi^\alpha = \chi^\alpha\psi_\alpha = \chi\psi \\ \bar\psi\bar\chi &\equiv \bar\psi_{\dot\alpha}\bar\chi^{\dot\alpha} = \bar\chi\bar\psi \\ (\psi\chi)^\dagger &= \bar\chi_{\dot\alpha}\bar\psi^{\dot\alpha} = \bar\chi\bar\psi = \bar\psi\bar\chi \end{aligned} \qquad (91)$$

SUSY algebra

Without central charges,

$$\boxed{\{Q_\alpha{}^I, \bar Q_{\dot\alpha J}\} = 2\,\sigma^a{}_{\alpha\dot\alpha}\,P_a\,\delta^I{}_J} \qquad (92)$$

Massive Irreps.

Use Wigner's method. choose $P^a = (M,0,0,0)$ and define:

$$a_1{}^I = Q^I{}_1/\sqrt{2M}, \quad a_2{}^I = Q_2{}^I/\sqrt{2M}$$

Then (92) is equivalent to

$$\{a_1{}^I,(a_1{}^J)^+\} = \delta^{IJ} \qquad \{a_2{}^I,(a_2{}^J)^+\} = \delta^{IJ} \qquad (93)$$

If $I,J=1,\dots,N$ the irrep. contains at least 2^{2N} states (without incorporating CPT invariance). If the Clifford vacuum is a single, $a_\alpha^I|\Omega\rangle=0$. If $|\Omega\rangle$ has spin j the irrep has dimension $(2j+1)\,2^{2N}$.
For instance, $N=1$ we have:

$$|\Omega\rangle, \quad a_\alpha^+|\Omega\rangle \quad \frac{1}{\sqrt2}\,\varepsilon^{\alpha\beta}a_\alpha^+a_\beta^+|\Omega\rangle \qquad (94)$$

From SU(2) point of view if $|\Omega\rangle$ has spin j, the irreps involved are:

$$(\ j,\ j+1/2,\ j-1/2,\ j\ ) \qquad (95)$$

For $j=0$ $(0,1/2,0)$ and we have a Weyl (or Majorana) spinor and a complex scalar (λ,ϕ) (chiral superfield). For $j=1/2$ we have $(1/2,1,0,1/2)$ which could be represented by A_μ, a Dirac fermion and a scalar (all massive with same mass M) so that we have 4 boson + 4 fermion degrees of freedom)

For $\underline{N=2}$ the multiplet has at least $2^4 = 16$ states

$$|\Omega\rangle,\ (a_\alpha^I)^+|\Omega\rangle,\ a_{\alpha_1}^{I_1+} a_{\alpha_2}^{I_2+}|\Omega\rangle \dots\ (a^+)^3|\Omega\rangle \quad (a^+)^4|\Omega\rangle$$

total antisymmetry in $(\alpha_1, I_1) \dots (\alpha_n, I_n)$ $n = 2, 3, 4 \dots$ is manifest. The highest spin is $N/2$ (or $j + N/2$ for $|\Omega\rangle$ of spin j), and we can work out the different populations as in atomic physics. If $|\Omega\rangle$ has spin 0, we have.

$$1 \cdot (j=1) \oplus 4\,(j=1/2) \oplus 5\,(j=0)$$

all other irreps include spin > 1 $(3/2, 2, \dots)$.

For $\underline{N=4}$ a massive multiplet contains $2^{2\cdot 4} = 256$ states, and even if $j=0$ for $|\Omega\rangle$ we always reach spin 2.

Massless Irreps.

Take now $P^\mu = M(1,0,0,1)$, then

$$\{Q_\alpha^I, \bar{Q}_{\dot\alpha J}\} = \begin{pmatrix} 0 & 0 \\ 0 & 4M \end{pmatrix} \delta^I{}_J \qquad (9)6$$

One of the Clifford algebras is trivially realized in a unitary theory. For the other define:

$$a^I = \frac{1}{2\sqrt{M}}\, Q_2^I \qquad \{a^I, (a^J)^+\} = \delta^{IJ} \qquad (9)7$$

For $I = 1, 2, \dots, N$ there are in total 2^N states in the multiplet. The state of highest helicity has $\lambda + N/2$ $\quad a^I|\Omega_\lambda\rangle = 0$. Hence

$\underline{N=1}$ $\quad |\lambda\rangle,\ |\lambda - 1/2\rangle$

$\underline{N=2}$ $\quad |\lambda\rangle,\ 2|\lambda - 1/2\rangle,\ |\lambda - 1\rangle$

$\underline{N=4}$ $\quad |\lambda\rangle\ \ 4|\lambda - 1/2\rangle,\ 6|\lambda - 1\rangle,\ 4|\lambda - 3/2\rangle\ \ 1|\lambda - 2\rangle$

Since the irreps are not necessarily CPT invariant, sometimes we need to double them with the CPT conjugate:

$N=1$. $\quad |1/2\rangle, |0\rangle \xrightarrow{CPT} |-1/2\rangle, |0\rangle$

$\quad |1\rangle, |1/2\rangle \rightarrow |-1\rangle, |-1/2\rangle$

$N=2$ $\quad |1\rangle\ \ 2|1/2\rangle\ \ 1|0\rangle \rightarrow |-1\rangle,\ 2|-1/2\rangle,\ 1|0\rangle$

$N=4$ $\quad |1\rangle,\ 4|1/2\rangle,\ 6|0\rangle,\ 4|-1/2\rangle,\ |-1\rangle$ is CPT self-conjugate

$$A_\mu^a,\ \chi_1^a, \chi_2^a,\ \underbrace{\phi_1^a, \phi_2^b, \phi_3^c}_{\text{complex scalars}}$$

Central charges.

When we have central charges the ireps change considerably.

$$\left.\begin{aligned}\{Q^I_\alpha, \bar{Q}_{\dot\beta J}\} &= 2\,\sigma^A{}_{\alpha\dot\beta} P_A \delta^J{}_J \\ \{Q^I_\alpha, Q_\beta{}^J\} &= \epsilon_{\alpha\beta} Z^{IJ} \\ \{\bar{Q}_{\dot\alpha I}, \bar{Q}_{\dot\beta J}\} &= \epsilon_{\dot\alpha\dot\beta} Z^*{}_{IJ}\end{aligned}\right\}$$

Take $I, J, .. = 1, ..., 2n$. Using a unitary transformation we can skewdiagonalize Z $\quad \tilde{Z}^{IJ} = u^I{}_A u^J{}_B Z^{AB} \quad Z = \epsilon \otimes D$. By a further chiral rotation we may choose the eigenvalues of D to be real. Once we have skew-diagonalize it suffices to consider just $N=2$ SUSY.

$$\left.\begin{aligned}\{Q^a_\alpha, (Q^b_\beta)^+\} &= 2(\sigma^A)_{\alpha\dot\beta} P_A \delta^a{}_b \\ \{Q^a_\alpha, Q_\beta^b\} &= \epsilon_{\alpha\beta}\epsilon^{ab} Z\end{aligned}\right\}$$

For massive representations go to the rest frame

$$\{Q^a_\alpha, (Q^b_\beta)^+\} = 2M\,\delta_{\alpha\dot\beta}\,\delta^a{}_b$$

$$\{Q^a_\alpha, Q^b_\beta\} = \epsilon_{\alpha\beta}\epsilon^{ab} Z$$

write $Z = |Z| e^{i2\alpha}$, by a chiral rotation we can set $\alpha = 0$. Therefore since Z is central we can fix it to be the eigenvalue for the given representation. Define:

$$a_\alpha = \frac{1}{\sqrt2}\{Q^1_\alpha + \epsilon_{\alpha\beta}(Q^2_\beta)^+\}$$

$$b_\alpha = \frac{1}{\sqrt2}\{Q^1_\alpha - \epsilon_{\alpha\beta}(Q^2_\beta)^+\}$$

$$\{a_\alpha, a^+_\lambda\} = \frac12\{Q^1_\alpha + \epsilon_{\alpha\beta}(Q^2_\beta)^+,\ Q^{1+}_\lambda + \epsilon_{\lambda\mu}(Q^2_\mu)^*\}$$
$$= \frac12\left(2M\,\delta_{\alpha\lambda} + 2M\,\epsilon_{\alpha\beta}\epsilon_{\lambda\mu}\delta_{\beta\mu} + \epsilon_{\alpha\beta}\epsilon_{\lambda\beta}|Z| + \epsilon_{\lambda\mu}\epsilon_{\alpha\mu}|Z|\right)$$
$$= \delta_{\alpha\lambda}\,(2M + |Z|)$$

$\{b_\alpha, b^+_\lambda\} = \delta_{\alpha\lambda}\,(2M - |Z|)$; $\{a_\alpha, b_\lambda\} = \{a_\alpha, b^+_\lambda\} = 0$ etc.

This implies that $\boxed{2M \geq |Z|}$ When $|Z| = 2M$ the representation becomes much smaller, and we have a reduced multiplet. Note that for higher N similar arguments apply.

A reduced massive $N=2$ multiplet has the same number of states as a massless $N=1$ multiplet.

Local Representations.

Our conventions are such that

$$\xi\cdot Q = \xi^\alpha Q_\alpha \qquad \bar\xi \bar Q = \bar\xi_{\dot\alpha}\bar Q^{\dot\alpha}$$

$$\delta_\xi A = (\xi\cdot Q + \bar\xi\cdot\bar Q)A$$

$$+ \cancel{[\delta_\xi,\delta_\xi]A = [\xi}$$

$$[\delta_\xi,\delta_\eta]A = [\xi Q + \bar\xi\cdot\bar Q,\ \eta\cdot Q + \bar\eta\cdot\bar Q]$$

$$= \left([\xi\cdot Q,\bar\eta\cdot\bar Q] + [\bar\xi\cdot\bar Q,\eta Q]\right)A = \left(\xi^\alpha\bar\eta^{\dot\alpha}\{Q_\alpha,\bar Q_{\dot\alpha}\} - \eta^\alpha\bar\xi^{\dot\alpha}\{Q_\alpha,\bar Q_{\dot\alpha}\}\right)A$$

$$[\delta_\xi,\delta_\eta]A = 2(\xi\sigma^a\bar\eta - \eta\sigma^a\bar\xi)P_a$$

The commutator of 2-SUSY's is a translation,

$$\{Q_\alpha,\bar Q_{\dot\alpha}\} = 2i\,\sigma^m{}_{\alpha\dot\alpha}\,\partial_m$$

We define a representation:

$$\left.\begin{aligned} Q_\alpha &= \frac{\partial}{\partial\theta^\alpha} - i\sigma^m{}_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\partial_m \\ \bar Q_{\dot\alpha} &= -\frac{\partial}{\partial\bar\theta^{\dot\alpha}} + i\sigma^m{}_{\alpha\dot\alpha}\theta^\alpha\partial_m \end{aligned}\right\} \quad \begin{aligned} &\{Q_\alpha,\bar Q_{\dot\alpha}\} = 2i\,\sigma^m{}_{\alpha\dot\alpha}\partial_m \\ &\{Q_\alpha,Q_\beta\} = 0 \end{aligned}$$

Similarly we have a super-derivative:

$$\left.\begin{aligned} D_\alpha &= \frac{\partial}{\partial\theta^\alpha} + i\sigma^m{}_{\alpha\dot\alpha}\partial_m \\ \bar D_{\dot\alpha} &= -\frac{\partial}{\partial\bar\theta^{\dot\alpha}} - i\sigma^m{}_{\alpha\dot\alpha}\theta^\alpha\partial_m \end{aligned}\right\} \quad \begin{aligned} &\{D_\alpha,\bar D_{\dot\alpha}\} = -2i\,\sigma^m{}_{\alpha\dot\alpha}\partial_m \\ &\{D_\alpha,D_\beta\} = \{D_\alpha,Q_\beta\} = \{D_\alpha,\bar Q_{\dot\beta}\} = 0 \end{aligned}$$

Useful identities

$$\theta^\alpha = \begin{pmatrix}\theta^1\\ \theta^2\end{pmatrix} \qquad \varepsilon^{\alpha\beta} = \begin{pmatrix}0&1\\-1&0\end{pmatrix} \qquad \varepsilon_{\alpha\beta} = \begin{pmatrix}0&-1\\1&0\end{pmatrix} \qquad \theta_\alpha = \varepsilon_{\alpha\beta}\theta^\beta$$

$$\theta_\alpha = \begin{pmatrix}-\theta^2\\ \theta^1\end{pmatrix} \qquad \theta\theta = \theta^\alpha\theta_\alpha = -2\theta^1\theta^2 \qquad \theta^\alpha\theta^\beta = -\tfrac12\varepsilon^{\alpha\beta}\theta\theta \qquad \theta_\alpha\theta_\beta = \tfrac12\varepsilon_{\alpha\beta}\theta\theta$$

Also $\bar\theta\bar\theta = \bar\theta_{\dot\alpha}\bar\theta^{\dot\alpha}$, then $\bar\theta^{\dot\alpha}\bar\theta^{\dot\beta} = \frac12\varepsilon^{\dot\alpha\dot\beta}\bar\theta\bar\theta$, $\bar\theta_{\dot\alpha}\bar\theta_{\dot\beta} = -\frac12\varepsilon_{\dot\alpha\dot\beta}\bar\theta\bar\theta$

Finally:

$$\theta\sigma^m\bar\theta\;\theta\sigma^n\bar\theta = \frac12\theta\theta\bar\theta\bar\theta\,\eta^{mn}. \quad \text{Take } m=n=0$$

$$\theta^\alpha\delta_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\;\theta^\beta\delta_{\beta\dot\beta}\bar\theta^{\dot\beta} = -\theta^\alpha\theta^\beta\bar\theta^{\dot\alpha}\bar\theta^{\dot\beta}\delta_{\beta\dot\beta}\delta_{\alpha\dot\alpha} = \frac14\theta^2\bar\theta^2\varepsilon^{\alpha\beta}\varepsilon^{\dot\alpha\dot\beta}\delta_{\beta\dot\beta}\delta_{\alpha\dot\alpha} =$$

$$= \frac12\theta^2\bar\theta^2$$

Summarizing:

$$\boxed{\begin{aligned}
&\theta^\alpha\theta^\beta = -\frac{1}{2}\varepsilon^{\alpha\beta}\theta\theta \\
&\theta_\alpha\theta_\beta = \frac{1}{2}\varepsilon_{\alpha\beta}\theta\theta \\
&\bar\theta^{\dot\alpha}\bar\theta^{\dot\beta} = \frac{1}{2}\varepsilon^{\dot\alpha\dot\beta}\bar\theta\bar\theta \\
&\bar\theta_{\dot\alpha}\bar\theta_{\dot\beta} = -\frac{1}{2}\varepsilon_{\dot\alpha\dot\beta}\bar\theta\bar\theta \\
&\theta\sigma^m\bar\theta\ \theta\sigma^n\bar\theta = \frac{1}{2}\theta\theta\bar\theta\bar\theta\,\eta^{mn}
\end{aligned}}$$

These formulae are the basis for Fierz rearrangements. Similarly we will need:

$$\chi\sigma^n\bar\psi = -\bar\psi\bar\sigma^n\chi \quad :$$

$$\chi\sigma^n\bar\psi = \chi^\alpha\sigma^n{}_{\alpha\dot\alpha}\bar\psi^{\dot\alpha} = -\bar\psi^{\dot\alpha}\sigma^n{}_{\alpha\dot\alpha}\chi^\alpha = -\bar\psi_{\dot\lambda}\varepsilon^{\dot\alpha\dot\lambda}\sigma^n{}_{\alpha\dot\alpha}\varepsilon^{\alpha\lambda}\chi_\lambda$$
$$= -\bar\psi_{\dot\lambda}(\bar\sigma^n)^{\dot\lambda\lambda}\chi_\lambda = -\bar\psi\bar\sigma^n\chi$$

Similarly for other relations:

$$\boxed{\begin{aligned}
\chi\sigma^n\bar\psi &= -\bar\psi\bar\sigma^n\chi \\
(\chi\sigma^n\bar\psi)^+ &= \psi\sigma^n\bar\chi \\
\chi\sigma^m\bar\sigma^n\psi &= \psi\sigma^n\bar\sigma^m\chi \\
(\chi\sigma^m\bar\sigma^n\psi)^+ &= \bar\psi\bar\sigma^n\sigma^m\bar\chi
\end{aligned}}
\qquad
\boxed{\begin{aligned}
\psi\chi &= \chi\psi \\
\bar\psi\bar\chi &= \bar\chi\bar\psi \\
(\psi\chi)^+ &= \bar\psi\bar\chi
\end{aligned}}$$

Chiral Superfield.

The $N=1$ scalar multiplet is represented by a superfield with one constraint

$$\bar D_{\dot\alpha}\Phi = 0 \qquad -\bar D_{\dot\alpha} = \frac{\partial}{\partial\bar\theta^{\dot\alpha}} + i\theta^\alpha\sigma^m{}_{\alpha\dot\alpha}\partial_m$$

Then:

$$\bar D_{\dot\alpha}\theta^\lambda = 0, \quad \bar D_{\dot\alpha}\left(x^a + i\theta\sigma^a\bar\theta\right) = 0 \qquad y^a = x^a + i\theta\sigma^a\bar\theta$$

Hence

$$\bar D_{\dot\alpha}\Phi = 0 \Rightarrow \Phi = \Phi(y,\theta)$$

$$\boxed{\Phi = A(y) + \sqrt{2}\,\theta\psi(y) + \theta\theta F(y)} \quad \text{(CSF)} \quad \text{Chiral Superfield}$$

An antichiral SF satisfies $D_\alpha\Phi^+ = 0$, $\quad y^+ = x^a - i\theta\sigma^a\bar\theta$

$$\boxed{\Phi^+ = A(y^+) + \sqrt{2}\,\bar\theta\bar\psi(y^+) + \bar\theta\bar\theta F(y^+)} \quad \text{(ACSF)}$$

The product of CSF's is a CSF.

$$\Phi_i \Phi_j = (A_i + \sqrt{2}\,\theta\psi_i + \theta^2 F_i)(A_j + \sqrt{2}\,\theta\psi_j + \theta^2 F_j) =$$
$$= A_i A_j + \sqrt{2}\,\theta(A_i\psi_j + A_j\psi_i) + \theta^2(A_i F_j + A_j F_i - \psi_i\psi_j)$$

where we used

$$\theta\psi_i\,\theta\psi_j = \theta^\alpha\psi_\alpha\,\theta^\beta\psi_\beta = -\theta^\alpha\theta^\beta\psi_\alpha\psi_\beta = \frac{1}{2}\varepsilon^{\alpha\beta}\psi_{i\alpha}\psi_{j\beta}\theta^2 = -\frac{1}{2}\theta^2\psi_i\psi_j$$

$$\boxed{\theta\psi_i\,\theta\psi_j = -\frac{1}{2}\theta\theta\,\psi_i\psi_j}$$

$$\Phi_i\Phi_j\Phi_k = A_iA_jA_k + \sqrt{2}\,\theta(A_iA_j\psi_k + A_iA_k\psi_j + A_jA_k\psi_i) +$$
$$+\theta\theta\left(A_iA_jF_k + A_jA_kF_i + A_kA_iF_j - A_i\psi_j\psi_k - A_j\psi_k\psi_i - A_k\psi_i\psi_j\right)$$

Note that.

$$\psi\chi = \psi^\alpha\chi_\alpha = -\psi_\alpha\chi^\alpha = \chi^\alpha\psi_\alpha \qquad \psi\chi = \chi\psi$$
$$(\psi\chi)^* = (\psi^\alpha\chi_\alpha)^+ = \bar\chi_{\dot\alpha}\bar\psi^{\dot\alpha} = \bar\chi\bar\psi = \bar\psi\bar\chi$$

For an arbitrary function (superpotential)

$$W(\Phi_i) = W(A_i + \sqrt{2}\,\theta\psi_i + \theta\theta F_i) =$$
$$= W(A_i) + \frac{\partial W}{\partial A_i}(\sqrt{2}\,\theta\psi_i + \theta^2 F_i) + \frac{1}{2}\frac{\partial^2 W}{\partial A_i\partial A_j}\,2\,\theta\psi_i\,\theta\psi_j$$
$$= W(A_i) + \sqrt{2}\frac{\partial W}{\partial A_i}\theta\psi_i + \theta\theta\left(\frac{\partial W}{\partial A_i}F_i - \frac{1}{2}\frac{\partial^2 W}{\partial A_i\partial A_j}\psi_i\psi_j\right)$$

$$\boxed{W[\Phi] = W(A_i) + \frac{\partial W}{\partial A_i}\sqrt{2}\,\theta\psi_i + \theta\theta\left(\frac{\partial W}{\partial A_i}F_i - \frac{1}{2}\frac{\partial^2 W}{\partial A_i\partial A_j}\psi_i\psi_j\right)}$$

However a CSF × ACSF is a general vector superfield. Go to the original variables

$$A(y) = A(x^a + i\theta\sigma^a\bar\theta) = A(x) + i\theta\sigma^a\bar\theta\,\partial_a A - \frac{1}{2}\theta\sigma^a\bar\theta\,\theta\sigma^b\bar\theta\,\partial^2_{ab}A =$$
$$= A(x) + i\theta\sigma^a\bar\theta\,\partial_a A - \frac{1}{4}\theta^2\bar\theta^2\,\Box A$$

$$\sqrt{2}\,\theta\psi(y) = \sqrt{2}\,(\theta\psi + i\theta\sigma^a\bar\theta\,\partial_a\theta\psi) = \sqrt{2}\,(\theta\psi - \frac{i}{2}\theta\theta\,\partial_a\psi\,\sigma^a\bar\theta)$$

$$\theta^\alpha\sigma^a_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\,\theta^\beta\partial_a\psi_\beta = \frac{1}{2}\theta^2\varepsilon^{\alpha\beta}\sigma^a_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\partial_a\psi_\beta = -\frac{1}{2}\theta^2\partial_a\psi\,\sigma^a\bar\theta$$

$$\theta^2 F(y) = \theta^2 F(x)$$

Therefore:

$$\left.\begin{aligned}\Phi(y,\theta) = \; & A(x) + i\theta\sigma^a\bar\theta\,\partial_a A - \tfrac{1}{4}\theta^2\bar\theta^2\,\Box A \\ & + \sqrt2\,\theta\psi(x) - \tfrac{i}{\sqrt2}\theta\theta\;\partial_a\psi\,\sigma^a\bar\theta \\ & + \theta\theta F(x)\end{aligned}\right\}$$

$$\left.\begin{aligned}\Phi^+(y,\theta) = \; & A^*(x) - i\,\theta\sigma^a\bar\theta\,\partial_a A^* - \tfrac{1}{4}\theta^2\bar\theta^2\,\Box A^* \\ & + \sqrt2\,\bar\theta\bar\psi(x) + \tfrac{i}{\sqrt2}\bar\theta\bar\theta\;\theta\sigma^a\partial_a\bar\psi \\ & + \bar\theta\bar\theta F^*(x)\end{aligned}\right\}$$

We are interested for the time being in the terms $\theta^2\bar\theta^2$ in $\Phi_i^+\Phi_j$

$$\begin{aligned}\Phi_i^+\Phi_j\big|_{\theta^2\bar\theta^2} = \; & \theta^2\bar\theta^2\left(-\tfrac14 A_i^*\Box A_j - \tfrac14\Box A_i^* A_j + F_i^*F_j\right) \\ & + (\theta\sigma^a\bar\theta)(\theta\sigma^b\bar\theta)\,\partial_a\bar A_i\,\partial_b A_j \; + \\ & + i\bar\theta^2\,\theta\sigma^a\partial_a\bar\psi_i\,\theta\psi_j - i\theta^2\,\bar\theta\bar\psi_i\,\partial_a\psi_j\sigma^a\bar\theta \\ = \; & \theta^2\bar\theta^2\left(-\tfrac14\bar A_i\Box A_j - \tfrac14\Box\bar A_i A_j + F_i^*F_j + \tfrac12\partial_a\bar A_i\partial^a A_j\right)\end{aligned}$$

$$\begin{aligned}\theta\sigma^a\partial_a\bar\psi_i\,\theta\psi_j &= \theta^\alpha\sigma^a_{\alpha\dot\alpha}\,\partial_a\bar\psi_i^{\dot\alpha}\,\theta^\beta\psi_{j\beta} = \tfrac12\theta^2\epsilon^{\alpha\beta}\sigma^a_{\alpha\dot\alpha}\partial_a\bar\psi_i^{\dot\alpha}\psi_{j\beta} \\ &= -\tfrac12\theta^2\,\psi_j\sigma^a\partial_a\bar\psi_i\end{aligned}$$

$$\begin{aligned}-\bar\theta\bar\psi_i\,\partial_a\psi_j\sigma^a\bar\theta &= -\bar\psi_{i\dot\alpha}\bar\theta^{\dot\alpha}\partial_a\psi_j^\alpha(\sigma^a)_{\alpha\dot\alpha}\bar\theta^{\dot\alpha} = \tfrac12\bar\psi_{i\dot\alpha}\epsilon^{\dot\alpha\dot\beta}\partial_a\psi_j^\alpha\sigma^a_{\alpha\dot\alpha}\bar\theta^2 \\ &= \tfrac12\bar\theta^2\,\partial_a\psi_j\sigma^a\bar\psi_i\end{aligned}$$

$$\boxed{\begin{aligned}\Phi_i^+\Phi_j\big|_{\theta^2\bar\theta^2} = & -\tfrac14\bar A_i\Box A_j - \tfrac14\Box\bar A_i A_j + F_i^*F_j + \tfrac12\partial_a\bar A_i\partial^a A_j \\ & - \tfrac{i}{2}\psi_j\sigma^a\partial_a\bar\psi_i + \tfrac{i}{2}\partial_a\psi_j\sigma^a\bar\psi_i\end{aligned}}$$

For $i=j$ summing:

$$\left.\begin{aligned}\Phi_i^+\Phi_i\big| &= \partial_a\bar A_i\partial^a A_i + \bar F_iF_i + i\,\partial_a\psi_i\sigma^a\bar\psi_i - \tfrac{i}{2}\partial_a(\psi_i\sigma^a\bar\psi_i) \\ &= \partial_a\bar A_i\partial^a A_i + \bar F_iF_i - i\,\bar\psi_i\bar\sigma^a\partial_a\psi_i\end{aligned}\right\}$$

The general SUSY-invariant Lagrangian with scalars superfields is:

$$\mathcal{L} = K(\Phi_i, \bar{\Phi}_j)\big|_{\theta^2\bar{\theta}^2} + W(\Phi)\big|_{\theta^2} + \bar{W}(\bar{\Phi})\big|_{\bar{\theta}^2}$$

$\int d^2\theta\, \theta^2 = 1$ $\quad \int d^2\theta\, d^2\bar{\theta}\, \theta^2\bar{\theta}^2 = 1$, we can write instead:

$$\boxed{\mathcal{L} = \int d^4\theta\, K(\Phi, \bar{\Phi}) + \int d^2\theta\, W(\Phi) + \int d^2\bar{\theta}\, \bar{W}(\bar{\Phi})}$$

R-Symmetry.

It acts on SF's as follows:

$$\left.\begin{aligned} R\,\Phi(x,\theta) &= e^{-2in\alpha}\,\Phi(x, e^{+i\alpha}\theta) \\ R\,\bar{\Phi}^{+}(x,\theta) &= e^{+2in\alpha}\,\Phi(x, e^{-i\alpha}\bar{\theta}) \end{aligned}\right\}$$

In components:

$$\left.\begin{aligned} A &\to e^{-2in\alpha} A \\ \psi &\to e^{-2i(n-1/2)\alpha}\,\psi \\ F &\to e^{-2i(n-1)\alpha}\, F \end{aligned}\right\}.$$

Since $\theta \to e^{+i\alpha}\theta$ $\quad d^2\theta \to e^{-2i\alpha} d^2\theta$, $\Sigma n = 1$ in each interaction term to have invariance. Similarly K should be R-neutral. The vector multiplet is real and it has no natural R-symmetry.

Vector Multiplet.

It is represented by a real superfield, but ~~now~~ we need some gauge invariance to eliminate unwanted degrees of freedom. With Wess-Bagger conventions:

$$V(x,\theta,\bar{\theta}) = C + i\theta\chi - i\bar{\theta}\bar{\chi} + \frac{i}{2}\theta^2(M+iN) - \frac{i}{2}\bar{\theta}^2(M-iN)$$
$$- \theta\sigma^m\bar{\theta}\, v_m + i\theta^2\bar{\theta}\left(\bar{\lambda} + \frac{i}{2}\bar{\sigma}^m\partial_m\chi\right)$$
$$- i\bar{\theta}^2\theta\left(\lambda + \frac{i}{2}\sigma^m\partial_m\bar{\chi}\right) + \frac{1}{2}\theta^2\bar{\theta}^2\left(D - \frac{1}{2}\Box C\right)$$

The abelian gauge symmetry is $V \to V + \Phi + \Phi^{+}$, Φ CSF. Choose WZ-gauge $C = M = N = \chi = 0$

$$V = -\theta\sigma^m\bar{\theta}\, v_m + i\theta^2\bar{\theta}\bar{\lambda} - i\bar{\theta}^2\theta\lambda + \frac{1}{2}\theta^2\bar{\theta}^2 D, \text{ in this gauge}$$
$$V^2 = \frac{1}{2} v_m v^m\, \theta^2\bar{\theta}^2, \qquad V^3 = 0$$

The field strength is ~~defined as~~ :

$$W_\alpha = -\frac{1}{4}\bar{D}^2 D_\alpha V \qquad\qquad D\cdot W = \bar{D}\bar{W}$$
$$\bar{W}_{\dot\alpha} = -\frac{1}{4} D^2 \bar{D}_{\dot\alpha} V$$

W_α is gauge invariant, hence we can compute it in WZ gauge.
In the non-abelian theory:

$$W_\alpha = -\frac{1}{4}\bar{D}^2 e^{-V} D_\alpha e^{V} \qquad V = V_A T^A, \quad (T^A)^+ = T^A$$

with gauge transformation

$$e^V \to e^{-i\Lambda^+} e^V e^{i\Lambda} \qquad \Lambda \text{ chiral}, \quad \Lambda = \Lambda_A T^A$$

The gauge invariant kinetic term is:

$$\Phi^+ e^{2gV}\Phi$$

and for gauge fields:

$$\frac{1}{4g^2}\operatorname{Tr} WW\big|_{\theta^2} + \frac{1}{4g^2}\operatorname{Tr}\bar{W}\bar{W}\big|_{\bar\theta^2}$$

$$W_\alpha = -\frac{1}{4}\bar{D}^2\left(e^{-V}D_\alpha e^V\right) = -\frac{1}{4}\bar{D}^2\left(D_\alpha V - \frac{1}{2}[V, D_\alpha V]\right)$$

Next we want to work out the form of the component Lagrangian. It is convenient to change variables to y, θ.

$$v_m(y) = v_m(x^a + i\theta\sigma^a\bar\theta) =$$
$$= v_m(x^a) + i\theta\sigma^a\bar\theta\,\partial_a v_m + \dots$$

$$\theta\sigma^m\bar\theta\, v_m(x) = \theta\sigma^m\bar\theta\left(v_m(y) - i\theta\sigma^a\bar\theta\,\partial_a v_m\right) -$$
$$= \theta\sigma^m\bar\theta\, v_m(y) - \frac{1}{2}\theta^2\bar\theta^2\, i\,\partial_a v^a \qquad \Rightarrow$$

$$V = -\theta\sigma^m\bar\theta\, v_m(y) + i\theta^2\bar\theta\bar\lambda - i\bar\theta^2\theta\lambda + \frac{1}{2}\theta^2\bar\theta^2\left(D + i\partial_a v^a\right)$$

Furthermore

$$\left.\begin{array}{l}\{D_\alpha, \bar{D}_{\dot\alpha}\} = -2i\sigma^a{}_{\alpha\dot\alpha}\partial_a, \\ \bar{D}^2 D_\alpha = \bar{D}_{\dot\lambda}\bar{D}^{\dot\lambda} D_\alpha = \epsilon^{\dot\lambda\dot\mu}\bar{D}_{\dot\lambda}\bar{D}_{\dot\mu} D_\alpha = D_\alpha\bar{D}^2 + 4i\,\sigma^a{}_{\alpha\dot\mu}\partial_a\bar{D}^{\dot\mu}\end{array}\right\}$$

Working in the (y,θ)-basis,

$$y^a = x^a + i\theta\sigma^a\bar\theta$$

$$\bar D_{\dot\alpha} = -\frac{\partial}{\partial\bar\theta^{\dot\alpha}} \qquad D_\alpha = \frac{\partial}{\partial\theta^\alpha} + 2i\sigma^a{}_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\frac{\partial}{\partial y^a}$$

$$\bar D^2\bar\theta^2 = \bar D_{\dot\alpha}\bar D^{\dot\alpha}\bar\theta_{\dot\lambda}\bar\theta^{\dot\lambda} = \varepsilon^{\dot\alpha\dot\beta}\bar D_{\dot\alpha}\bar D_{\dot\beta}\,\varepsilon_{\dot\lambda\dot\mu}\bar\theta^{\dot\mu}\bar\theta^{\dot\lambda}$$

$$= \varepsilon^{\dot\alpha\dot\beta}\frac{\partial}{\partial\bar\theta^{\dot\alpha}}\frac{\partial}{\partial\bar\theta^{\dot\beta}}\varepsilon_{\dot\lambda\dot\mu}\bar\theta^{\dot\mu}\bar\theta^{\dot\lambda} = 2\varepsilon^{\dot\alpha\dot\beta}\frac{\partial}{\partial\bar\theta^{\dot\alpha}}\varepsilon_{\dot\lambda\dot\beta}\bar\theta^{\dot\lambda} =$$

$$= 2\varepsilon^{\dot\alpha\dot\beta}\varepsilon_{\dot\alpha\dot\beta} = -4$$

Thus

$$\bar D^2 V = \bar D^2\left(-i\bar\theta^2\theta\lambda + \frac{1}{2}\theta^2\bar\theta^2(D + i\partial\cdot v)\right) =$$

$$= 4i\,\theta\lambda(y) - 2\theta^2(D + i\partial\cdot v)$$

$$D_\alpha\bar D^2 V = \left(\frac{\partial}{\partial\theta^\alpha} + 2i\sigma^a_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\frac{\partial}{\partial y^a}\right)\left(4i\,\theta\lambda(y) - 2\theta^2(D + i\partial\cdot v)\right)$$

$$= 4i\lambda_\alpha(y) - 4\theta_\alpha(D + i\partial\cdot v)$$

Next,

$$4i\sigma^a_{\alpha\dot\mu}\partial_a\varepsilon^{\dot\mu\dot\lambda}\left(-\frac{\partial}{\partial\bar\theta^{\dot\lambda}}\right)\Big(-\theta\sigma^m\bar\theta v_m + i\theta^2\bar\theta\bar\lambda - i\bar\theta^2\theta\lambda$$
$$+\frac{1}{2}\theta^2\bar\theta^2(D + i\partial_a\cdot v^a)\Big)$$

look only for chiral terms

$$4i\sigma^a_{\alpha\dot\mu}\partial_a\varepsilon^{\dot\mu\dot\lambda}\left(-(\theta\sigma^m)_{\dot\lambda}v_m + i\theta^2\bar\lambda_{\dot\lambda}\right)$$

$$= -4\theta^2(\sigma^a\partial_a\bar\lambda)_\alpha + 4i(\sigma^a\bar\sigma^m\theta)_\alpha\partial_a v_m$$

$$W_\alpha = -\frac{1}{4}\bar D^2 D_\alpha V = -i\lambda_\alpha(y) + \theta_\alpha(D + i\partial\cdot v)$$
$$+\theta^2(\sigma^a\partial_a\bar\lambda)_\alpha - i(\sigma^a\bar\sigma^m\theta)_\alpha\partial_a v_m$$

$$\boxed{W_\alpha = -i\lambda_\alpha(y) + \theta_\alpha D - \frac{i}{2}(\sigma^a\bar\sigma^m\theta)_\alpha F_{am} + \theta^2(\sigma^a\partial_a\bar\lambda)_\alpha}$$

Next we compute $W^\alpha W_\alpha|_{\theta^2}$.

$$\gamma^a = \begin{pmatrix} 0 & \sigma^a \\ \bar\sigma^a & 0 \end{pmatrix}$$

$$\gamma^a\gamma^b\gamma^c\gamma^d = \begin{pmatrix} 0 & \sigma^a \\ \bar\sigma^a & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^b \\ \bar\sigma^b & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^c \\ \bar\sigma^c & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^d \\ \bar\sigma^d & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \sigma^a\bar\sigma^b\sigma^c\bar\sigma^d & 0 \\ 0 & \bar\sigma^a\sigma^b\bar\sigma^c\sigma^d \end{pmatrix}$$

$$\mathrm{Tr}\,\gamma^a\gamma^b\gamma^c\gamma^d = 4\,(\eta^{ab}\eta^{cd} - \eta^{ac}\eta^{bd} + \eta^{ad}\eta^{bc})$$

$$\mathrm{Tr}\,\gamma^5\gamma^a\gamma^b\gamma^c\gamma^d = -i\,\mathrm{Tr}\,\underbrace{\gamma^0\gamma^1\gamma^2\gamma^3}\gamma^a\gamma^b\gamma^c\gamma^d = 4i\varepsilon^{abcd}$$

$$\varepsilon^{abcd}\,\gamma^0\gamma^1\gamma^2\gamma^3$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma_x \\ -\sigma_x & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma_y \\ -\sigma_y & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma_z \\ -\sigma_z & 0 \end{pmatrix} = \begin{pmatrix} \sigma_x & 0 \\ 0 & -\sigma_x \end{pmatrix}\begin{pmatrix} -\sigma_y\sigma_z & \\ & -\sigma_y\sigma_z \end{pmatrix} = -i\begin{pmatrix} \cancel{\sigma_x}\,1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma_x \\ \sigma_x & \end{pmatrix} \qquad -i\,(\;) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\mathrm{Tr}\left(\frac{1-\gamma_5}{2}\right)\gamma^a\gamma^b\gamma^c\gamma^d$$

$$W^\alpha W_\alpha\Big| = \left(-i\lambda^\alpha + \theta^\alpha D - \frac{i}{2}(\sigma^a\bar\sigma^m\theta)^\alpha F_{am} + \theta^2(\sigma^a\partial_a\bar\lambda)^\alpha\right)$$
$$\cdot\left(-i\lambda_\alpha + \theta_\alpha D - \frac{i}{2}(\sigma^r\bar\sigma^s\theta)_\alpha F_{rs} + \theta^2(\sigma^b\partial_b\bar\lambda)_\alpha\right)$$

$$= \theta^2\left(-2i\,\lambda\sigma^a\partial_a\bar\lambda + D^2\right)$$
$$-\frac{1}{4}(\sigma^a\bar\sigma^m\theta)^\alpha(\sigma^r\bar\sigma^s\theta)_\alpha\, F_{am}F_{rs}$$

$$\varepsilon^{\alpha\beta}(\sigma^a\bar\sigma^m)_\beta{}^\lambda\,\theta_\lambda\,(\sigma^r\bar\sigma^s)_\alpha{}^\rho\,\theta_\rho\, F_{am}F_{rs} =$$
$$= \varepsilon^{\alpha\beta}(\sigma^a\bar\sigma^m)_\beta{}^\lambda\,(\sigma^r\bar\sigma^s)_\alpha{}^\rho\,\frac{1}{2}\varepsilon_{\lambda\rho}\,\theta^2 F_{am}F_{rs}$$
$$= \underbrace{\mathrm{Tr}\,(i\sigma_y)(\sigma^a\bar\sigma^m)(-i\sigma_y)(\sigma^r\bar\sigma^s)^T}\ \frac{1}{2}\theta^2 F_{am}F_{rs}$$

$$\mathrm{Tr}\ \sigma^a\bar\sigma^m\sigma_y\bar\sigma^{sT}\sigma^{rT}\sigma_y = \mathrm{Tr}\ \sigma^a\bar\sigma^m\sigma^s\bar\sigma^r$$

$$= \mathrm{Tr}\ \sigma^a\bar\sigma^m\sigma^s\bar\sigma^r\ \frac{1}{2}\theta^2 F_{am}F_{rs} =$$
$$= \left(\eta^{am}\eta^{sr} - \eta^{as}\eta^{mr} + \eta^{ar}\eta^{ms} \pm i\,\varepsilon^{amsr}\right)\theta^2 F_{am}F_{rs}$$
$$\theta^2\left(2F_{am}F^{am} \pm i\,\varepsilon^{amsr}F_{am}F_{rs}\right)$$

$$\boxed{W^\alpha W_\alpha\Big| = -2i\,\lambda\sigma^a\partial_a\bar\lambda + D^2 - \frac{1}{2}F^2 \pm \frac{i}{4}\varepsilon^{abcd}F_{ab}F_{cd}}$$

In the non-abelian case we have a few more terms needed in the computation

$$W_\alpha = -\frac{1}{4}\bar D^2 e^{-V} D_\alpha e^V = -\frac{1}{4}\bar D^2\left(D_\alpha V - \frac{1}{2}[V, D_\alpha V]\right)$$

It's effect is simply to covariantize derivatives

$$V = V^a T^a \qquad (T^a)^+ = T^a \qquad \mathrm{Tr}\,T^aT^b = \delta^{ab}$$

$$[V, D_\alpha V]^a = i f^{abc} V^b D_\alpha V^c$$

$$\boxed{\begin{aligned} W_\alpha &= T^a\left(-i\lambda^a_\alpha + \theta_\alpha D^a - \frac{i}{2}(\sigma^n\bar\sigma^m\theta)_\alpha F^a_{nm} + \theta^2\sigma^m D_m\bar\lambda^a\right)\\ F^a_{nm} &= \partial_n A^a_m - \partial_m A^a_n - \frac{1}{2}f^{abc}A^b_n A^c_m\\ D_m\bar\lambda^a &= \partial_m\bar\lambda^a - \frac{1}{2}f^{abc}v^b_m\bar\lambda^c \end{aligned}}$$

Finally we have the change in the matter kinetic term

$$\bar{\Phi}^+ e^V \Phi = \Phi^+\Phi + \Phi^+ V \Phi + \frac{1}{2}\Phi^+ V^2 \Phi$$

Once again this covariantizes derivatives, and induces a coupling between the gaugino and the "quark" fields.

$$D_m A = \left(\partial_m A + i\frac{1}{2} T^a v_m^a A\right)$$

$$D_m \psi = \left(\partial_m \psi + \frac{i}{2} T^a v_m^a \psi\right)$$

and we also have the terms:

$$F_i^* F_i + \frac{1}{2} D^c \bar{A} T^a A + \frac{i}{\sqrt{2}} \bar{A} T^a \lambda^a \psi - \frac{i}{\sqrt{2}} \bar{\psi} T^a A \bar{\lambda}^a$$

The gauge coupling is not standard, we rescale:

$$\begin{pmatrix} v \\ \lambda \\ D \end{pmatrix}^a \longrightarrow 2g \begin{pmatrix} v \\ \lambda \\ D \end{pmatrix}$$

$$D_m(A,\psi) \rightarrow (\partial_m + igT^a v_m^a)(A,\psi)$$

$$\frac{1}{2} D^a \bar{A} T^a A \rightarrow g D^a \bar{A} T^a A$$

$$\frac{i}{\sqrt{2}}\left(\bar{A} T^a \lambda^a \psi - \bar{\psi} T^a A \bar{\lambda}^a\right) \rightarrow ig\sqrt{2}\left(\bar{A} T^a \lambda^a \psi - \bar{\psi} T^a A \bar{\lambda}^a\right)$$

For the gauge part

$$W^\alpha W_\alpha| = -\frac{1}{2} F^a F^a - 2i\lambda^a \sigma^m D_m \bar{\lambda}^a + (D^a)^2 + \frac{i}{4}\varepsilon^{abcd} F_{ab} F_{cd}$$

$$\downarrow$$

$$\cancel{(4g)}\, 4g^2\left(-\frac{1}{2} F^{a2} - 2i\lambda^a \sigma^m D_m \bar{\lambda}^a + D^{a2} + \frac{i}{4}\varepsilon^{\alpha\beta\gamma\delta} F^a_{\alpha\beta} F^a_{\gamma\delta}\right)$$

To normalise properly:

$$W^\alpha W_\alpha\Big|_{\theta^2} = \bar{W}_{\dot\alpha} \bar{W}^{\dot\alpha}\Big|_{\bar\theta^2}$$

$$\cancel{\frac{1}{8g^2}} \quad \frac{1}{16g^2}\left(W^\alpha W_\alpha| + \bar{W}\bar{W}|\right) \qquad \text{is the gauge kinetic term.}$$

$$W_\alpha = -\frac{1}{4}\bar{D}^2 e^{-V} D_\alpha e^V$$

The most general $N=1$ coupling is:

$$\boxed{\begin{aligned}\mathcal{L} = & -\frac{1}{4} F^a_{mn} F^{a\,mn} - i \lambda^a \sigma^m D_m \bar\lambda^a \oplus \frac{i}{8} \varepsilon^{mnrs} F_{mn} F_{rs} \\ & + |\partial_m A + i g T^a v_m^a A|^2 - i \bar\psi \bar\sigma^m (\partial_m \psi + i g T^a v_m^a \psi) \\ & + \frac{1}{2} D^a D^a + g D^a \bar A T^a A + i g \sqrt{2}\, \bar A T^a \lambda^a \psi - i g \sqrt{2}\, \bar\psi T^a A \bar\lambda^a \\ & + F_i^* F_i + \frac{\partial W}{\partial A_i} F_i + \frac{\partial \bar W}{\partial \bar A_i} F_i^* - \frac{1}{2} \frac{\partial^2 W}{\partial A_i \partial A_j} \psi_i \psi_j - \frac{1}{2} \frac{\partial^2 \bar W}{\partial \bar A_i \partial \bar A_j} \bar\psi_i \bar\psi_j\end{aligned}}$$

To ~~make~~ include a θ-parameter, note that for a complex coupling constant

$$z = \theta + i\eta$$

$$(\theta + i\eta)\left(\tfrac{1}{2}\mathrm{Re}\, W^2| + \tfrac{1}{2} i\, \mathrm{Im}\, W^2|\right)$$

$$= i\left(\eta \tfrac{1}{2}\mathrm{Re}\, W^2| + \theta \tfrac{1}{2}\mathrm{Im}\, W^2\right) + \cdots$$

$$\frac{1}{8g^2} W^\alpha W_\alpha\Big|_{\theta^2} = -\frac{1}{4} F^2 - i \lambda^a \sigma^m D_m \bar\lambda^a + \frac{1}{2}(D^a)^2 + \frac{i}{8} \varepsilon^{mnrs} F_{mn} F_{rs}$$

$$z = \theta + \frac{i}{4g^2} \quad \Rightarrow \quad \mathrm{Im}\, z\, W^\alpha W_\alpha\Big|_{\theta^2}$$ contain the usual coupling together with the θ-term.

The scalar potential is obtained after eliminating the auxiliary fields.

$$F_i = -\frac{\partial \bar W}{\partial \bar A_i} \qquad D^a + g \bar A T^a A = 0$$

$$V = \sum_i \left|\frac{\partial W}{\partial A_i}\right|^2 + \frac{1}{2} g^2 (\bar A T^a A)^2$$

In $N=2$ SUSY gauge theories in $d=4$ the central charge is generated by the electric and magnetic charges

We can write the normalization of the gauge kinetic term more carefully. It is sometimes convenient to write the matter kinetic term as:

$$\frac{1}{g^2} \bar\Phi^+ e^{2V} \Phi \Big|_{\theta^2 \bar\theta^2}$$

Choose

$$\mathrm{Tr}\, T^a T^b = \kappa_2\, \delta^{ab}$$

$$W_\alpha = -\frac{1}{4} \bar{D}^2 e^{-2V} D_\alpha e^{2V} = W_\alpha^a T^a$$

Then

$$\frac{1}{\kappa_2} \mathrm{Tr}\, W^\alpha W_\alpha \Big|_{\theta^2} = 4 \left(-\frac{1}{2} (F^a)^2 - 2i\, \lambda \cdot \sigma \cdot D \bar{\lambda} + D^2 + \frac{i}{2} F^a \cdot \tilde{F}^a \right)$$

Now consider

$$\frac{1}{8\kappa_2} \mathrm{Im}\; \tau\; \mathrm{Tr}\, W^\alpha W_\alpha \Big|_{\theta^2} =$$

$$= \mathrm{Im}\,(A + iB) \left(-\frac{1}{4} F^2 - i\, \lambda \sigma \cdot D \bar{\lambda} + \frac{1}{2} D^2 + \frac{i}{4} F\tilde{F} \right) =$$

$$= B \left(-\frac{1}{4} F^2 - i\, \lambda \sigma \cdot D \bar{\lambda} + \frac{1}{2} D^2 \right) + \frac{i}{4} A F\tilde{F}$$

Take $B = 1/g^2$

$$\boxed{\frac{1}{8\kappa_2} \frac{1}{4\pi} \mathrm{Im} \left(\frac{\theta^{QCD}}{2\pi} + \frac{4\pi i}{g^2} \right) W^\alpha W_\alpha \Big|_{\theta^2}}$$

$$\boxed{\tau_{cl} \equiv \frac{\theta}{2\pi} + i \frac{4\pi}{g^2}}$$

Seiberg-Witten-(I).

We analyze the $N=2$ pure $SU(2)$ theory following Seiberg and Witten. $N=2$ has powerful Ward identities which together with some physical input lead to interesting conclusions like monopole condensation and after $N=2$ is softly broken to $N=1$, to confinement due to condensation of the massless monopoles. The chain of reasoning is long and elaborate, but along the way we will also try to generalize wherever possible to groups of higher rank. For the theories under consideration the ~~Bogom~~ BPS bound is

$$\left.\begin{aligned} &M \geq \sqrt{2}\,|Z| \\ &Z = v\left(n_e + \frac{i}{\alpha}\, n_m\right) \qquad \alpha = \frac{g^2}{4\pi} \end{aligned}\right\}$$

Z is symmetric under $n_e \leftrightarrow n_m$ $\alpha \leftrightarrow 1/\alpha$ $v \leftrightarrow v/\alpha$. This is some evidence for Monton-Olive, however later we will see that Z gets an interesting renormalization. For $N=4$ the conjecture seems to ~~not~~ hold. For $N=2$, there is an action of $SL(2,\mathbb{Z})$ on $\tau = \theta/2\pi + i4\pi/g^2$ which embodies the ~~weaker~~ weak form of Montonen-Olive for $N=2$.

From previous chapters we are familiar with the irreps of massless $N=2$ SUSY. The first one we consider is the vector multiplet A_μ, two Weyl λ, ψ and a scalar ϕ all in the adjoint rep of G:

$$\begin{pmatrix} & A_\mu & \\ \lambda & & \psi \\ & \phi & \end{pmatrix}$$

$SU(2)_R$ acts by rotating (λ, ψ). In an $N=1$ formalism we have a vector superfield W_α (A, λ) and a chiral superfield $\Phi(\varphi, \psi)$. Of $SU(2)_R$ only $U(1)_J$ is manifest. We also have ~~an~~ R-symmetry

$$U(1)_R : \Phi \to e^{2i\alpha}\, \Phi(e^{-i\alpha}\theta)$$

$$U(1)_J : \Phi \to \Phi(e^{-i\alpha}\theta) \qquad (\lambda \text{ also transforms})$$

Since we will not study yet coupling to hypermultiplets we do not spell out their local form and R-properties, just to say that hypermultiplets from $N=1$ point of view come in pair of chiral superfields in conjugate representations $Q, \tilde{Q}$, and the

The allowed superpotential is:

$$W = \sqrt{2}\,\tilde{Q}\,\Phi\, Q$$

Classically we have the global $SU(2)_R \times U(1)_R$ symmetry, however $U(1)_R$ is broken to a discrete subgroup due to anomalies.

$$U(1)_R \qquad \begin{aligned} Q &\to Q(e^{-i\alpha}\theta) \\ \tilde{Q} &\to \tilde{Q}(e^{-i\alpha}\theta) \\ \Phi &\to e^{2i\alpha}\,\Phi(e^{-i\alpha}\theta) \end{aligned}$$

We also need to make the gluino field transform. This follows from the 3 trilinear couplings.

$$\psi_q \to e^{-i\alpha}\psi_q \qquad \psi_{\tilde{q}} \to e^{-i\alpha}\psi_{\tilde{q}} \qquad \psi \to e^{i\alpha}\psi$$

We have the couplings $\bar{q}\,T^a \psi \lambda^a$ etc. $\Rightarrow$ $\lambda \to e^{i\alpha}\lambda$. To compute the anomaly for $SU(N_c)$, note that by the index theorem in the presence of an instanton there is 1 zero mode for left moving fermion in the fundamental or antifundamental, $2N_c$ in the adjoint for left-handed fermion. Hence the first non-vanishing correlator is:

$$G = \langle \lambda(x_1)\ldots\lambda(x_{2N_c})\,\psi(y_1)\ldots\psi(y_{2N_c})\,\psi_q(z_1)\ldots\psi_q(z_{N_f})\,\psi_{\tilde{q}}(w_1)\ldots\psi_{\tilde{q}}(w_{N_f})\rangle$$

N_f is # of flavors. Then under the R-symmetry G transforms as

$$G \to e^{i\alpha(4N_c - 2N_f)}\, G$$

Hence $U(1)_R \to \mathbb{Z}_{4N_c - 2N_f}$. If we have no flavors then $U(1)_R \to \mathbb{Z}_{4N_c}$. The global symmetry group is $SU(2)_R \times \mathbb{Z}_{4N_c}$. The center of $SU(2)_R$ is contained in $\mathbb{Z}_{4N_c}$, hence the true group is:

$$SU(2)_R \times \mathbb{Z}_{4N_c} / \mathbb{Z}_2$$

The field φ ~~is~~ in the $N=2$ multiplet has couplings of the form: $\bar{\varphi}\,T^a \psi \lambda^a$, since φ has charge 2 with respect to $\mathbb{Z}_{4N_c}$, once φ acquires a VEV $\mathbb{Z}_{4N_c}$ will be broken. For pure $N=2$ the classical potential is just.

$$\frac{1}{g^2}\,\mathrm{Tr}\,[\Phi, \Phi^\dagger]^2$$

Hence ~~generally~~ $[\Phi, \Phi^\dagger] = 0$ means that Φ can be chosen in the Cartan subalgebra $\Phi \sim a_i H^i$, and generically $a_i \neq a_j$, $i \neq j$.

There is ~~however~~ still some residual gauge invariance given by the Weyl group, and the correct description of the vacuum is in terms of Weyl invariants. These we can obtain by ~~solving~~ looking at the characteristic equation:

$$\det(\lambda - \phi) = 0$$

The roots are the quantities $a_1, \dots, a_r$ (r = rank of G), and the coefficients are Weyl invariant. The discriminant of this polynomial indicates the areas where two eigenvalues coincide and therefore the unbroken group jumps from $U(1)^r$ to something at least as big as $U(1)^{r-1} \times SU(2)$. The gauge invariant description of the moduli space is therefore in terms of the coefficients of the characteristic polynomials.

$$\phi \sim \begin{pmatrix} a_1 & & & \\ & a_2 & & \\ & & \ddots & \\ & & & a_N \end{pmatrix} \quad \text{for } SU(N) \quad , \quad \sum a_i = 0 .$$

$$\lambda^N + \lambda^{N-2} \sum_{i<j} a_i a_j - \lambda^{N-3} \sum_{i<j<k} a_i a_j a_k + \cdots + (-1)^N \prod a_i = 0$$

$SU(2)$ is simple $\quad \phi \sim \frac{1}{2} a \sigma_3 \qquad \mathrm{Tr}\,\phi^2 = \frac{1}{2} a^2$

For $SU(3)$ we can write

$$\phi \sim \begin{pmatrix} a_1 & & \\ & a_2 & \\ & & a_3 \end{pmatrix} \qquad a_3 = -(a_1 + a_2)$$

thus

$$(\lambda - a_1)(\lambda - a_2)(\lambda - a_3) = \lambda^3 - \lambda^2 \sum a_i + \lambda\left(a_1 a_2 + a_1 a_3 + a_2 a_3\right) - a_1 a_2 a_3 = 0$$

$$\left.\begin{aligned} \mathrm{Tr}\,\phi^2 &= a_1^2 + a_2^2 + a_3^2 \\ \mathrm{Tr}\,\phi^3 &= a_1^3 + a_1^3 + a_3^3 \end{aligned}\right\} \qquad a_3 = -(a_1 + a_2) \quad \text{we have}$$

$$a_1 a_2 + a_1 a_3 + a_2 a_3 = a_1 a_2 - (a_1 + a_2)^2 = -\left(a_1^2 + a_2^2 + a_1 a_2\right) = -\frac{1}{2}\mathrm{Tr}\,\phi^2$$

$$a_1 a_2 a_3 = -a_1 a_2 (a_1 + a_2) = \frac{1}{3}\mathrm{Tr}\,\phi^3$$

$$\left(\sum a_i\right)^2 = 0 = a_1^2 + a_2^2 + a_3^2 + 2\sum_{i<j} a_i a_j \quad ; \quad (\mathrm{Tr}\,\phi)^2 = \mathrm{Tr}\,\phi^2 + 2\sum_{i<j} a_i a_j ,$$

$$\sum_{i<j} a_i a_j = -\frac{1}{2}\mathrm{Tr}\,\phi^2$$

$$\mathrm{Tr}\,\phi^3 = -3a_1^2 a_2 - 3a_2^2 a_1 = -3a_1 a_2 (a_1 + a_2) = 3 a_1 a_2 a_3$$

$$\boxed{\lambda^3 - \lambda \frac{\mathrm{Tr}\,\phi^2}{2} - \frac{1}{3}\mathrm{Tr}\,\phi^3 = 0} \qquad \boxed{\begin{aligned} u &= \mathrm{Tr}\,\phi^2/2 = a_1^2 + a_2^2 + a_1 a_2 \\ v &= -\mathrm{Tr}\,\phi^3/3 = a_1 a_2 (a_1 + a_2) \end{aligned}}$$

For $SU(N)$ in general similar formulae can be worked out.
The coefficients of the charact. polynomial are the "chern" classes of ϕ, the symmetric polynomials

$$\det(\lambda-\phi) = \lambda^N - \lambda^{N-1} c_1(\phi) + \lambda^{N-2} c_2(\phi) \dots (-1)^j \lambda^{N-j} c_j(\phi) \dots = 0$$

$$c_j(\phi) = \sum_{n_1<\dots<n_j} a_{n_1} a_{n_2} \dots a_{n_j}$$

$$\det(\lambda-\phi) = \prod(\lambda-a_i) = \lambda^N \prod(1-a_i/\lambda) =$$

$$= \lambda^N \exp \sum_i \ln(1-a_i/\lambda) =$$

$$= \lambda^N \exp -\sum_i \sum_n \frac{a_i^n}{n\lambda^n} = \lambda^N \exp\left(-\sum_{n\geq 1} \frac{\mathrm{Tr}\,\phi^n}{n\lambda^n}\right)$$

This provides all polynomial relations between different invariants.

$SU(3)$ $\quad \lambda^3 \exp\left(-\frac{\mathrm{Tr}\,\phi^2}{2\lambda^2} - \frac{1}{3\lambda^3}\mathrm{Tr}\,\phi^3 - \dots\right)$

Under the unbroken discrete symmetry $\mathbb{Z}_{4N}$, recall that $\phi \to \omega^2\phi$ $\quad\omega$ is a $4N$-th root of 1.

$$u_2 = \frac{1}{2}\mathrm{Tr}\,\phi^2 \to \omega^4 u_2$$

$$u_3 = \frac{1}{3}\mathrm{Tr}\,\phi^3 \to \omega^6 u_3 \quad \text{etc.}$$

For $SU(2)$ $\quad u = +\frac{1}{2}a^2 \quad u$ has charge 4 hence there is ~~still~~ $\mathbb{Z}_8 \to \mathbb{Z}_4$. ~~$\mathbb{Z}_2$ bro~~ The broken $\mathbb{Z}_2$ acts by $u \to -u$.

For $SU(3)$ $\quad u \sim \frac{1}{2}\mathrm{Tr}\,\phi^2 \quad v \sim \mathrm{Tr}\,\phi^3$, have resp. charges $(4,6)$. The residual discrete symmetry is $\mathbb{Z}_{12}$ $\quad u \to \omega^4 u$, $v \to \omega^6 v$. The unbroken symmetry must satisfy $\omega^4 = \omega^6 = 1$.

$$\exp 2\pi i \frac{n4}{12} = \exp 2\pi i \frac{n6}{12} = 1 \Rightarrow \exp 2\pi i \frac{n}{3} = \exp 2\pi i \frac{n}{2} = 1 \qquad n \in \mathbb{Z}/\mathbb{Z}_{12}.$$

Since $n \equiv 0 \pmod 3$ $\quad n \equiv 0 \pmod 2 \to n \equiv 0 \pmod 6$ hence $\mathbb{Z}_{12} \to \mathbb{Z}_2$.
~~the~~ Only $\mathbb{Z}_2$ survives and ~~$\mathbb{Z}$~~ $\mathbb{Z}_6$ should act in the (u,v) moduli surface. For $SU(4)$ for instance we have $\mathbb{Z}_{16}$, and the invariants are $\mathrm{Tr}\,\phi^2, \mathrm{Tr}\,\phi^3, \mathrm{Tr}\,\phi^4$ with charges $4, 6, 8$; hence nothing in $\mathbb{Z}_{16}$ leaves them invariant. For $SU(N)$ we have $\mathrm{Tr}\,\phi^2, \mathrm{Tr}\,\phi^3, \dots, \mathrm{Tr}\,\phi^N$ with charges $4, 6, 8, \dots, 2N$ and $U(1)_R \to \mathbb{Z}_{4N}$.
Generically $\mathbb{Z}_{4N} \to 1$, no discrete axial symmetry survives.

For $SU(2)$ $u = \frac{1}{2}a^2 = \mathrm{Tr}\,\phi^2$ $\phi = \frac{1}{2} a \sigma^3$, u has charge 4, $\mathbb{Z}_4$ is unbroken, and $\mathbb{Z}_8/\mathbb{Z}_4 \simeq \mathbb{Z}_2$ acts on the u-moduli space; $u \to -u$.

Low Energy Action

Far from the points where two eigenvalues of $\langle\phi\rangle$ coincide the only massless fields are the vector supermultiplets associated to the unbroken subgroup $U(1)^r$. $N=2$ is not broken and the effective action (the part with at most two derivatives) is given by a prepotential depending only on r-vector supermultiplets. For $SU(2)$ and in $N=1$ superspace language, the LEEA is:

$$\frac{1}{4\pi}\,\mathrm{Im}\left[\int d^4\theta\, \frac{\partial \mathcal{F}(A)}{\partial A}\bar{A} + \int d^2\theta\, \frac{1}{2}\frac{\partial^2\mathcal{F}}{\partial A^2} W_\alpha W^\alpha\right]$$

More generally:

$$\frac{1}{4\pi}\,\mathrm{Im}\left[\int d^4\theta\, \frac{\partial \mathcal{F}}{\partial A^i}\bar{A}^i + \int d^2\theta\, \frac{1}{2}\frac{\partial^2\mathcal{F}}{\partial A_i \partial A_j} W_\alpha^i W^{j\alpha}\right]$$

1) As $|a| \to \infty$ AF takes over and the theory is weakly coupled. The vacuum degeneracy is not lifted. The Kähler potential is determined by the function $\mathcal{F}(A)$.

$$K = \mathrm{Im}\,\frac{\partial \mathcal{F}(A)}{\partial A_i}\bar{A}_i$$

and the metric becomes

$$ds^2 = \mathrm{Im}\,\frac{\partial^2\mathcal{F}}{\partial A_i \partial A_j}\, dA^i d\bar{A}^j$$

In the classical theory $\mathcal{F}(A) = \frac{1}{2}\tau_{cl} A^2$; $\tau_{cl} = \frac{\theta}{2\pi} + i\frac{4\pi}{g^2}$. Replacing θ, g by running coupling for large $|a|$ we obtain a good approximation to the quantum behavior of $\mathcal{F}(a)$. For fixed a, we can always absorb θ in phase redefinition of the fermion fields.

2). We should be allowed to use any coordinates for ds^2.

3). $\tau(a) = \frac{\partial^2\mathcal{F}}{\partial a^2}$

4). For the full $SU(2)$ theory, $\mathcal{F}(\sqrt{A\cdot A}) = \mathcal{H}(A\cdot A)$.

$$S = \frac{1}{2\pi}\,\mathrm{Im}\left[\int d^4\theta\, \mathcal{H}' A^a (e^V)_{ab} A^b + \int d^2\theta\, \frac{1}{2}(\mathcal{H}'\delta_{ab} + 2\mathcal{H}'' A_a A_b)\cdot W^a_\alpha W_b^{\alpha}\right]$$

The 1-loop quantum corrections determining $\mathcal{F}(A)$ can be computed without much difficulty. For SU(2):

$$\mathcal{F} = \frac{i}{2\pi} A^2 \ln \frac{A^2}{\Lambda^2}$$

For other groups we note that for every root α, $\alpha > 0$ we have fields $W^{\alpha}, W^{-\alpha}$. For simply laced groups $\alpha^2 = 2$. Then:

$$\mathcal{F} = \frac{i}{4\pi} \sum_{\alpha>0} \frac{1}{2} (\vec{\alpha}\cdot\vec{a})^2 \ln \frac{(\vec{\alpha}\cdot\vec{a})^2}{2\Lambda^2}$$

$\alpha^2/2 = 1$ for SU(2). The coefficient of the logarithm follows from the 1-loop β-function and it also leads to the ~~con~~ unbroken global $U(1)_R$-symmetry. We write

$$\vec{a} = a_i \vec{\lambda}^{(i)}, \qquad 2\vec{\lambda}^{(i)}\alpha_{(j)} / \alpha_{(j)}^2 = \delta_{ij},$$

for SU(2) $\quad \vec{\alpha}\cdot\vec{a} = a$

The form of $\mathcal{F}$ is determined by looking at the renormalization of the gauge coupling constant. If the group is not simply laced, presumably the correct normalization is $(2\vec{\alpha}\cdot\vec{a}/\alpha^2)$ instead of $\vec{\alpha}\cdot\vec{a}$ everywhere in $\mathcal{F}(a)$ (check this). There are in principle instanton corrections to $\mathcal{F}$. ~~These corrections are one of the goals of~~ The solution of Seiberg Witten determines them in a rather precise way.

$$SU(2): \qquad \vec{a} = a\begin{pmatrix}\theta \\ 2\end{pmatrix} \qquad \vec{a}\cdot\vec{\alpha} = a$$

$$SU(3). \qquad \vec{a} = a_1\lambda^{(1)} + a_2\lambda^{(2)} \qquad \left\{ \begin{array}{ll} \lambda^{(1)}\cdot\alpha_{(1)} = 1 & \lambda^{(1)}\cdot\alpha_{(2)} = 0 \\ \lambda^{(2)}\alpha_{(2)} = 1 & \lambda^{(2)}\cdot\alpha_{(1)} = 0 \end{array} \right.$$

Duality.

As a consequence of $N=2$ SUSY the metric on moduli space is

$$ds^2 = \mathrm{Im}\,\tau(a)\, da\, d\bar{a} \qquad \tau = \partial^2\mathcal{F}/\partial a^2$$

$$\mathcal{F}_{1\,loop} = \frac{i}{2\pi} a^2 \ln\frac{a^2}{\Lambda^2} \qquad \mathcal{F}' = \frac{i}{2\pi}\left(2a\ln\frac{a^2}{\Lambda^2} + 2a\right) = \frac{i}{\pi}\left(a\ln\frac{a^2}{\Lambda^2} + a\right)$$

$$\mathcal{F}'' = \frac{i}{\pi}\left(\ln\frac{a^2}{\Lambda^2} + 3\right)$$ Hence $\tau(a)$ is a multivalued function whose $\mathrm{Im}\,\tau(a)$ is single valued and positive. However if $\mathrm{Im}\,\tau(a)$ is globally defined it cannot be positive definite because a harmonic f'n has no minimum. ds^2 is only valid locally. Define:

$$\boxed{a_D \equiv \frac{\partial\mathcal{F}}{\partial a}}$$

The metric is.

$$ds^2 = \operatorname{Im} da_D d\bar{a} = -\frac{i}{2}\left(da_D d\bar{a} - da\, d\bar{a}_D\right)$$

(a_D, a) are local parameters. Introduce an arbitrary local holo. coordinate u in a complex manifold $\mathcal{M}$. $X \cong \mathbb{C}^2 (a_D, a)$. On X ~~has the~~ choose the symplectic form $\omega = \operatorname{Im} da_D \wedge d\bar{a}$. The functions $(a_D(u), a(u))$ give a map $f: \mathcal{M} \to X$.

$$ds^2 = \operatorname{Im} \frac{da_D}{du}\frac{d\bar{a}}{d\bar{u}} du d\bar{u} = -\frac{i}{2}\left(\frac{da_D}{du}\frac{d\bar{a}}{d\bar{u}} - \frac{d\bar{a}_D}{d\bar{u}}\frac{da}{du}\right) du d\bar{u}$$

For arbitrary (a_D, a) the metric is not positive. Choosing $u = a$ we get back to the original formula. Write $a^\alpha = (a_D, a)$ $\alpha = 1, 2$.

$$ds^2 = -\frac{i}{2}\epsilon_{\alpha\beta}\frac{da^\alpha}{du}\frac{d\bar{a}^\beta}{d\bar{u}} du d\bar{u}$$

$v = \begin{pmatrix} a_D \\ a \end{pmatrix}$ $\quad v \to Mv + c, \quad M \in SL(2,\mathbb{R}), \quad c \in \mathbb{R}^2$ leaves ds^2 invariant. This is duality in moduli space.

W/o matter fields $c = 0$. In fact only $SL(2,\mathbb{Z})$ survives. $SL(2,\mathbb{R})$ is generated by $S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $S^2 = -1$. $T_b = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$. However only integer b is allowed, S, T generate $SL(2,\mathbb{Z})$.

Higher Dimensions. If the group has rank r:

$$ds^2 = \operatorname{Im} \frac{\partial^2 \mathcal{F}}{\partial a_i \partial a_j} da_i d\bar{a}_j$$

Introduce $X \cong \mathbb{C}^{2r}$ with coordinates a^i, $a_{Dj} = \partial\mathcal{F}/\partial a^j$. Endow X with the symplectic form $\omega = \frac{i}{2}\sum_i (da^i \wedge d\bar{a}_{D,i} - da_{D,i} \wedge d\bar{a}^i)$ and the holomorphic 2-form:

$$\omega_h = \sum_i da^i \wedge da_{D,i}$$

u^s $s = 1, \ldots, r$ coordinates on $\mathcal{M}$ $\quad f: \mathcal{M} \to X$ $\{a^i(u), a_{Dj}(u)\}$, such that $f^*(\omega_h) = 0$; this ensures that $a_{Dj} = \partial\mathcal{F}/\partial a_j$, then

$$ds^2 = \operatorname{Im} \frac{\partial a_{Di}}{\partial u^r}\frac{\partial \bar{a}^i}{\partial \bar{u}^r} du^r d\bar{u}^s$$

$v = \begin{pmatrix} a_D \\ a \end{pmatrix}$ $\quad v \to Mv + c \quad M \in Sp(2r, \cancel{\mathbb{Z}}\mathbb{R})$ which again reduces to $Sp(2r, \mathbb{Z})$ w/o matter fields. The allowed monodromy then lives in $Sp(2r, \mathbb{Z})$ whose generators can be written in terms of Dehn twists for a Riemann surface of genus $\cancel{g}$ r.

The transformations S, T can be interpreted in terms of duality.

Interpretation in terms of duality:

In Minkowski space $F_{\mu\nu}^2 = -(^*F)^2_{\mu\nu}$, $^*(^*F) = -F$. The relevant terms are.

$$\frac{1}{32\pi} \mathrm{Im} \int \tau(a) \,(F + i\,^*F)^2 = \frac{1}{16\pi} \mathrm{Im} \int \tau(a)\,(F^2 + i\,^*FF)$$

$dF=0$ implemented by adding a Lagrange multiplier vector V_D. $U(1) \subset SU(2)$ is normalized so that all $SO(3)$ fields have $\mathbb{Z}$-valued charges. A magnetic pole corresponds to $\varepsilon^{0\mu\nu\rho}\partial_\mu F_{\nu\rho} = 8\pi\,\delta^{(3)}(\vec{x})$

$$\frac{1}{8\pi} \int V_{D\mu}\, \varepsilon^{\mu\nu\rho\sigma} \partial_\nu F_{\rho\sigma} = \frac{1}{8\pi} \int {}^*F_D F =$$

$$= \frac{1}{16\pi} \mathrm{Re} \int (^*F_D - i F_D)(F + i\,^*F)$$

integrate over F :

$$\cancel{\frac{1}{8\pi}}\ \frac{1}{32\pi} \mathrm{Im}\left(-\frac{1}{\tau}\right)(F_D + i\,^*F_D)^2 = \frac{1}{16\pi} \mathrm{Im}\left(-\frac{1}{\tau}\right)\left(F_D^2 + i\,^*F_D F_D\right)$$

Similarly in $N=1$ superspace

$$\frac{1}{8\pi} \mathrm{Im} \int d^2\theta\ \tau(A)\, W^2 .$$

$dF=0$ is replaced by $\mathrm{Im}\, \mathcal{D}W = 0$, which is implemented by a vector superfield V_D:

$$\frac{1}{4\pi} \mathrm{Im} \int d^4x\, d^4\theta\ V_D \mathcal{D}W = \frac{1}{4\pi} \mathrm{Re} \int d^4x\, d^4\theta\ ,\mathcal{D}V_D\, W = -\frac{1}{4\pi} \mathrm{Im} \int d^4x\, d^2\theta\, W_D W$$

$$\Downarrow$$

$$\frac{1}{8\pi} \mathrm{Im} \int d^2\theta \left(-\frac{1}{\tau(A)} W_D^2\right)$$

We also need to transform the $N=1$ chiral multiplet A to A_D. The kinetic term is:

$$\mathrm{Im} \int d^4\theta\ h(A)\,\bar{A} \qquad\qquad h(A) = \partial\mathcal{F}/\partial A.$$

$$= \mathrm{Im} \int d^4\theta\ A_D(A)\,\bar{A}$$

Similarly for arbitrary $SL(2,\mathbb{Z})$ transformations. $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is a symmetry but $S \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ is a different description of the same theory. We present a general argument which will be useful later in the computation of monodromies.

& Consider an $SL(2,\mathbb{Z})$ transformation.

$$\begin{pmatrix} a_D \\ a \end{pmatrix} \to \begin{pmatrix} q_m & q_e \\ r & s \end{pmatrix} \begin{pmatrix} a_D \\ a \end{pmatrix} \qquad \begin{matrix} q_m s - q_e r = 1 \\ (q_m, q_e) = 1 . \end{matrix}$$

$$\left. \begin{aligned} a_D(q) &= q_m a_D + q_e a \\ a(q) &= r a_D + s a \end{aligned} \right\}$$

Duality invariance for the Kähler potential of the A-field means that

$$\int d^4\theta \ \mathrm{Im}\, A_D(q) \overline{A(q)} = \int d^4\theta \ \mathrm{Im}\, A_D \bar{A}$$

This is satisfied if $\begin{pmatrix} q_m & q_e \\ r & s \end{pmatrix} \in SL(2,\mathbb{Z})$. To determine where the coupling constants behave as they should

$$\left. \begin{aligned} h_q(a_q) &= q_m h(a) + q_e a \\ a_q &= r\, h(a) + s a \end{aligned} \right\} \Rightarrow$$

$$\left. \begin{aligned} \rightarrow \quad d h_q(a_q) &= (q_m h'(a) + q_e)\, da \\ d a_q &= (r\, h'(a) + s)\, da \end{aligned} \right\} \quad \text{since } \tau(a) = h'(a),$$

$$\tau_q(a_q) = \frac{d h_q(a_q)}{d a_q} = \frac{q_m \tau(a) + q_e}{r \tau(a) + s}$$

As it should be. Later we generalize these formulae to rank r gauge groups.

Dyon Masses.

For $SU(2)$ the BPS bound is $M \geq \sqrt{2}\,|Z|$, with

$$Z = a(n_e + \tau_{cl}\, n_m), \qquad \tau_{cl} = \frac{\theta}{2\pi} + i\frac{4\pi}{g^2}$$

This formula is derived in general for $N=2$ using Olive-Witten argument. When the bound is saturated $M = \sqrt{2}\,|Z|$ we obtain small representations of the $N=2$ algebra. For $a \neq 0$ the fields $M, \tilde{M}$ are massive. We can add two $N=1$ chiral multiplets $M, \tilde{M}$ with charge n_e. $N=2$ fixed the coupling.

$$\sqrt{2}\, n_e A M \tilde{M}$$

$$\boxed{Z = a n_e + a_D n_m} \quad \text{for dyons}$$

In the rank-r case we have:

$$Z = a^i n_{e,i} + h_i(a) n_m^i = a \cdot n_e + a_D \cdot n_m .$$

The mass formula should be monodromy invariant. $v = \begin{pmatrix} a_D \\ a \end{pmatrix} \to M \begin{pmatrix} a_D \\ a \end{pmatrix}$
$M \in Sp(2r, \mathbb{R})$. If $v \to Mv$, the column vector $w = (n_m, n_e)$
$\to w M^{-1}$. Since (n_m, n_e) are integers, M^{-1} is an integral matrix, since $\det M = 1$. Thus $M \in Sp(2r, \mathbb{Z})$. If $M = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$, then $\tau(A) \to (A\tau + B)(C\tau + D)^{-1}$, which is generated by $\tau \to \tau + B$, and $\tau \to -\tau^{-1}$. Hence the monodromy in moduli space is parametrized by elements in $Sp(2r, \mathbb{Z})$; this makes it reasonably appealing that our r-dim. moduli space $\mathcal{M}_r$ describes some special genus r curves. If Ω_r is the modular group of genus r surfaces,
$1 \to \mathrm{Tor} \to \Omega_r \to Sp(2r, \mathbb{Z}) \to 1$ is an exact sequence of groups

SU(2) - Moduli Space Monodromy.

(a). Monodromy at ∞. For this all we need is the form of $\mathcal{F}$ for large a.

$$\mathcal{F}(a) = \frac{i}{2\pi} a^2 \ln \frac{a^2}{\Lambda^2} \qquad |a| \text{ large}$$

$$a_D = \frac{\partial \mathcal{F}}{\partial a} = \frac{2ia}{\pi} \ln\left(\frac{a}{\Lambda}\right) + \frac{ia}{\pi}.$$

In the u plane $\ln u \to \ln u + 2\pi i$, $\ln a \to \ln a + i\pi$, hence:

$$\left.\begin{array}{l} a_D \to -a_D + 2a \\ a \to -a \end{array}\right\} \boxed{M_\infty = PT^{-2} = \begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix}} \text{ monodromy at infinity.}$$

$$P = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

There must be additional monodromy somewhere else in the u-plane. If $\mathcal{M}' = \mathcal{M} - \{\text{singularities}\}$ it must have at least ~~two sing.~~ three singular points. If the monodromy is abelian, a^2 is a good global coordinate and we lose positivity. There must be at least two extra punctures exchanged by the broken discrete symmetry $u \to -u$. The singularities could be interpreted as the appearance of some additional massless states.

Let us determine the monodromy assuming that some dyon acquires zero mass at a given point in moduli space. In Seiberg-Witten one analyzes what happens at $a_D(u_0) = 0$. The 1-loop β-function implies.

$$\tau_D \simeq -\frac{i}{\pi} \ln a_D \qquad a_D \simeq c_0 (u - u_0)$$

The detailed argument is as follows.

$$\left.\begin{aligned} a_D(q) &= \alpha\, a_D + \beta a \\ a(q) &= \gamma\, a_D + \delta a \end{aligned}\right\}$$

If ~~$a_D(q)$ condensa~~, $a(q)$ vanishes,

$$\tau(a_q) \simeq -\frac{i}{\pi} \lg a(q) \Rightarrow \quad \int \tau(a_q)\, da_q \simeq -\frac{i}{\pi} a(q) \ln a(q) + a_0 = a_D(q)$$

For the monopole which acquires zero mass, $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

then $a_D(q) = a \quad a(q) = -a_D$, then

$$a \simeq a_0 + \frac{i}{\pi} a_D \ln a_D$$

$$a_D$$

a_D is a good coordinate near the massless monopole, $a_D \to e^{2\pi i} a_D$

$$a \to a_0 + \frac{i}{\pi} a_D \ln a_D - 2a_D \qquad \begin{pmatrix} a_D \\ a \end{pmatrix} \to \begin{pmatrix} a_D \\ a - 2a_D \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} a_D \\ a \end{pmatrix}$$

The generalization now follows.

$$h_D = \int I_D \, da_D = -\frac{i}{\pi}\left(a_D \ln a_D - a_D\right)$$

Similarly from our analysis of $h_D(a_D)$, a_D:

$$a(u) \simeq a_0 + \frac{i}{\pi} a_D \ln a_D$$

Since $a_D \simeq c_0 (u - u_0)$ a_D is a good coordinate near u_0, and under monodromy: $\ln a_D \to \ln a_D + 2\pi i$:

$$\left.\begin{array}{l} a_D \longrightarrow a_D \\ a \longrightarrow a - 2a_D \end{array}\right\} \quad M_1 = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}$$

Imagine now some other charge becoming zero $a_D(q) = 0$, with $(q_m, q_e) = 1$ (coprime for stability or marginal stability). Then:

$$\left.\begin{array}{l} a_D(q) \to a_D(q) \\ a(q) \to a(q) - 2a_D(q) \end{array}\right\}$$

Since $\begin{pmatrix} a_D(q) \\ a(q) \end{pmatrix} = \underbrace{\begin{pmatrix} q_m & q_e \\ r & s \end{pmatrix}}_{\rho_q} \begin{pmatrix} a_D \\ a \end{pmatrix} \to \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} a_D(q) \\ a(q) \end{pmatrix}$

$$\begin{pmatrix} a'_D \\ a' \end{pmatrix} = \rho_q^{-1} \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \rho_q \begin{pmatrix} a_D \\ a \end{pmatrix}, \quad \text{and}$$

$$M(q) = \begin{pmatrix} s & -q_e \\ -r & q_m \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} q_m & q_e \\ r & s \end{pmatrix} = \begin{pmatrix} 1 + 2q_m q_e & 2q_e^2 \\ -2q_m^2 & 1 - 2q_m q_e \end{pmatrix}$$

$$\boxed{M(q) = \begin{pmatrix} 1 + 2q_m q_e & 2q_e^2 \\ -2q_m^2 & 1 - 2q_m q_e \end{pmatrix}}$$

$\mathrm{Tr}\, M(q) = 2$, it is always parabolic

Since $M(\infty) = \begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix}$, we can solve the equation:

$$M(m,n)\, M(m',n') = M(\infty)$$

$$\begin{pmatrix} 1 + 2mn & 2n^2 \\ -2m^2 & 1 - 2mn \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 - 2m'n' & -2n'^2 \\ 2m'^2 & 1 + 2m'n' \end{pmatrix}$$

leading to the equations:

$$1+mn = m'n' + 2m'^2 \qquad a)$$
$$m^2 = m'^2 \qquad b)$$
$$n^2 = n'^2 + 1 + 2m'n' \qquad c)$$
$$1-mn = -m'n' \qquad d)$$

a)+d) $\Rightarrow$ $m' = \pm 1$ then b) $\Rightarrow$ $m = \pm 1$. There are four cases $(m',m) = (1,1), (1,-1), (-1,1), (-1,-1)$

$m' = +1$ $\quad n^2 = n'^2 + 1 + 2n' = (n'+1)^2 \Rightarrow n' = \pm n - 1$

$$1 - mn = -(\pm n - 1) = 1 \mp n \quad \begin{cases} n' = +n-1 & m = 1 \\ n' = -n-1 & m = -1 \end{cases}$$

$(m',n') = (1, n-1) \qquad (m,n) = (1,n)$
$(m',n') = (1, -n-1) \qquad (m,n) = (-1,n)$

$m' = -1$ $\quad n^2 = (n'-1)^2 \Rightarrow n' = \pm n + 1$

$$1 - mn = \pm n + 1 \quad \begin{cases} n' = n+1 & m = -1 \\ n' = -n+1 & m = 1 \end{cases}$$

$(m',n') = (-1, n+1) \qquad (m,n) = (-1, n)$
$(m',n') = (-1, -n+1) \qquad (m,n) = (1, n)$

Hence we have the solutions:

$$\boxed{\begin{array}{lllll} (m',n') & (1,n-1) & (1,-n-1) & (-1,n+1) & (-1,-n+1) \\ (m,n) & (1,n) & (-1,n) & (-1,n) & (1,n) \end{array}}$$

There do not seem to be any solutions where M_∞ is factorized into a product of more than two such parabolic $M(m,n)$.

$$M(1,n) = \begin{pmatrix} 1+2n & 2n^2 \\ -2 & 1-2n \end{pmatrix}$$

Hence $M(1,0) = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}$ $\qquad M(1,-1) = \begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix}$

$$M(-1,0) = \begin{pmatrix} +1 & 0 \\ -2 & 1 \end{pmatrix}$$

Fixed values:

$$(q_m, q_e)\begin{pmatrix} 1+2mn & 2n^2 \\ -2m^2 & 1-2mn \end{pmatrix} = (q_m, q_e)$$

$$\left.\begin{aligned} q_m(1+2mn) - 2m^2 q_e &= q_m \Rightarrow & 2mn\, q_m - 2m^2 q_e = 0 \\ 2n^2 q_m + q_e - 2mn\, q_e &= q_e \Rightarrow & 2n^2 q_m - 2mn\, q_e = 0 \end{aligned}\right\}$$

a) If $m \neq 0$ $\quad n q_m - m q_e = 0$

b) If $n \neq 0$ $\quad n q_m - m q_e = 0$

Since $(q_m, q_e) = 1$ $(n,m) \neq (0,0)$ then $n \propto q_e$, $m \propto q_m$. $q_m = M m$, $q_e = M n$

For the two monodromies we have:

$(1,0)$ at $u=1$ $\qquad M_1 = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}$

$(1,-1)$ at $u=-1$ $\qquad M_{-1} = \begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix}$

$$\boxed{\begin{aligned} M_1 &= S T^2 S^{-1} \\ M_{-1} &= TST^2(TS)^{-1} \\ M_\infty &= PT^{-2} \end{aligned}}$$

The monodromy at ∞ shifts the electric charge by 2 units.

$$(1,n)\begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix} = (-1, -n+2) \;; \quad (1,n)\begin{pmatrix} -1 & -2 \\ 0 & -1 \end{pmatrix} = (-1, -n-2)$$

$$(-1,-n)\begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix} = (1, n-2) \;; \quad (-1,-n)\begin{pmatrix} -1 & -2 \\ 0 & -1 \end{pmatrix} = (1, n+2)$$

Hence electric charge is really defined mod 2 in the points where condensation takes place. Semiclassically only $(\pm 1, n)$ dyons exist, and it is interesting to see that the monodromy at ∞ does not allow the condensation of other types of dyons. We can conjugate the representation of the fundamental group by M_∞^n.

U(1) β-function.

If we have Weyl fermions with charge Q_f and complex scalars with charge Q_S, their contributions to the U(1) β-functions are

$$\mu \frac{d}{d\mu} g = \frac{g^3}{16\pi^2}\left(\sum_f \frac{2}{3} Q_f^2 + \sum_s \frac{1}{3} Q_s^2 \right)$$

If we consider a reduced multiplet of $N=2$ with spin $\leq 1/2$, we'll have 2 Weyl fermions and two complex scalars with the same charge, we can obtain rather simple formulae.

$$\mu \frac{dg}{d\mu} = b g^3 \Rightarrow \mu \frac{d}{d\mu} g^2 = 2bg^4 \rightarrow \mu \frac{d}{d\mu}\alpha = 8\pi b \alpha^2$$

$$\boxed{\mu \frac{d}{d\mu}\left(\frac{1}{\alpha}\right) = - 8\pi b}$$

In $N=1$ language we have $M, \tilde{M}$ 2 chiral superfields

$$b = \frac{1}{16\pi^2} Q^2\left(2 \cdot \frac{2}{3} + 2 \cdot \frac{1}{3}\right) = \frac{1}{8\pi^2} Q^2$$

Thus

$$\mu \frac{d}{d\mu} \frac{1}{\alpha} = - \frac{Q^2}{\pi} \Rightarrow \mu \frac{d}{d\mu} \tau = - \frac{i}{\pi} Q^2$$

In case of a monopole condensing, this means:

$$\boxed{\tau \simeq - \frac{i}{\pi} \ln a_D}$$ which was used to construct the monodromy.

The interpretation of the singularities is related to the condensation of magnetic monopoles or dyons.

Extension to other Groups.

Without getting into interpretational issues we'd like to extend the previous material to other groups G of rank $r > 1$. In ~~this~~ the case of $SU(N_c)$ we already know that

$$U(1)_R \rightarrow \mathbb{Z}_{4N_c}$$

The field ϕ gets a VEV $\phi \sim \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_{N_c} \end{pmatrix}$ $\sum a_i = 0$; and ϕ has charge 2. Since the Weyl group acts on ϕ, the gauge invariant parameters describing the moduli are taken to be Weyl invariants: $u_2 = \mathrm{Tr}\,\phi^2/2$, $u_3 = \mathrm{Tr}\,\phi^3/3, \ldots, u_{N_c} = \mathrm{Tr}\,\phi^{N_c}/N_c$ with charges $(4, 6, 8, \ldots, 2N_c)$. $U(1)_R \rightarrow \mathbb{Z}_{4N_c}$, consequently in general only $\mathbb{Z}_2$ is unbroken. Since $\mathbb{Z}_{4N_c}/\mathbb{Z}_2 \simeq \mathbb{Z}_{2N_c}$, we have an action of $\mathbb{Z}_{2N_c}$ on the moduli space coordinates. If we look at $SU(3)$, $\mathbb{Z}_{12} \simeq \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$

$$u_2 = \mathrm{Tr}\,\phi^2/2 \rightarrow \omega^4 u_2$$
$$u_3 = \mathrm{Tr}\,\phi^3/3 \rightarrow \omega^6 u_3$$

We have two generators for the unbroken group

\buildrel {\alpha} \over \rightarrow

$$(u_2,u_3)\xrightarrow{\alpha}(\omega u_2,u_3) \qquad \alpha^3=1$$
$$(u_2,u_3)\xrightarrow{\beta}(u_2,-u_3) \qquad \beta^2=1$$
$$\left.\right\} \quad 1,\alpha,\alpha^2 \quad \beta,\beta\alpha,\beta\alpha^2$$

$SU(4) \qquad U(1)_R \to \mathbb{Z}_{16} \qquad (u_2,u_3,u_4)$ charges $(4,6,8)$. Again $\mathbb{Z}_2$ is unbroken $\mathbb{Z}_{16}/\mathbb{Z}_2 \simeq \mathbb{Z}_8$ is broken.

$\omega^{16}=1$ ω primitive, $u_2\to\omega^4 u_2$, $u_3\to\omega^6 u_3$ $u_4\to\omega^8 u_4$.

~~Take~~ For $SU(N)$

$$u_2\to\omega^4 u_2 \qquad u_j\to\omega^{2j}u_j \qquad j=1,\dots,N_c \quad ; \quad \omega^{4N_c}=1.$$

$$T=\begin{pmatrix}\omega^4 & & \\ & \omega^6 & \\ & & \omega^8\end{pmatrix} = \omega^4\begin{pmatrix}1 & & \\ & \omega^2 & \\ & & \omega^4\end{pmatrix}$$

$$T^4=\begin{pmatrix}1 & & \\ & \omega^8 & \\ & & 1\end{pmatrix}=\begin{pmatrix}1 & & \\ & -1 & \\ & & 1\end{pmatrix} \qquad T^8=1.$$

$$T_{SU(4)}=\omega^4\begin{pmatrix}1 & & \\ & \omega^2 & \\ & & \omega^4\end{pmatrix} \qquad T_{SU(3)}=\omega^4\begin{pmatrix}1 & \\ & \omega^2\end{pmatrix} \qquad T_{SU(5)}=\omega^4\begin{pmatrix}1 & & & \\ & \omega^2 & & \\ & & \omega^4 & \\ & & & \omega^6\end{pmatrix}$$

$$T_{SU(6)}=\omega^4\begin{pmatrix}1 & & & & \\ & \omega^2 & & & \\ & & \omega^4 & & \\ & & & \omega^6 & \\ & & & & \omega^8\end{pmatrix} \qquad T_{SU(N)}=\omega^4\begin{pmatrix}1 & & & \\ & \omega^2 & & \\ & & \ddots & \\ & & & \omega^{2(N-2)}\end{pmatrix}$$

$$T^{2N}=\omega^{8N}\begin{pmatrix}1 & & \\ & \omega^{4N} & \\ & & \ddots\end{pmatrix}=1.$$

The low energy action for the $U(1)^r$ theory is:

$$\frac{1}{4\pi}\,\mathrm{Im}\left[\int d^4\theta\,\frac{\partial\mathcal{F}}{\partial A^i}\bar{A}^i+\int d^2\theta\,\frac{1}{2}\frac{\partial^2\mathcal{F}}{\partial A_i\partial A_j}W^{i\alpha}W^j_\alpha\right]$$

With Kähler potential:

$$\left.\begin{aligned} K &= \mathrm{Im}\,\frac{\partial\mathcal{F}(A)}{\partial A^i}\bar{A}^i \\ ds^2 &= \mathrm{Im}\,\frac{\partial^2\mathcal{F}}{\partial A^i\partial A^j}\,dA^i d\bar{A}^j\end{aligned}\right\}$$

The coupling constants are $\tau_{ij}(A)=\dfrac{\partial^2\mathcal{F}}{\partial A^i\partial A^j}$. Define as before

$$A_{D\,i}=\frac{\partial\mathcal{F}}{\partial A^i}\equiv h_i(A)$$

Monopole Condensation + Confinement.

The $N=2$ field A decomposes in $N=1$ as W_α, Φ. Breaking $N=2$ down to $N=1$, one can add $W = m \, Tr\, \Phi^2$, for the chiral multiplet. This gives a ~~bar~~ bare mass to Φ. The low-energy theory has Z_4 chiral symmetry. The Theory is believed to have a mass gap with confinement and spontaneous breaking $Z_4 \to Z_2$. In the $N=2$ Theory the massless spectrum in the semiclassical limit contains only The abelian chiral multiplet A. Let us analyze The effect of turning on m. $Tr\, \Phi^2$ in the low-E Theory is represented by a chiral superfield U. Its 1st component is ~~$\mathrm{Tr}\, \Phi^2$~~ $u = tr\, \varphi^2$. A holomorphic f'n of moduli space. For small m we should add to the low-E Lagrangian $W_{eff} = m\, U$. To get a mass for The gauge field we need either i) extra light gauge fields $\Rightarrow$ strongly coupled non-abelian theory ii) light charged fields $\Rightarrow$ Higgs mechanism. Somewhere on $\mathcal{M}$ extra massless states must appear. Take option ii) with The light fields being monopoles or dyons Near the point with massless monopoles we can use $N=1$ language $M, \tilde{M}$ chiral fields as long as we describe the gauge field by the dual of the original field A_D:

$$\hat{W} = \sqrt{2}\, A_D M \tilde{M} + m\, U(A_D)$$

The low energy vacuum structure is easy to analyze. Vacua correspond to solutions of

$$d\hat{W} = 0$$

satisfying $|M| = |\tilde{M}|$ so that the D-term vanishes, For $m=0$ $M = \tilde{M} = 0$, a_D arbitrary. if $m \neq 0$,

$$\left.\begin{aligned} &\sqrt{2}\, M\tilde{M} + m \frac{du}{da_D} = 0 \\ &a_D M = a_D \tilde{M} = 0 \end{aligned}\right\}$$

assume $du/da_D \neq 0$, then $M, \tilde{M} \neq 0 \Rightarrow a_D = 0$, and

$$M = \tilde{M} = \left(-m\, u'(0)/\sqrt{2}\right)^{1/2}$$

about this vacuum there is a mass gap. The gauge field gets a mass by the Higgs mechanism. This is a magnetic Higgs mechanism $\Rightarrow$ confinement of charge.

The Solution of the Model

$\mathcal{M}$ is the u-plane with singularities at $1, -1, \infty$ and $\mathbb{Z}_2$-symmetry $u \to -u$. Over the punctured plane there is a flat $SL(2,\mathbb{Z})$ bundle V with the monodromies.

$$M_\infty = \begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix} \qquad M_1 = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \qquad M_{-1} = \begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix}$$

$(a_D(u), a(u))$ are a holomorphic sections of $V \otimes \mathbb{C}$.

$$u \simeq \infty \quad \begin{cases} a \simeq \sqrt{2u} \\ a_D \simeq i \dfrac{\sqrt{2u}}{\pi} \ln u \end{cases}$$

$$u = 1 \quad \begin{cases} a_D \simeq c_0 (u-1) \\ a \approx a_0 + \dfrac{i}{\pi} a_D \ln a_D \end{cases} \qquad a_0, c_0 \text{ constant}$$

$u = -1$ (same but a_D replaced by $a - a_D$)

$$ds^2 = \mathrm{Im}(\tau)\, |da|^2, \qquad \tau = \frac{da_D/du}{da/du}$$

$\mathrm{Im}\,\tau$ should be +ve definite.

$\{M_\infty, M_{\pm 1}\}$ generate $\Gamma(2)$. u-plane $/ \{1, -1, \infty\}$ is $H/\Gamma(2)$ with three cusps. The family of curves $H/\Gamma(2)$ can be written as:

$$E_u \cdot \{ y^2 = (x-1)(x+1)(x-u) \}$$

E_u is a genus 1-curve.

$$w: \begin{cases} u \to -u \\ x \to -x \\ y \to \pm i y \end{cases}$$ generates a $\mathbb{Z}_4$ symmetry, only $\mathbb{Z}_2$ acts on u.
$$I = w^2 \quad (u, x, y) \to (u, x, -y)$$

On E_u consider $V_u = H^1(E_u, \mathbb{C})$. (a_D, a) is a section of V_u. Pick two cycles γ_1, γ_2 on E_u.

$$\gamma_1 \cdot \gamma_2 = 1$$

$H^1(E_u, \mathbb{C})$ can be thought of as the space of meromorphic $(1,0)$-forms on E_u with vanishing residue modulo exact forms. If λ is such one form, it can be paired with 1-cycles

$$\gamma \to \oint_\gamma \lambda$$

Since $\mathrm{Res}\,\lambda = 0 \Rightarrow$ this is an invariant pairing. $H^1(E_u, \mathbb{C})$ has dim $= 2$, a basis is

$$\lambda_1 = \frac{dx}{y} \qquad \lambda_2 = \frac{x\,dx}{y}$$

λ_1 is holomorphic

$$b_i = \oint_{\gamma_i} \lambda_1 \quad i=1,2 \Rightarrow \quad b_1/b_2 = \tau_u$$

An arbitrary section of V_u is

$$\lambda = a_1(u)\,\lambda_1 + a_2(u)\,\lambda_2$$

$$a_D = \oint_{\gamma_1} \lambda \qquad a = \oint_{\gamma_2} \lambda$$

$$\tau = \frac{da_D/du}{da/du} \ni \quad \mathrm{Im}\,\tau > 0,$$

$$\frac{da_D}{du} = \oint_{\gamma_1} \frac{d\lambda}{du} \qquad \frac{da}{du} = \oint_{\gamma_2} \frac{d\lambda}{du}$$

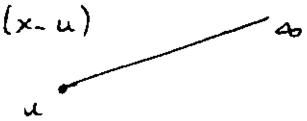
Suppose $\dfrac{d\lambda}{du} = f(u)\,\lambda_1 = f(u)\,\dfrac{dx}{y} \Rightarrow$

$$\Rightarrow \frac{da_D}{du} = f(u)\,b_1 \qquad \frac{da}{du} = f(u)\,b_2 \Rightarrow \tau = \frac{b_1}{b_2} = \tau_u \qquad \mathrm{Im}\,\tau_u > 0$$

If $f = -\sqrt{2}/4\pi$ everything works.

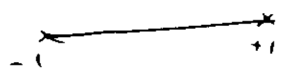
$$\frac{d\lambda}{du} = -\frac{\sqrt{2}}{4\pi}\,\frac{dx}{y} = -\frac{\sqrt{2}}{4\pi}\,\frac{dx}{[(x^2-1)(x-u)]^{1/2}} \Rightarrow$$

$$\Rightarrow \lambda = \frac{\sqrt{2}}{2\pi}\,\frac{dx\sqrt{x-u}}{\sqrt{x^2-1}} = \frac{\sqrt{2}}{2\pi}\,\frac{dx\,y}{x^2-1} = \frac{\sqrt{2}}{2\pi}\,\frac{dx}{y}\,(x-u)$$



λ is odd under I.

Computation then shows that the degenerating limit $u \to +1, -1, \infty$ satisfy the correct conditions.



Hyperelliptic Curves.

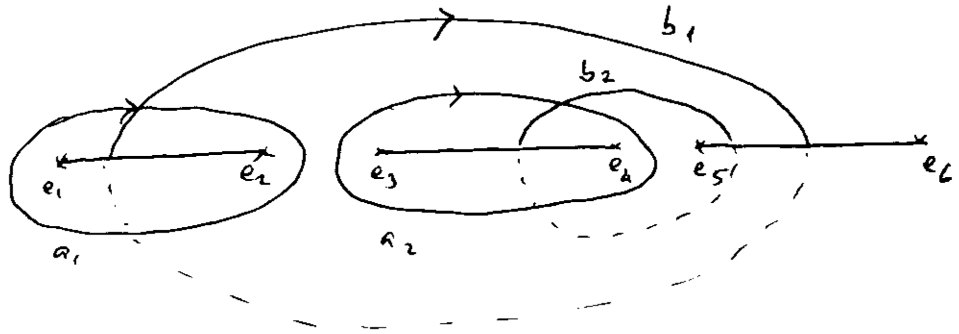
A curve is hyperelliptic if it admit a meromorphic function with exactly two poles. In this case the ramification points have branch number 1, and by Riemann-Hurwitz the number of branch points and the genus are related by $B = 2g+2$. We can describe such curves as double covers over the plane branched at $2g+2$ points. Choose a function $z: M \to \mathbb{P}^1$ of degree 2 on M. Any other function with that property is Möbius transform of z. Let $P_1, \dots, P_{2g+2}$ be the branch points of z, assume $z(P_j) \neq \infty$; $j = 1, \dots, 2g+2$. Consider the function:

$$w = \sqrt{\prod_{i=1}^{2g+2} (z - z(P_i))}$$

w defines a meromorphic function on M with divisor:

$$(w) = \frac{P_1 \cdots P_{2g+2}}{(Q_1 Q_2)^{g+1}}$$

Since all branch point of z have ramification number 2, close to P_i: $z - z(P_i) = \zeta_i^2$ is a good local coordinate, hence $w(P_i) = 0$. close to the poles $w \sim \left(\frac{1}{z - Q_i}\right)^{g+1}$, thus (w) has the divisor above. A good choice of marking is (for $g=2$)



$$a_i \cdot b_j = \delta_{ij} \qquad a_i \cdot a_j = b_i \cdot b_j = 0.$$

Abelian differentials. They are given by:

$$\frac{z^j\, dz}{w} \qquad j = 0, \dots, g-1$$

$$(z) = \frac{Q_3 Q_4}{Q_1 Q_2} \qquad z(P_j) \neq 0. \qquad (dz) = \frac{P_1 \cdots P_{2g+2}}{Q_1^2 Q_2^2}$$

because $z - e_j = \zeta_j^2$ close to $P_j \Rightarrow dz = 2\zeta_j d\zeta_j \to 0 \quad \zeta_j \to 0$.

$$\left(\frac{z^j dz}{w}\right) = \frac{(Q_3 Q_4)^j}{(Q_1 Q_2)^j} \frac{P_1 \cdots P_{2g+2}}{Q_1^2 Q_2^2} \frac{Q_1^{g+1} Q_2^{g+1}}{P_1 \cdots P_{2g+2}} = (Q_3 Q_4)^j (Q_1 Q_2)^{g-1-j}$$

which is positive for $j = 0, 1, \ldots, g-3$. A basis for the quadratic differentials is:

$$\left.\begin{array}{ll} \dfrac{z^j (dz)^2}{w^2} & j = 0,1,\ldots,2g-2 \\ z^j \dfrac{dz^2}{w} & j = 0,1,\ldots,g-3 \end{array}\right\} \quad 2g-1+g-2 = 3g-3 \checkmark$$

The theory of hyperelliptic surfaces is useful in generalizing Seiberg-Witten to the A-D-E case, using $y^2 = W^2_{A,D,E} - \Lambda^{2h}$, where A, D, E is the Landau-Ginzburg potential (singularity polynomial).

Instantons SUSY Effective Potentials

Some F- terms are induced by instantons and they help understand the vacuum structure of SUSY QCD for massless quarks. The BPST instanton is a solution to the self-duality equation $F = \pm * F$. Some simplification occur w.r.t. the usual 't Hooft computation because the non-zero modes cancel. The instanton action is $8\pi^2/g^2$. The number of fermion zero modes is given by AS theorem:

$$\text{ind}\, \not{D}_A = k\, \hat{T}_2(R) \; ; \quad \begin{cases} T_2(N) = 1 & SU(N) \\ \hat{T}_2(adj) = 2N & \text{"} \end{cases}$$

k is the instanton number. For $SU(2)$ the instanton solution is:

$$A_\mu^a = \frac{2}{g}\, \eta^a{}_{\mu\nu} \frac{(x-x_0)^\nu}{(x-x_0)^2+\rho^2}$$

$\eta^a{}_{\mu\nu}$ are the 't Hooft symbols. If $g = x^0 + i\vec{x}\cdot\vec{\sigma}$, $g^\dagger g = 1 \Rightarrow$
$\Rightarrow (x^0)^2 + \vec{x}^2 = 1$, then $(g^{-1}dg) = \frac{1}{2i}\sigma_a \omega^a$, $\omega^a \sim \eta^a{}_{\mu\nu} x^\nu dx^\nu$.
By conformal symmetry of the the YM equation, we have the full conformal group $SO(4,2)$ acting on the space of solutions. The generators of $SO(4,2)$ are P_μ, $M_{\alpha\beta}$, K_μ, D_α; also $SU(2)$ acts on A_μ. Only $SU(2)$, P_μ, D act on A_μ^a, the other generators can be compensated by $SU(2)$, P_μ, D, hence for $SU(2)$ there are $8k$ parameters for instantons of instanton number k. For $SU(N)$ we have to count the different ways of embedding $SU(2)$ in $SU(N)$. This is given by the dimension of the homogeneous space $SU(N)/SU(N-2)\times U(1)$ which is $N^2-1-(N-2)^2+1-1 = 4N-5$, hence there are $4N-5+5 = 4N$ collective coordinates. The action of $SU(2)$ is already included in our way of counting. If we ignore fermions and SUSY, the collective coordinate measure for $SU(2)$ is:

$$d^4x_0\, d\rho\, \sin\theta\, d\varphi\, d\psi\, d\theta\, M^8 \left(\sqrt{S_0}\right)^8 \rho^3$$

which can be understood as follows, we compute $\det^{1/2}(L+\Delta L+M^2)$. $(\det(L+\Delta L))^{-1/2}$, where $L = (\delta^2 S_{YM}/\delta A^2)|_{inst.}$, ΔL is gauge fixing, and we envisage Pauli-Villars regularization. Hence each zero mode gives a power of M (hence M^8), as is standard in the treatment of collective coordinates (See e.g. S. Coleman: Aspects of Symmetry) there is a normalization factor $S_0^{1/2}$, and the volume of the group is $(\rho^3 d^3\Omega)$. Thus:

$$\cancel{\beta}\ d^4x_0 \frac{d\rho}{\rho^5} (\rho M)^8 \left(\frac{8\pi^2}{g^2}\right)^7 e^{-8\pi^2/g^2}$$

is the contribution of the measure + classical action. For $SU(N)$ we'll have a factor ρ^{4N-5} from gauge rotations, M^{4N} from zero modes, therefore the measure is

$$d^4x_0 \frac{d\rho}{\rho^5} (d\Omega) (M\rho)^{4N} \left(\frac{8\pi^2}{g^2}\right)^{2N} e^{-8\pi^2/g^2} \times (\text{non-zero modes})$$

For SUSY, once the fermion zero modes are included, there are important changes in the collective ~~co~~ coordinate measure. One of the most interesting results in SUSY + instantons is:

<u>Russian β-function</u>.

The claim is that for n-extended susy:

$$M \frac{d}{dM} \alpha = \beta(\alpha) = \frac{\alpha^2}{2\pi} \frac{(n-4)N}{1 - \frac{2-n}{2\pi} N \alpha_s}$$

The result is only unexpected for $n = 1$. For $n = 2, 4$ we know that respectively the theory has only 1-loop corrections ($n=2$) or it is finite ($n=4$). To make susy manifest it is convenient to write the collective measure ~~e~~ out of the instanton vector superfield:

$$V(x_0, \rho, \alpha, \bar\beta) = e^{iP\cdot x_0} e^{-i(\alpha\cdot Q + \bar\beta\cdot\bar S)} V_{inst}(x_0 = 0, \rho) \cdot e^{-iP\cdot x_0} e^{i(\alpha\cdot Q + \bar\beta \bar S)}$$

The equations are superconformally invariant, hence Q, S generate fermionic zero modes (Zumino), which accounts in $SU(2)$ for all fermionic zero modes. In this form the susy zero modes are normalized differently. For $SU(2)$ we have

$$I(x_0, \rho, \alpha, \bar\beta) = c \left(\frac{2\pi}{\alpha_{s0}}\right)^2 e^{-2\pi/\alpha_{s0}} (M\rho)^6\, d^4x_0 \frac{d\rho}{\rho^5} (d^2\alpha)(d^2\beta)$$

For adjoint fermions we are computing:

$$\det{}^{1/2} \not{D} \,/\, \det{}^{1/2}(\not{D} + iM)$$

For boson zero modes we have $(\sqrt{S_0})^8$, for fermions $(\sqrt{S_0})^{-4}$ and for each fermion zero mode a factor $M^{-1/2}$. Hence for $SU(N)$ we can write for the instanton amplitude:

$$I \sim \left(\frac{2\pi}{\alpha_s}\right)^{(n_b-n_f)/2} e^{-2\pi/\alpha_s} (M\rho)^{n_b - n_f/2}\, d^4x_0 \frac{d\rho}{\rho^5} (d\alpha\, d\beta)$$

For $SU(N)$ and n-extended SUSY

$$n_b = 4N \qquad n_f = 2Nn$$

$$\left(\frac{2\pi}{\alpha}\right)^{(4N-2Nn)/2} e^{-2\pi/\alpha} (M\rho)^{4N-Nn} \left(1+\text{const.}\,\alpha+\dots\right)$$

Taking logs and $M\frac{d}{dM}$, we obtain:

$$-(2-n)N\frac{1}{\alpha}M\frac{d\alpha}{dM}+\frac{2\pi}{\alpha^2}M\frac{d\alpha}{dM}+N(4-n)+\text{const.}\,M\frac{d\alpha}{dM}\frac{1}{1+\text{const.}\,\alpha}=0$$

i.e.

$$\beta(\alpha)\left(-\frac{(2-n)N\alpha}{2\pi}+1+\text{const.}\,\frac{\alpha^2}{2\pi}(1+O(\alpha))\right)=N(n-4)\frac{\alpha^2}{2\pi}$$

If const $=0$

$$\boxed{\beta(\alpha)=\frac{N(n-4)\,\alpha^2/2\pi}{1-(2-n)\frac{N\alpha}{2\pi}}}$$

For $n=2,4$ const. $=0$. For $n=1$ the russians (Shifman et al) have an argument which suggests const $=0$. Only the first two coefficients are meaningful, and they do not depend on the value of const.

$$\beta(\alpha)=N(n-4)\frac{\alpha^2}{2\pi}+\frac{N^2(2-n)(n-4)}{4\pi^2}\alpha^3+O(\alpha^4)$$

It is remarkable that the 1st. two coefficients of β are given by purely classical arguments. In the SUSY case the non-zero part of the determinants cancel in the presence of an instanton by supersymmetry. For $n=1$

$$\beta(\alpha)_{n=1}=-\frac{3N\alpha^2}{2\pi}\left(1-\frac{N\alpha}{2\pi}\right)^{-1}$$

which obviously agrees with the standard 1-loop computation:

$$\beta(g)=-\frac{g^3}{16\pi^2}\Big(\underbrace{\tfrac{11}{3}C_2(adj)}_{\text{gluon + ghost}}-\underbrace{\tfrac{2}{3}C_2(adj)}_{\text{gluino}}-\underbrace{\sum_R T_2(R)}_{\text{each chiral superfield}}\Big)$$

$$\frac{g\beta(g)}{2\pi}=-\frac{\alpha^2}{2\pi}\Big(3N-\sum_R T_2(R)\Big)\ ; \qquad \boxed{\mu\frac{d\alpha}{d\mu}=\frac{g}{2\pi}\beta(g)}$$

The non-renormalization theorem is due to the fact that the instanton induced term is an ~~F~~ F-term. Using a chiral source field, the vacuum energy is

$$\int d^2\theta\, d^2\bar\theta\, d^4x \;\; f(x_{ch}-x_0,\, \theta-\theta_0), \quad x_{ch} = x + i\theta\sigma\bar\theta$$

$$f = \chi + \sqrt{2}\,\theta\psi + \theta\theta F$$

$$\chi = \chi(x) + \partial_\mu \chi\; \theta\sigma^\mu\bar\theta + \frac{1}{2}\,\partial^2_{\mu\nu}\chi\;\; \theta\sigma^\mu\bar\theta\;\; \theta\sigma^\nu\bar\theta =$$

$$= \chi(x) + \partial_\mu\chi\;\theta\sigma^\mu\bar\theta - \frac{1}{4}\,\Box\chi\;\;\theta^2\bar\theta^2 .$$

The other terms

$$\sqrt{2}\,\theta\,(\psi + \partial_\mu\psi\;\theta\sigma^\mu\bar\theta) + \theta\theta F(x)$$

χ gives a total derivative and for the other terms there are not enough $\bar\theta$'s to saturate $\int d^2\bar\theta$, hence the vacuum energy vanishes if χ decreases fast enough. Using this the russians agreed (?) that the non-renormalization theorem applies.

N = 1 SQCD. Non-Perturbative F-terms.

The generic form of the ~~Lag~~ Lagrangian for massive quarks is

$$\int d^4\theta\,(\bar Q e^V Q_i + \bar{\tilde Q}_i e^{\tilde V}\tilde Q_i) + \int d^2\theta\, \sum m_i\, \tilde Q_i Q^i + \text{h.c.}$$

$$+ \frac{1}{4\pi}\,\mathrm{Im}\int d^2\theta\; W^\alpha W_\alpha\,, \qquad \tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2}$$

With massive quarks Witten showed that $\mathrm{Tr}(-1)^F = N$ $(G = SU(N))$. As $m_i \to 0$ the vacua move to infinity as we will show later. When N_f/N = integer, Cohen & Gomez used special twisted B.C.'s to show that $\mathrm{Tr}(-1)^F \neq 0$. In other cases with $m_i = 0$ it is an interesting question whether susy breaks or not. For $N = 1$ only the holomorphicity of W is guaranteed. The D term is

$$D^a \sim \sum_i \bar Q_i T^a Q_i + \bar{\tilde Q}_i \tilde T^a \tilde Q_i$$

Since $\tilde Q_i$ is in the $\bar N$ irrep, $\tilde T^a = -{}^tT^a$, hence

$$D^a \sim \sum_i (\bar Q_i T^a Q_i - \tilde Q_i T^a \tilde Q_i^*)$$

When $m_i = 0$, the vacuum structure is given by D^a. If $m_i \neq 0\ \forall i$, the perturbative vacuum is at $Q_i = \tilde Q_i = 0$. The Lagrangian has a number of R-symmetries. Take for the time being $m_i = 0$.

The simplest R-symmetry is:

$$\lambda^a \to e^{-i\alpha}\lambda^a \qquad (\psi^i, \tilde\psi^i) \to e^{i\alpha}(\psi^i,\tilde\psi^i)$$

$$\left.\begin{aligned} W^a &\to e^{-i\alpha} W^a(x, e^{i\alpha}\theta) \\ Q_i, \tilde Q_i &\to (Q_i, \tilde Q_i)(x, e^{i\alpha}\theta) \end{aligned}\right\} \quad \begin{aligned} &\text{the coupling is} \\ &(\bar q^i T^a \psi^i \lambda^a + h.c. \\ &\tilde{\bar q}^i \tilde T^a \tilde\psi^i \lambda^a + h.c \end{aligned}$$

Mass terms do not respect this symmetry. It is also ~~affted~~ affected by anomalies.

$$\partial_\mu j^\mu_R = 2(N_f - N)\left(\frac{F\tilde F}{32\pi^2}\right) \leftarrow \text{zero mode density}$$

A fermion λ^a in the adjoint has $2N$ zero modes in the presence of an instanton, and ~~ϕ~~ $\psi, \tilde\psi$ has one each. Hence the symmetry breaks.

$$U(1)_R \longrightarrow \mathbb{Z}_{2|N-N_f|}$$

We can construct also R-symmetries only broken by the m_j mass. These $U(1)$'s are crucial to determine the mass dependence of special $N=1$ amplitudes (see below). The action of $U(1)^{(j)}$ is:

$$\left.\begin{aligned} W(x,\theta) &\to e^{-i\alpha/2N} W(x, e^{i\alpha/2N}\theta) \\ Q_j(x,\theta) &\to e^{i\alpha(N-1)/2N} Q_j(x, e^{i\alpha/2N}\theta) \\ Q_{j'}(x,\theta) &\to e^{-i\alpha/2N} Q_{j'}(x, e^{i\alpha/2N}\theta) \end{aligned}\right\} \quad \begin{aligned} &U(1)_R^{(j)} \\ &(\text{same for } \tilde Q_j, \tilde Q_{j'}) \end{aligned}$$

or in components

$$\left.\begin{aligned} &\lambda \to e^{-i\alpha/2N}\lambda \\ &\psi_j \to e^{i\alpha/2}\psi_j \qquad\qquad \varphi_j \to e^{i\alpha(N-1)/2N}\varphi_j \\ &\psi_{j'} \to \psi_{j'} \qquad\qquad\quad \varphi_{j'} \to e^{-i\alpha/2N}\varphi_{j'} \end{aligned}\right\}$$

$$\left.\begin{aligned} m_{j'}\tilde Q_{j'} Q_{j'} &\to m_{j'} e^{-2i\alpha/2N}\tilde Q_{j'} Q_{j'}(x, e^{i\alpha/2N}\theta) \\ m_j \tilde Q_j Q_j &\to e^{i\alpha} m_j e^{-2i\alpha/2N}\tilde Q_j Q_j(x, e^{i\alpha/2N}\theta) \end{aligned}\right\}$$

If we think of m_j as an external chiral superfield, then we can assign to it R-charge -1, then the superpotential is R_j-invariant. The R_j-symmetry is anomaly free by construction.

Useful Ward Identities.

Let $\mathcal{O}(y) = X(y) + \sqrt{2}\,\theta\psi(y) + \theta^2 F(y)$ be a chiral superfield, then:

$$\{\bar{Q}_{\dot\alpha}, \psi_\alpha\} \sim \sigma^m{}_{\alpha\dot\alpha}\partial_m X \qquad [\bar{Q}_{\dot\alpha}, X] = 0$$

Consider the Green's function:

$$G(x_1, \ldots, x_n) = \langle 0|\, T\, X(x_1) \ldots X(x_n)\,|0\rangle,$$

then

$$\frac{\partial}{\partial x_j{}^\mu} G(x_1, \ldots, x_n) = 0 \qquad j = 1, \ldots, n.$$

$$\sigma^m{}_{\alpha\dot\alpha}\frac{\partial}{\partial x_j{}^m} G = \langle 0|\, T\, X(x_1) \ldots \{\bar{Q}_{\dot\alpha}, \psi_\alpha\} \ldots |0\rangle$$

If $\bar{Q}_{\dot\alpha}|0\rangle = \langle 0|\bar{Q}_{\dot\alpha} = 0$, then since $[\bar{Q}_{\dot\alpha}, X] = 0$ we obtain $\partial G/\partial x_j{}^m = 0$. Some of the simplest examples are:

$$\left.\begin{aligned} \tilde{Q}_i Q^j &= \tilde{q}_i q^j + \cdots \\ WW &= \lambda\lambda + \cdots \end{aligned}\right\}$$

Konishi anomaly.

Using point splitting is it possible to show that

$$\frac{1}{2\sqrt{2}}\{\bar{Q}_{\dot\alpha}, \bar{\psi}^{\dot\alpha i}(x)\,\varphi_j(x)\} = -m_i\tilde{\varphi}^i\varphi_j + \frac{g^2}{32\pi^2}\lambda\lambda\,\delta^i{}_j$$

These identities together with the $U(1)^{(j)}_R$ - symmetries allow us to determine the mass dependence of G. First of all G is holomorphic in m,

$$\frac{\partial}{\partial m^*} G = \langle 0|\, T\, X_1 \ldots X_n \int d^4y\, F^*\, |0\rangle = 0$$

since $F^* \sim \{\bar{Q}_{\dot\alpha}, \bar{\psi}\}$ and $[\bar{Q}_{\dot\alpha}, X] = 0$, $\bar{Q}_{\dot\alpha}|0\rangle = 0$. If $m_j = |m_j|\, e^{i\alpha_j}$, then

$$\partial^\mu J^{(j)}_{\mu R} = i\, m_j\, \tilde{\psi}_j \psi_j + \text{h.c.}, \quad \text{and}$$

$$m_j \frac{\partial}{\partial m_j} G^{(p,q)} = \left(m_j\frac{\partial}{\partial m_j} - m_j^*\frac{\partial}{\partial m_j^*}\right) G^{(p,q)} = \frac{1}{i}\frac{\partial}{\partial \alpha_j} G^{(p,q)}, \leftarrow \text{axial rotation}$$

where

$$G^{(p,q)} = \langle 0|\, T\, \tilde{\varphi}^{i_1}\varphi_{j_1}(x_1) \ldots \tilde{\varphi}^{i_p}\varphi_{j_p}(x_p)\, \frac{g^2}{32\pi^2}\lambda\lambda(x_{p+1}) \ldots \frac{g^2}{32\pi^2}\lambda\lambda(x_{p+q})\,|0\rangle$$

The total charge of the correlator is:

$$q \text{ pairs of } \lambda\lambda \longrightarrow -\frac{q}{N}$$

$$p \ \tilde{\varphi}\varphi \text{ pairs} \longrightarrow -\frac{p}{N} + \frac{1}{2}\sum_{l=1}^{p}\left(\delta_{i_l j} + \delta_{j_l j}\right)$$

Hence

$$m_j \frac{\partial}{\partial m_j} G^{(p,q)} = q^{(j)} G^{(p,q)}$$

$$q^{(j)} = \frac{p+q}{N} - \frac{1}{2}\sum_{l=1}^{p}\left(\delta_{i_l j} + \delta_{j_l j}\right)$$

which integrates to

$$\prod_{l=1}^{p}\left(m_{i_l} m_{j_l}\right)^{1/2} G^{(p,q)} = C^{(p,q)\, i_1\cdots}_{\quad\ \ j_1\cdots}(\mu, g)\left(\prod_{l=1}^{N_f} m_l\right)^{\frac{p+q}{N}}$$

The μ-dependence follows from dimensional analysis

$$[G^{(p,q)}] = 2p + 3q$$

$$\left[\prod_{l=1}^{p}(m_{i_l} m_{j_l})^{1/2}\right] = p \qquad \left[\left(\prod^{N_f} m_l\right)^{\frac{p+q}{N}}\right] = N_f \frac{p+q}{N}$$

$$C^{(p,q)}(\mu, g) = \mu^{\alpha}\, c^{(p,q)}(g), \quad \text{then}$$

$$2p + 3q + p = \alpha + N_f \frac{p+q}{N} \Rightarrow \alpha = (p+q)(3 - N_f/N)$$

$$C^{(p,q)}(\mu, g) = c^{(p,q)}(g)\, \mu^{(p+q)(3 - N_f/N)}$$

To first order in the instanton approximation everything can be written in terms of m_{inv} and Λ.

$$\Lambda = \mu \exp - \int^{g} dg'/\beta(g'), \quad [m]_{inv} = m \exp - \int^{g} \frac{\gamma_m(g')\, dg'}{\beta(g')}$$

$\underline{U(1)_X}$ Another anomaly free R-symmetry of some use.

$$U(1)_X \quad \begin{cases} W_\alpha(\theta) \longrightarrow e^{-i\alpha} W(e^{i\alpha}\theta) \\ (Q, \tilde{Q}) \longrightarrow e^{i\alpha(N-N_f)/N_f}(Q, \tilde{Q})(e^{i\alpha}\theta) \\ (\varphi, \tilde{\varphi}) \longrightarrow e^{i\alpha N/N_f}(\varphi, \tilde{\varphi}) \end{cases}$$

only possible if $N_f \neq 0$.

Possible form of F-term.

We take $N_f \le N-1$. For $N_f = N-1$ the term we consider is induced by instantons. The global symmetry is $SU(N_f)_L \times SU(N_f)_R \times U(1)_V \times U(1)_X$. Analyticity of the superpotential + gauge invariance together with $SU(N_f)_L \times SU(N_f)_R$ symmetry imply:

$$F \sim \int d^2\theta \; f\left(\det_{r,r'} \tilde{Q}_{\alpha r} Q^{\alpha}{}_{r'} \right) \qquad (\alpha = \text{color index})$$

Under $U(1)_X$ symmetry the function f must be:

$$(\det \tilde{Q}Q)^{-1/N-N_f} \longrightarrow \left(e^{i\alpha(N-N_f)/N_f} \right)^{-\frac{2N_f}{N-N_f}} (\det \hat{Q}Q)^{-\frac{1}{N-N_f}} (e^{i\alpha}\theta)$$

$$= e^{-2i\alpha} (\det \tilde{Q}Q)^{-1/N-N_f} (e^{i\alpha}\theta)$$

by dimensional analysis

$$[\det \tilde{Q}Q]^{-1/N-N_f} = -\frac{2N_f}{N-N_f} \qquad [d^2\theta] = \left[\frac{\partial^2}{\partial\theta\partial\theta}\right] = +1 \qquad [\theta] = -1/2$$

$$[\Lambda^\alpha \int d^2\theta \, (\det \tilde{Q}Q)^{-1/N-N_f}] = \alpha + 1 - \frac{2N_f}{N-N_f} = 4 \Rightarrow \alpha = \frac{3N-N_f}{N-N_f}$$

$$\boxed{F = b\,\Lambda^{(3N-N_f)/N-N_f} \int d^2\theta \left(\det_{r r'} \tilde{Q}_{\alpha r} Q^{\alpha}{}_{r'} \right)^{-\frac{1}{N-N_f}}}$$

This makes no sense for $N_f > N$, because $\det \tilde{Q}_{\alpha r} Q^{\alpha}{}_{r'}$ vanishes identically due to Bose statistics of $Q, \tilde{Q}$. ~~So~~ The difficult part is the computation of b. Some further checks on F:

1) $U(1)_R \longrightarrow \mathbb{Z}_{2|N_f-N|}$. If $N_f = N-1$, $U(1)_R \to \mathbb{Z}_2$ which is consistent with F. $(\det \tilde{Q}Q)^{-1/(\,)} \to (\det \tilde{Q}Q)^{-1/(\,)}$

2) $N_f = N-1$, the instanton mass term has the form:

$$m \sim v \exp - \frac{8\pi^2}{g^2(v)}$$

$$\mu \frac{d}{d\mu} g = -\frac{g^3}{16\pi^2}\left(3C_2(adj) - 2N_f T_2(N) \right) = -\frac{g^3}{16\pi^2}(3N - N_f)$$

$$\frac{8\pi^2}{g^2} = (3N - N_f) \ln \frac{v}{\Lambda} \Rightarrow m \sim v\, e^{-8\pi^2/g^2(v)} \sim$$

$$m \sim \Lambda^{3N-N_f} v^{1-3N+N_f}$$

Input VEV's in F, then after some computation:

$$Q_{ir} = v_r \delta_{ir} + q_{ir}$$
$$\tilde{Q}_{ir} = v_r \delta_{ir} + \tilde{q}_{ir}$$

$$\det{}^{\alpha}(1+M) = 1 + \alpha \, \mathrm{Tr}\, M + \frac{\alpha}{2}(\mathrm{Tr}\, M)^2 - \frac{\alpha}{2} \mathrm{Tr}\, M^2 + \frac{\alpha(\alpha-1)}{2}(\mathrm{Tr}\, M)^2 + \dots$$

$$\tilde{Q}_r Q_{r'} = v^2 \left(\delta_{rr'} + \frac{1}{v} \tilde{q}_{rr'} + \frac{1}{v} q_{rr'} + \frac{1}{v^2} \tilde{q}_{ir} q_{ir'} \right)$$

$$(\det \tilde{Q} Q)^{-\frac{1}{N-N_f}} \simeq (v^2)^{-\frac{N_f}{N-N_f}} \left(1 + \frac{1}{v(N_f-N)} \mathrm{Tr}(q+\tilde{q}) + \frac{1}{v^2} \frac{\tilde{q}_{ir} q_{ir}}{N_f - N} + \dots \right)$$

the quadratic term has the form

$$(v^2)^{-1} (v^2)^{-\frac{N_f}{N-N_f}} (\tilde{q} q) = (v^2)^{-\frac{N}{N-N_f}} (\tilde{q} q)$$

The fermion masses are:

$$\left. \frac{\partial^2 W}{\partial \tilde{Q} \partial Q} \right|_{\langle VEV \rangle} \tilde{Q} Q$$

$$m_F \sim \Lambda^{(3N-N_f)/(N-N_f)} \, v^{-\frac{2N}{N-N_f}}$$

for $N_f = N-1$ $\left\{ \begin{array}{l} m_{inst} \sim \Lambda^{2N+1} v^{-2N} \\ m_F \sim \Lambda^{2N+1} v^{-2N} \end{array} \right.$

For $N_f < N-1$, we can add ~~masses~~ heavy masses for quarks at $N_f = N-1$ to obtain a similar expression for $N_f < N-1$. Using χSB arguments it can be argued that the F-term is generated.

$m \neq 0$

Add now a small mass for $N_f < N$, for simplicity take $m_i = m$.

$$W = b\Lambda^{\frac{3N-N_f}{N-N_f}} \left(\det_{k,k'} \tilde{Q}_{ik} Q^{i}{}_{k'} \right)^{-\frac{1}{N-N_f}} + \frac{1}{g^2} m \tilde{Q}_{ik} Q^{i}{}_{k}$$

The vanishing of D-term implies:

$$\langle Q^{ik} \rangle = \left\{ \begin{array}{ll} v_k \delta^{ik} & 1 \leq i \leq N_f \\ 0 & \text{otherwise} \end{array} \right\}$$

$$\langle \tilde{Q}^{+}{}_{ik} \rangle = \left\{ \begin{array}{ll} u_k \delta^{ik} & 1 \leq i \leq N_f \\ 0 & \text{otherwise} \end{array} \right.$$

then: $\partial W / \partial \tilde{Q} | = \partial W / \partial Q | = 0$ become.

$$-\frac{b}{N-N_f} \Lambda^{\frac{3N-N_f}{N-N_f}} \left(\det \tilde{Q} Q \right)^{-\frac{1}{N-N_f}} \frac{1}{\langle Q^{ik} \rangle} + \frac{m}{g^2} \langle \bar{Q}^{ik} \rangle = 0$$

meaningful only for $i = k$, in which case $v_k u_k^*$ is independent of k

$$-\frac{g^2 b}{(N-N_f)m} \Lambda^{\frac{3N-N_f}{N-N_f}} (vu^*)^{-\frac{N_f}{N-N_f}} + vu^* = 0 \Rightarrow$$

$$\Rightarrow \boxed{vu^* = \Lambda^2 \left(\frac{m(N-N_f)}{bg^2 \Lambda} \right)^{-\frac{N-N_f}{N}} e^{2\pi i n/N}} \quad n = 0, 1, \ldots, N-1$$

There are N SUSY vacua. As $m \to 0$ $|vu^*| \to \infty$. Since we assume equal masses, on top of the usual $SU(N_f)_V \times U(1)$ symmetry there is a global $\mathbb{Z}_{2N}$ symmetry

$$W \to e^{-i\pi n/N} W(e^{i\pi n/N}\theta) \qquad \lambda \to e^{-i\pi n/N} \lambda$$

$$(Q, \tilde{Q}) \to e^{-i\pi n/N} (\theta, \tilde{\theta})(e^{i\pi n/N}\theta) \qquad (q, \tilde{q}) \to e^{-i\pi n/N} (q, \tilde{q})$$

The gauge invariant order parameter $\tilde{q}q$ transforms as $\tilde{q}q \to e^{2\pi i n/N}\tilde{q}q$, hence $\mathbb{Z}_{2N} \to \mathbb{Z}_2$. $\mathbb{Z}_{2N}/\mathbb{Z}_2 \simeq \mathbb{Z}_N$ which rotates among the N-vacua described by Witten. As $m \to 0$ the vevs move to ∞ and in principle one might have susy breaking ~~??~~ happening dynamically.

Λ - For Different Flavors.

Since $\mu \frac{d}{d\mu} g = -\frac{g^3}{16\pi^2}(3N - N_f)$, $\frac{1}{g^2_{N_f}} = \frac{3N-N_f}{16\pi^2} \ln \frac{\mu^2}{\Lambda^2_{N_f}}$

If the N_f flavor becomes heavy the B.C. for decoupling gives

$$g^2_{N_f}(m^2) \sim g^2_{N_f}(m)$$

$$\frac{3N-N_f}{16\pi^2} \ln \frac{m^2}{\Lambda^2_{N_f}} \sim \frac{3N-N_f+1}{16\pi^2} \ln \frac{m^2}{\Lambda^2(N_f-1)}$$

$$\left(\frac{m^2}{\Lambda^2_{N_f}}\right)^{3N-N_f} \sim \left(\frac{m^2}{\Lambda^2_{N_f-1}}\right)^{3N-N_f+1}.$$

or:

$$\boxed{\Lambda(N_f-1) = \left(m\,\Lambda^{3N-N_f}\right)^{\frac{1}{3N-N_f+1}}}$$

Imagine we start with N_f flavor and add a mass m for the N_f quark. Below m we have effectively $N_f - 1$ quarks. Freezing the N_f-quark to its VEV

$$\left(\frac{\Lambda^{(3N-N_f)/(N-N_f)}}{(\det \tilde{Q}Q)}\right)^{1/N-N_f} \longrightarrow \left(\frac{\Lambda_L^{(3N-N_f+1)/N-N_f+1}}{(\det' \tilde{Q}Q)}\right)^{1/(N-N_f+1)}$$

using the eq'n of motion for $Q_{N_f}, \tilde{Q}_{N_f}$ leads as expected to $\Lambda_L = \Lambda(N_f - 1)$ as in the equation above. For $N_f \geq N$ we have to use different methods. In fact there are new operators to be considered, the baryon operators. To prove this statement consider the superpotential

$$W = \Lambda^{\frac{3N-N_f}{N-N_f}} (\det \tilde{Q}Q)^{-\frac{1}{N-N_f}} + \frac{m^2}{g^2}\tilde{Q}_{N_f} Q_{N_f}$$

Differentiate w.r.t. $Q^{\alpha}_{N_f}$, since the derivative of a determinant is a sum of ~~the derivatives~~ determinants where we differentiate only a row or a column,

$$\frac{\partial}{\partial \tilde{Q}^{\alpha}_{N_f}} W = -\frac{1}{N-N_f} \Lambda^{\frac{3N-N_f}{N-N_f}} (\det \tilde{Q} Q)^{-\frac{1}{N-N_f}-1} \cdot \begin{vmatrix} \tilde{Q}_1 Q_1 \cdots & \tilde{Q}_{1\alpha} \\ \vdots & \vdots \\ \tilde{Q}_{N_f} Q_1 & \tilde{Q}_{N_f\alpha} \end{vmatrix}$$

$$+ \frac{m^2}{g^2} \tilde{Q}_{N_f \alpha} = 0$$

From this it follows that $\tilde{Q}_{N_f} \cdot Q_1 = \tilde{Q}_{N_f} \cdot Q_2 = \cdots = \tilde{Q}_{N_f} \cdot Q_{N_f-1} = 0$ because the determinant will vanish. ~~a~~ ~~Contra~~ Contract with $Q^{\alpha}_{N_f}$ to obtain:

$$-\frac{g^2 \Lambda^{(3N-N_f)/N-N_f}}{m^2(N-N_f)} \cdot (\det{}' \tilde{Q} Q)^{-\frac{1}{N-N_f}} \cdot (\tilde{Q}_{N_f} Q_{N_f})^{-\frac{1}{N-N_f}} + \tilde{Q}_{N_f} Q_{N_f} = 0$$

$$(\tilde{Q}_{N_f} Q_{N_f})^{\frac{N-N_f+1}{N-N_f}} = \frac{g^2 \Lambda^{\frac{3N-N_f}{N-N_f}}}{m^2(N-N_f)} (\det{}' \tilde{Q} Q)^{-\frac{1}{N-N_f}}$$

At the minimum ~~the~~ W becomes

$$(N-N_f+1) \frac{m^2}{g^2} (\tilde{Q}_{N_f} Q_{N_f})$$

$$= (N-N_f+1) \frac{m^2}{g^2} \left[\frac{g^2 \Lambda^{\frac{3N-N_f}{N-N_f}}}{m^2(N-N_f)} \right]^{\frac{N-N_f}{N-N_f+1}} (\det{}' \tilde{Q} Q)^{-\frac{1}{N-N_f+1}}$$

$$= \cancel{c(N-N_f+1)\left(m^2 \Lambda^{3N-N_f}\right)}$$

$$= \text{const.} \, (\Lambda_{N_f-1})^{\frac{3N-N_f+1}{N-N_f+1}} (\det{}' \tilde{Q} Q)^{-\frac{1}{N-N_f+1}}$$

where const: $(N-N_f+1) \times$ (g- factors at m^2). More explicitly

$$(N-N_f+1) \left(\frac{1}{N-N_f} \right)^{\frac{N-N_f}{N-N_f+1}} (g^2 \det{}' \tilde{Q} Q)^{-1/N-N_f+1} .$$

$$\underbrace{\cdot \left[(m^2 \Lambda_{N_f}^{3N-N_f})^{\frac{1}{N-N_f}} \right]^{\frac{N-N_f}{N-N_f+1}}}_{\cancel{(\Lambda_{N_f-1})^{\frac{1}{N-N_f+1}}} \; (\Lambda_{N_f-1})^{\frac{3N-N_f+1}{N-N_f+1}}}$$

(see page 65)

Hence starting with $N_f = N-1$ we can integrate out massive quark multiplets to recover the non-perturbative F-term for any $N_f \leq N-1$.

D. term zeroes

When $m_i = 0$ it is important to know the amount of vacuum degeneracy in the theory. To do this we need to find the zeroes of D. terms. The scalar potential takes the form.

$$V(q,\tilde{q}) = \frac{1}{2g^2}(D^a)^2$$
$$2D^a = \sum (q_i^+ \lambda^a q_i - {}^t\tilde{q}_i \lambda^a \tilde{q}_i^*)$$

The minima of V are given by those configurations of $q_i, \tilde{q}_i$ where $D^a = 0$. We can distinguish two cases $N_f < N$, $N_f \geq N$.
Think of $q_\alpha^{(i)}$ as a set of N vectors in $\mathbb{C}^{N_f}$, and construct the matrix of scalar products

$$\sum_i q_\alpha^{(i)} \bar{q}_\beta^{(i)} = q_\alpha \cdot \bar{q}_\beta =: (QQ^+)_{\alpha\beta}$$

Then
$$\sum_i q_i^+ \lambda^a q_i = \mathrm{Tr}\, QQ^+ \lambda^a$$

Similarly we construct the matrix $\tilde{q}_\alpha^* \cdot \tilde{q}_\beta := (\tilde{Q}^* \tilde{Q}^t)$. Hence

$$D^a = \mathrm{Tr}(QQ^+ - \tilde{Q}^* \tilde{Q}^t)\lambda^a$$

Since λ^a is an irrep. of $SU(N)$, $QQ^+ - \tilde{Q}^*\tilde{Q}^t = c^2 \mathbb{1}_N$ give the vanishing condition for D^a

1) $N_f < N$. Now the matrix QQ^+ has rank N_f, thus it has $N - N_f$ zero eigenvalues, and the same applies to $\tilde{Q}^*\tilde{Q}^t$. By using $SU(N) \times SU(N_f)_L \times U(1)$ rotations QQ^+ can be diagonalized, hence $\tilde{Q}^*\tilde{Q}^t$ must also be diagonal

$$Q = \begin{pmatrix} v_1^{(1)} & & 0 \\ 0 & v_2^{(2)} & \\ & & \ddots & \\ & & & v_{N_f}^{(N_f)} \\ & & 0 & \end{pmatrix} \qquad \tilde{Q} = \begin{pmatrix} \tilde{v}_1^{(1)} & & 0 \\ 0 & \ddots & \\ & & v_{N_f}^{(N_f)} \\ & 0 & \end{pmatrix}$$

implying that $c = 0$, and $v_i^{(i)} = \tilde{v}_i^{(i)}$. $2NN_f - N_f^2$ quark superfields become heavy and N_f^2 are Goldstones of the various broken symm.

2). $N_f \geq N$ In this case QQ^+ has rank N; generically its eigenvalues are $\neq 0$, and $c \neq 0$

$$Q = \begin{pmatrix} v_1^{(1)} & & & \\ & \ddots & & 0 \\ & & v_N^{(N)} & \end{pmatrix} \qquad \tilde{Q} = \left(\begin{array}{ccc|c} \tilde{v}_1^{(1)} & & & \\ & \ddots & & 0 \\ & & \tilde{v}_N^{(N)} & \end{array}\right)$$

and $\tilde{v}_i^{(i)} = \sqrt{|v_i^{(i)}|^2 - c^2}$

For $N_f < N$ part of the gauge group survives $SU(N-N_f)$. If $N_f = N-1$ $SU(N)$ is completely broken. For $N_f \geq N$ the gauge group is also completely broken, and depending on the values of $v_i^{(i)}$ the pattern of χSB is quite complicated with many possibilities. For $N_f > N$ there are surviving R-symmetries.

N = 2 Currents.

We want to check in detail the BPS condition for $N=2$ theories. For this we need to construct the supercurrents. To warm up start with the matter lagrangian:

$$\mathcal{L} = \int d^4\theta\, \bar{\Phi}^+\Phi + \int d^2\theta\, W(\bar{\Phi}) + \int d^2\theta\, W(\Phi)$$

$y^a = x^a + i\theta\sigma^a\bar{\theta}$, then

$$Q_\alpha = \frac{\partial}{\partial\theta^\alpha} \qquad \bar{Q}_{\dot{\alpha}} = -\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} + 2i(\theta\sigma^a)_{\dot{\alpha}}\frac{\partial}{\partial y^a}$$

$\Phi = \varphi + \sqrt{2}\,\theta\psi + \theta\theta F$, then:

$\delta_\epsilon = \epsilon^\alpha Q_\alpha + \bar{\epsilon}_{\dot{\alpha}}\bar{Q}^{\dot{\alpha}} = \epsilon^\alpha Q_\alpha - \bar{\epsilon}^{\dot{\alpha}}\bar{Q}_{\dot{\alpha}}$, and

$$\left.\begin{aligned} \delta_\epsilon\varphi &= \sqrt{2}\,\epsilon\psi \\ \delta_\epsilon\psi &= \sqrt{2}\,\epsilon F + i\sqrt{2}\,\sigma^m\bar{\epsilon}\,\partial_m\varphi \\ \delta_\epsilon F &= i\sqrt{2}\,\bar{\epsilon}\,\bar{\sigma}^m\partial_m\psi \end{aligned}\right\} \quad \begin{aligned} \delta_\epsilon\bar{\varphi} &= \sqrt{2}\,\bar{\epsilon}\bar{\psi} \\ \delta_\epsilon\bar{\psi} &= -i\sqrt{2}\,\epsilon\sigma^m\partial_m\bar{\varphi} + \sqrt{2}\,\bar{F}\bar{\epsilon} \\ \delta_\epsilon\bar{F} &= i\sqrt{2}\,\epsilon\sigma^m\partial_m\bar{\psi} \end{aligned}$$

It is crucial to compute the total derivative induced in $\mathcal{L}$ under δ_ϵ. Start with the W-contribution

$$\delta\left(W'F - \frac{1}{2}\frac{\partial^2 W}{\partial A^2}\psi\psi\right) = W''\delta A\,F + W'\delta F - \frac{1}{2}W'''\delta A\,\psi\psi - W''\psi\delta\psi =$$

(with several fields $W'''\delta A\,\psi\psi = W'''\sqrt{2}(\epsilon\psi)\psi\psi = 0$ (W''' totally symmetric))

$$= W'' \sqrt{2}\,\epsilon\psi F + i\sqrt{2} W' \bar\epsilon \bar\sigma^m \partial_m \psi - W'' \psi (i\sqrt{2}\,\sigma^m \bar\epsilon \partial_m \varphi + \cancel{\sqrt{2} F \epsilon})$$

$$= \partial_m \left(i\sqrt{2} \frac{\partial W}{\partial \varphi_j} \bar\epsilon \bar\sigma^m \psi_j \right)$$

Thus

$$\boxed{\delta \left(\int d^2\theta\, W + \int d^2\bar\theta\, \bar W \right) = i\sqrt{2}\, \partial_m \left(\frac{\partial W}{\partial \varphi_j} \bar\epsilon \bar\sigma^m \psi^j - \frac{\partial \bar W}{\partial \bar\varphi_j} \bar\psi_j \bar\sigma^m \epsilon \right)}$$

Next the kinetic term.

$$\mathcal{L}_D = \partial_a \bar\varphi \partial^a \varphi + \bar F F - \frac{i}{2} \bar\psi \bar\sigma^a \partial_a \psi + \frac{i}{2} \partial_a \bar\psi \bar\sigma^a \partial_a \psi$$

A ~~slightl~~ slightly lengthy computation yields:

$$\left. \begin{aligned} \delta \mathcal{L}_D = & -\frac{i}{\sqrt{2}} \partial_m (F \bar\psi \bar\sigma^m \epsilon) + \frac{i}{\sqrt{2}} \partial_m (\epsilon\psi \partial^m \bar\varphi - \epsilon \sigma^{am} \psi \partial_a \bar\varphi) \\ & + \frac{i}{\sqrt{2}} \partial_m (\bar F \bar\epsilon \bar\sigma^m \psi) + \frac{1}{\sqrt{2}} \partial_m (\bar\epsilon \bar\psi \partial^m \varphi - \bar\psi \bar\sigma^{ma} \bar\epsilon \partial_a \varphi) \end{aligned} \right\}$$

$$\sigma^{am} = \frac{1}{2} (\sigma^a \bar\sigma^m - \sigma^m \bar\sigma^a) ; \quad \bar\sigma^{am} = \frac{1}{2} (\bar\sigma^a \sigma^m - \bar\sigma^m \sigma^a)$$

Since

$$J^\mu = \frac{\partial \mathcal{L}}{\partial \partial_\mu \Phi} \delta \Phi - K^\mu \qquad \text{(Noether's theorem)},$$

after some elementary manipulations we obtain:

$$\boxed{\begin{aligned} S^m = & \sqrt{2}\, \epsilon \sigma^a \bar\sigma^m \psi \partial_a \bar\varphi + i\sqrt{2} \frac{\partial \bar W}{\partial \bar\varphi} \bar\psi \bar\sigma^m \epsilon \\ & + \sqrt{2}\, \bar\psi \bar\sigma^m \sigma^a \bar\epsilon \partial_a \varphi - i\sqrt{2} \frac{\partial W}{\partial \varphi} \bar\epsilon \bar\sigma^m \psi \end{aligned}}$$

Next we consider the inclusion of gauge fields. We have two ~~ope~~ options

$$W_\alpha = -\frac{1}{4} \bar D^2 e^{-V} D_\alpha e^V$$

or to agree with standard normalizations:

$$\left\{ \begin{aligned} W_\alpha &= -\frac{1}{4} \bar D^2 e^{-2V} D_\alpha e^{2V} \qquad (\lambda, A, D) \longrightarrow 2(\lambda, A, D) \\ W_\alpha &= T^a W_\alpha^a \end{aligned} \right.$$

$$\left. \begin{aligned} F_{mn}^a &= \partial_m A_n^a - \partial_n A_m^a - f^{abc} A_m^b A_n^c \\ D_m \bar\lambda^a &= \partial_m \bar\lambda^a - f^{abc} A_m^b \bar\lambda^c \end{aligned} \right\}$$

$$\frac{1}{g^2} \bar\Phi^+ \Phi \longrightarrow \frac{1}{g^2} \bar\Phi^+ e^{2V} \Phi$$

If $\quad \mathrm{Tr}\, T^a T^b = \delta^{ab} k \quad$ (take $k = 1/2$ for the fundamental)

$$\frac{1}{k} \mathrm{Tr}\, W^\alpha W_\alpha = 4\left(-\frac{1}{2} F^2 - 2i\, \lambda \sigma \cdot D \bar{\lambda} + D^2 + \frac{i}{2} F^a \tilde{F}^a \right)$$

$$F^a \tilde{F}^a = \frac{1}{2} \varepsilon^{\alpha\beta\lambda\mu} F^a_{\alpha\beta} F^a_{\lambda\mu}$$

The gauge action is.

$$\frac{1}{8k} \frac{1}{4\pi} \mathrm{Im}\, z\, \mathrm{Tr}\, W^\alpha W_\alpha \Big|_{\theta^2} \qquad z = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2}$$

$$\mathcal{L} = \frac{1}{g^2}\left(-\frac{1}{4} F^a_{mn} F^{a\,mn} - i\, \lambda^a \sigma \cdot D \bar{\lambda}^a + \frac{1}{2} D^2 \right) + \frac{\theta}{32\pi^2} F^a \tilde{F}^a$$

The susy variation of the fields are: $\quad$ take $-\frac{i}{2} \bar{\lambda}^a \bar{\sigma} \cdot D \lambda^a + \frac{i}{2} D\bar{\lambda}^a \cdot \bar{\sigma} \lambda^a$

$$\left.\begin{aligned}
\delta A^a_n &= -i\, \bar{\epsilon} \bar{\sigma}_n \lambda^a + i\, \bar{\lambda}^a \bar{\sigma}_n \epsilon \\
\delta D^a &= \bar{\epsilon} \bar{\sigma}^m D_m \lambda^a + D_m \bar{\lambda}^a \bar{\sigma}^m \epsilon \\
\delta \lambda^a &= \frac{1}{2} \sigma^{mn} \epsilon F^a_{mn} + i \epsilon D^a \\
\delta \bar{\lambda}^a &= \frac{1}{2} \bar{\epsilon} \bar{\sigma}^{nm} F^a_{mn} - i \bar{\epsilon} D^a
\end{aligned}\right\}$$

The matter fields transformation need a little modification

$$\delta \varphi = \sqrt{2}\, \epsilon \psi$$

$$\delta \psi = \sqrt{2}\, \epsilon F + i\sqrt{2}\, \sigma^m \bar{\epsilon} D_m \varphi$$

$$\delta \bar{\psi} = \sqrt{2}\, \bar{\epsilon} \bar{F} - i\sqrt{2}\, \epsilon \sigma^m D_m \bar{\varphi}$$

$$\delta F = i\sqrt{2}\, \bar{\epsilon} \bar{\sigma}^m D_m \psi - 2i\, T^a \varphi\, \bar{\epsilon} \bar{\lambda}^a$$

The last term in δF is needed to cancel part of the variation of ψ in the term $\bar{\varphi} T^a \psi \lambda^a$ in the Lagrangian.

Useful identities:

$$\left.\begin{aligned}
\sigma^a \bar{\sigma}^b \sigma^c &= \eta^{ab} \sigma^c - \eta^{ac} \sigma^b + \eta^{bc} \sigma^a + i\, \varepsilon^{abcd} \sigma_d \\
\bar{\sigma}^a \sigma^b \bar{\sigma}^c &= \eta^{ab} \bar{\sigma}^c - \eta^{ac} \bar{\sigma}^b + \eta^{bc} \bar{\sigma}^a - i\, \varepsilon^{abcd} \bar{\sigma}_d
\end{aligned}\right\}$$

If we ignore the $\theta F\tilde{F}$ term, the supercurrent takes the form:

$$\boxed{S^{\rho} = -\frac{i}{2}\,\bar\lambda^a \bar\sigma^s \sigma^{mn}\epsilon\, \bar F^a_{mn} - \frac{i}{2}\,\bar\epsilon\, \bar\sigma^{mn} \bar\sigma^s \lambda^a F^a_{mn}}$$

For the theory coupled to matter, the supercurrent is not just the sum of the above plus the gauge-covariantized version of S_{matter}. We expect some contribution from the $D^a\,\bar\varphi T^a\varphi$ and $i\bar\varphi T^a\psi\lambda^a$ + h.c. terms. This determines the modification of δF and also the change in the supercurrent. The final result is:

$$\boxed{\begin{aligned} J^s = & -\frac{i}{2}\,\bar\lambda^a \bar\sigma^s \sigma^{mn}\epsilon F^a_{mn} - \frac{i}{2}\,\bar\epsilon\,\bar\sigma^{mn}\bar\sigma^s\lambda^a F^a_{mn} \\ & + \sqrt{2}\,\epsilon\sigma^m\bar\sigma^s\psi_j D_m\bar\varphi_j + i\sqrt{2}\,\frac{\partial\bar W}{\partial\bar\varphi_j}\,\bar\psi_j\bar\sigma^s\epsilon \\ & + \sqrt{2}\,\bar\psi_j\bar\sigma^s\sigma^m\bar\epsilon\, D_m\varphi_j - i\sqrt{2}\,\frac{\partial W}{\partial\varphi_j}\,\bar\epsilon\bar\sigma^s\psi_j \\ & - \bar\varphi T^a\varphi\;\epsilon\bar\sigma^s\lambda^a - \bar\lambda^a\bar\sigma^s\epsilon\;\bar\varphi T^a\varphi \end{aligned}}$$

Pure N=2 Gauge Theory.

In this case we consider matter in the adjoint representation. We have two adjoint Weyl fermions λ_1^a, λ_2^a a complex scalar φ^a and the gauge field A_m^a. The supercurrent constructed previously becomes:

$$\left.\begin{aligned} J^s_{(1)} = & -\frac{i}{2}\,\bar\lambda_1^a\bar\sigma^s\sigma^{mn}\epsilon F^a_{mn} - \frac{i}{2}\,\bar\epsilon\,\bar\sigma^{mn}\bar\sigma^s\lambda_1^a F^a_{mn} \\ & + \sqrt{2}\,\epsilon\sigma^m\bar\sigma^s\lambda_2^a D_m\bar\varphi^a + \sqrt{2}\,\bar\lambda_2^a\bar\sigma^s\sigma^m\bar\epsilon\, D_m\varphi^a \\ & - \bar\varphi T^a\varphi\;\bar\epsilon\bar\sigma^s\lambda_1^a - \bar\lambda_1^a\bar\sigma^s\epsilon\;\bar\varphi T^a\varphi \end{aligned}\right\}$$

The second current is given by exchanging λ_1 and λ_2

$$\left.\begin{aligned} J^s_{(2)} = & -\frac{i}{2}\,\bar\lambda_2^a\bar\sigma^s\sigma^{mn}\epsilon F^a_{mn} - \frac{i}{2}\,\bar\epsilon\,\bar\sigma^{mn}\bar\sigma^s\lambda_2^a F^a_{mn} \\ & - \sqrt{2}\,\epsilon\sigma^m\bar\sigma^s\lambda_1^a D_m\bar\varphi^a - \sqrt{2}\,\bar\lambda_1^a\bar\sigma^s\sigma^m\bar\epsilon\, D_m\varphi^a \\ & - \bar\varphi T^a\varphi\;\bar\epsilon\bar\sigma^s\lambda_2^a - \bar\lambda_2^a\bar\sigma^s\epsilon\;\bar\varphi T^a\varphi \end{aligned}\right\}$$

Next we want to compute the explicit form of the central charge in the N=2 algebra. ~~For~~ It suffices to examine the ϵ-parts of $J_{(1)}$, $J_{(2)}$.

Using the identities on the bottom of page 70 we obtain:

$$\left.\begin{aligned} g^2 J^s_{(1)}(\varepsilon) &= -i\bar\lambda^a_1 \bar\sigma_n \varepsilon F^{asn} - \bar\lambda_1 \bar\sigma_n \epsilon \, {}^*F^{sn} \\ &\quad + \sqrt{2}\, \varepsilon \sigma^m \bar\sigma^s \lambda^a_2 D_m \bar\varphi^a - \bar\lambda^a_1 \bar\sigma^s \varepsilon \, \bar\varphi T^a \varphi \\ g^2 J^s_{(2)}(\varepsilon') &= -i\bar\lambda^a_2 \bar\sigma_n \varepsilon F^{asn} - \bar\lambda_2 \bar\sigma_n \varepsilon' \, {}^*F^{cd} \\ &\quad - \sqrt{2}\, \varepsilon' \sigma^m \bar\sigma^s \lambda^a_1 D_m \bar\varphi^a - \bar\lambda_2 \bar\sigma^s \varepsilon' \, \bar\varphi T^a \varphi \end{aligned}\right\}$$

or:

$$\left.\begin{aligned} g^2 J^s_{(1)\alpha} &= i\sigma_{n\alpha\dot\alpha} \bar\lambda^{\dot\alpha}_1 F^{(a)sn} + \sigma_{n\alpha\dot\alpha} \bar\lambda^{\dot\alpha}_1 \, {}^*F^{sn} \\ &\quad + \sqrt{2}\, \sigma^m \bar\sigma^s \lambda^a_2 D_m \bar\varphi^a + \bar\sigma^s_{\alpha\dot\alpha} \bar\lambda^{\dot\alpha}_1 \, \bar\varphi T^a \varphi \\ g^2 J^s_{(2)\alpha} &= i\sigma_{n\alpha\dot\alpha} \bar\lambda^{\dot\alpha}_2 F^{(a)sn} + \sigma_{n\alpha\dot\alpha} \bar\lambda^{\dot\alpha}_2 \, {}^*F^{sn} \\ &\quad - \sqrt{2}\, \sigma^m \bar\sigma^s \lambda^a_1 D_m \bar\varphi^a + \bar\sigma^s_{\alpha\dot\alpha} \bar\lambda^{\dot\alpha}_2 \, \bar\varphi T^a \varphi \end{aligned}\right\}$$

The factor of g^2 appears because of the way we normalized the Lagrangian. To evaluate the central charges we are interested in

$$\left\{ \int d^3x \, J^0_{(1)\alpha}(\vec x, 0), \int d^3y \, J^0_{(2)\alpha}(\vec y, 0) \right\}$$

we only need to look for boundary terms to check the Olive-Witten result. Since these are given by electric and magnetic charges we can easily identify the terms leading to the boundary terms. Bring spinor indices down

$$\bar\lambda^{\dot\alpha}_1 = \varepsilon^{\dot\alpha\dot\beta} \bar\lambda_{1\dot\beta} = i\sigma_y \bar\lambda_1$$

and using vector notation

$$\left.\begin{aligned} g^2 J^0_{(1)\alpha} &= -i\,\vec\sigma\, i\sigma_y \, \vec F^{(a)} \bar\lambda_1 - \vec\sigma\, i\sigma_y \, {}^*\vec F \, \bar\lambda_1 \\ &\quad + \sqrt{2}\, \lambda^a_2 \partial_0 \bar\varphi^a + \sqrt{2}\, \vec\sigma \cdot \vec D \bar\varphi^a \lambda^a_2 + i\sigma_y \bar\lambda_1 \, \bar\varphi T^a \varphi \\ g^2 J^0_{(2)\alpha} &= -i\,\vec\sigma\, i\sigma_y \cdot \vec F \, \bar\lambda_2 - \vec\sigma\, i\sigma_y \cdot {}^*\vec F \, \bar\lambda_2 \\ &\quad - \sqrt{2}\, \lambda^a_1 \partial_0 \bar\varphi^a - \sqrt{2}\, \vec\sigma \cdot \vec D \bar\varphi^a \lambda^a_1 + i\sigma_y \bar\lambda_2 \, \bar\varphi T^a \varphi \end{aligned}\right\}$$

$$\vec F^{(a)} = F^{0i(a)} \qquad {}^*\vec F^{(a)} = {}^*F^{0i(a)}$$

We are looking in the commutator for terms of the form $\int d^3x \, \partial_i (\bar\varphi F^{0i} + \bar\varphi \, {}^*F^{0i})$. Those involving F^{0i} are more laborious because $D_i F^{0i(a)}$ appears only after the full commutator is computed together with the use of the equations of motion, what one obtains is ρ^a (the charge density); for the magnetic part however the total derivative appears easily once we take into account the Bianchi identity $\varepsilon^{ijk} D_i F_{jk} = 0$. The terms of interest are

$$g^2 J^0_{(1)\alpha} = -(\vec{\sigma}\, i\sigma_y)\bar{\lambda}_1^a \left(i\vec{F}^{(a)} + *\vec{F}^{(a)} \right) + \sqrt{2}\, \vec{\sigma}\, \vec{D}\, \bar{\varphi}^a \lambda_2^a$$

$$g^2 J^0_{(2)\alpha} = -(\vec{\sigma}, \sigma_y)\bar{\lambda}_2^a \left(i\vec{F}^{(a)} + *\vec{F}^{(a)} \right) - \sqrt{2}\, \vec{\sigma}\, \vec{D}\, \bar{\varphi}^a \lambda_1^a$$

Thus we compute.

$$\int d^3x d^3y \; i\sqrt{2}\,(\sigma^i\sigma_y)_{\alpha\mu}\, \sigma^j{}_{\beta\nu} \left\{ \lambda^{+a}_{1\mu}, \lambda^{a'}_{1\nu} \right\} \left(iF^{(a)i} + *F^{i(a)} \right) D^j \bar{\varphi}^{a'}$$

$$- i\sqrt{2}\, (\sigma^j)_{\alpha\mu} (\sigma^i\sigma_y)_{\beta\nu} \left\{ \lambda^{+a}_{2\mu}, \lambda^{a'}_{2\nu} \right\} \left(iF^{(a)i} + *F^{i(a)} \right) D^j \bar{\varphi}^{a'}$$

$$= i\sqrt{2}\int d^3x \left\{ (\sigma^i\sigma_y)_{\alpha\mu}\, \sigma^j{}_{\beta\mu} \left(iF^{(a)i} + *F^{i(a)} \right) D^j \bar{\varphi}^a \right.$$
$$\left. - (\sigma^i\sigma_y)_{\beta\mu}\, \sigma^j{}_{\alpha\mu} \left(iF^{(a)i} + *F^{i(a)} \right) D^j \bar{\varphi}^a \right\} g^2$$

$$(\sigma^i\sigma_y)_{\alpha\mu}\,\sigma^j{}_{\beta\mu} = (\sigma^i\sigma_y\sigma^{jT})_{\alpha\beta} = -\sigma_y\sigma^{iT}\sigma^{jT} =$$
$$= -\left[\sigma_y \left(\delta^{ij} - i\varepsilon^{ijk}\sigma^{kT} \right)\right]_{\alpha\beta}$$

$$(\sigma^i\sigma_y)_{\beta\mu}\,\sigma^j{}_{\alpha\mu} = -\left[\sigma_y\, \sigma^{iT}\sigma^{jT}\right]_{\beta\alpha} =$$
$$= -\left[\sigma_y \left(\delta^{ij} - i\varepsilon^{ijk}\sigma^{kT} \right)\right]_{\beta\alpha}$$

subtracting

$$-2(\sigma_y)_{\alpha\beta}\,\delta^{ij} + i\left((\sigma_y\sigma^{kT})_{\alpha\beta} - (\sigma_y\sigma^{kT})_{\beta\alpha} \right) \varepsilon^{ijk}$$

$$= -2(\sigma_y)_{\alpha\beta}\,\delta^{ij} + i\,\underbrace{\left(\sigma_y\sigma^{kT} + \sigma^k\sigma_y \right)_{\alpha\beta}}_{=0}\varepsilon^{ijk} \qquad \sigma_y\sigma^{kT}\sigma_y = -\sigma^k$$

also $(i\sigma_y)_{\alpha\beta} = -\epsilon_{\alpha\beta}$.

$$\boxed{g^2 \left\{ Q_{(1)\alpha}, Q_{(2)\beta} \right\} = 2\sqrt{2} \int d^3x\; \varepsilon_{\alpha\beta}\, \nabla_i \left[\bar{\varphi}^a \left(iF^{(a)i} + *F^{i(a)} \right) \right]}$$

$$\boxed{g^2 \left\{ \bar{Q}_{(1)\dot{\alpha}}, \bar{Q}_{(2)\dot{\beta}} \right\} = 2\sqrt{2} \int d^3x\; \varepsilon_{\dot{\alpha}\dot{\beta}}\, \nabla_i \left[\varphi^a \left(-iF^{(a)i} + *F^{i(a)} \right) \right]}$$

To find the effect of the θ-parameter we can either ~~x~~ modify $J^a_{(1,2)}$ by the term coming from θ (and also the canonical momenta), or simply use the Witten effect (pp 18-19). Due to our normalization condition ($1/g^2$ everywhere including the $\Phi^+ e^{2V}\Phi$ term except for the θ-term), we can finally write (in the case where electric and magnetic charges are integral, as is the case for SU(2) with adjoint fields) $SU(2) \to U(1)$

$$\{ \bar{Q}^{(1)}_{\dot\alpha}, \bar{Q}^{(2)}_{\dot\beta} \} = -2\sqrt{2}\, \varepsilon_{\dot\alpha\dot\beta}\, \xi\, a\,(n_e + \tau_{cl}\, n_m)$$

$$\tau_{cl} = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2} ,$$

the θ-term is due to ~~the~~ Witten's effect. Since $\{ Q_\alpha, \bar{Q}_{\dot\alpha} \} = 2M$ in the rest frame we conclude that

$$M \geq \sqrt{2}\, |Z| \quad ; \quad Z = a\,(n_e + \tau_{cl}\, n_m)$$

which at the level of the effective Lagrangian becomes

$$\boxed{Z = a n_e + a_D n_m} .$$

Similar constructions apply for arbitrary groups SU(N) with adjoint matter.

When matter in the fundamental representation is included there is an important change: the central charge receives contributions from the matter sector as well. A quark hypermultiplet contains two chiral superfields $Q, \tilde{Q}$ resp. in the $N, \bar{N}$ reps of SU(N). ~~See~~ $N=2$ ~~su~~ algebra contains and $SU(2)_R$ symmetry. Q contains fields q, ψ_q, $\tilde{Q}$: $\tilde{q}, \psi_{\tilde{q}}$, then $SU(2)_R$ acts such that $\psi_q, \psi^+_{\tilde{q}}$ are singlets, and $q, \tilde{q}^+$ forms a doublet. $q, \tilde{q}^+$ belong both to the same rep of SU(N) as it should be. The form of the superpotential in the $N=2$ theory is as follows

$$\sum_i \sqrt{2}\, \tilde{Q}_i \Phi Q_i + \sum_{i=1}^{N_f} m_i \tilde{Q}_i Q_i + h.c.$$

for N_f quark flavours, and Φ is the chiral adjoint superfield in the vector multiplet. In the vector multiplet $SU(2)_R$ acts on λ^a_1, λ^a_2. If we want to construct ~~both~~ the $N=2$ supercurrents now we have to exchange λ_1, λ_2 and $q, \tilde{q}^+$. However ~~at~~ rather than working the full $N=2$ algebra all we want is to track down the origin of the extra term in the ~~ch~~ central operators. It is crucial to include the mass term explicitly in the supercurrent. Recall the terms:

$$J^s = \dots + \sqrt{2}\,\varepsilon\sigma^m\bar\sigma^s\psi D_m\bar\varphi + i\sqrt{2}\,\frac{\partial W}{\partial\bar\varphi}\,\bar\psi\,\bar\sigma^s\varepsilon$$
$$+ \sqrt{2}\,\bar\psi\bar\sigma^s\sigma^m\bar\varepsilon D_m\varphi - i\sqrt{2}\,\frac{\partial W}{\partial\varphi}\,\bar\varepsilon\bar\sigma^s\psi \quad \dots$$

Consider only the ε-part:

$$\left.\begin{aligned} J^s_{(1)} = {} & \sqrt{2}\,\varepsilon\sigma^m\bar\sigma^s\psi_q D_m\bar q + i\sqrt{2}\,m\,\bar{\tilde q}\,\bar\psi_q\,\bar\sigma^s\varepsilon \\ & + \sqrt{2}\,\varepsilon\sigma^m\bar\sigma^s\psi_{\tilde q} D_m\bar{\tilde q} + i\sqrt{2}\,m\,\bar q\,\bar\psi_{\tilde q}\,\bar\sigma^s\varepsilon + \dots \end{aligned}\right\}$$

The same terms in the second current will look like:

$$J^s_{(2)} = \dots + \sqrt{2}\,\varepsilon\sigma^m\bar\sigma^s\psi_q D_m\tilde q - i\sqrt{2}\,m\,q\,\bar\psi_q\,\bar\sigma^s\varepsilon$$
$$- \sqrt{2}\,\varepsilon\sigma^m\bar\sigma^s\psi_{\tilde q} D_m q + i\sqrt{2}\,m\,\tilde q\,\bar\psi_{\tilde q}\,\bar\sigma^s\varepsilon + \dots$$

It is thus easy to see that in the anticommutator of $J_{(1)}, J_{(2)}$ we will obtain a term proportional to $m_i Q_i$, where Q_i is the i-th flavor quantum number, the other terms will again contribute to build up the whole charge ρ^a which is then replaced by $D_i F^{0i(a)}$. Indeed,

$$\boxed{Z = n_e a + n_m a_D + \sum_i \frac{1}{\sqrt{2}}\, m_i Q_i}$$

This formula is crucial in the Seiberg-Witten analysis of ~~SU~~ $N=2$ $SU(2)$ with quark multiplets.

"Explicit" Formulae for $a(u)$, $a_D(u)$ $N=2$

We collect here some useful formulae relating the integral representation of a and a_D as functions of u

$$a(u) = \frac{\sqrt{2}}{\pi} \int_{-1}^{+1} dt \, \frac{\sqrt{u-t}}{\sqrt{1-t^2}} \qquad a_D(u) = \int_1^u \frac{\sqrt{u-t}}{\sqrt{1-t^2}} \, dt$$

Recall that the complete elliptic integrals are ~~represented by~~ related to hypergeometric function

$$K(k) = \frac{\pi}{2} F(1/2, 1/2, 1; k^2) \qquad k' = \sqrt{1-k^2}$$

$$E(k) = \frac{\pi}{2} F(-\tfrac{1}{2}, \tfrac{1}{2}, 1; k^2)$$

$$K'(k) = K(k') = \frac{\pi}{2} F(\tfrac{1}{2}, \tfrac{1}{2}, 1; 1-k^2)$$

$$E'(k) = E(k')$$

$$F(\alpha, \beta, \gamma; z) \equiv \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} \int_0^1 dt \, t^{\beta-1} (1-t)^{\gamma-\beta-1} (1-zt)^{-\alpha}$$

$$= \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)} \sum_{n \geq 0} \frac{\Gamma(\alpha+n)\Gamma(\beta+n)}{\Gamma(\gamma+n)} \frac{z^n}{n!}$$

Take $\gamma = 2\beta$, $t = (1+x)/2$, and let $z = 2/(1+u)$ i.e. $u = (1-z/2)/(z/2)$, then

$$F(-\tfrac{1}{2}, \tfrac{1}{2}, 1; \tfrac{2}{1+u}) = \frac{(1+u)^{-1/2}}{\pi} \int_{-1}^{+1} dt \, (1-t^2)^{-1/2} (u-t)^{1/2}, \quad \text{and}$$

$$\boxed{a(u) = \sqrt{2}\,(1+u)^{1/2} F(-1/2, 1/2, 1; \tfrac{2}{1+u})}$$

Use the identity:

$$F(\alpha, \beta, 2\beta; z) = (1-z/2)^{-\alpha} F\left(\frac{\alpha}{2}, \frac{1+\alpha}{2}, \beta + 1/2; \left(\frac{z}{2-z}\right)^2\right)$$

$$F(-1/2, 1/2, 1; \tfrac{2}{1+u}) = \left(\frac{u}{1+u}\right)^{1/2} F(-\tfrac{1}{4}, \tfrac{1}{4}, 1; u^{-2})$$

$$\boxed{a(u) = (2u)^{1/2} F(-1/4, 1/4, 1; \tfrac{1}{u^2})}$$

If $\quad k^2 = \frac{2}{1+u}$, $\quad u = \frac{2-k^2}{k^2} = \frac{1+k'^2}{1-k'^2}$

$$\boxed{a(u) = \frac{4}{\pi k} E(k)} \qquad u \to \infty,\ k \to 0$$

$k \to 0 \quad E(k) = \frac{\pi}{2}\left(1 - \frac{1}{4}k^2 + \dots\right)$

$$a(u) = \sqrt{2u}\ \frac{1}{\Gamma(-1/4)\Gamma(1/4)} \sum_{n\geq 0} \frac{\Gamma(n-1/4)\Gamma(n+1/4)}{(n!)^2} \frac{1}{u^{2n}}$$

Now $a_D(u)$. Make the change of variables $t = (u-1)\xi + 1$

$$a_D(u) = \frac{?}{\pi}(u-1) \int_0^1 d\xi\ \xi^{-1/2}(1-\xi)^{1/2}\left(1 - \frac{1-u}{2}\xi\right)^{-1/2}$$

From the values $b = a = 1/2$, $c = 2$ in F:

$$a_D(u) = \frac{i}{\pi}(u-1)\,\Gamma\left(\tfrac{1}{2}\right)\Gamma\left(\tfrac{3}{2}\right) F\left(\tfrac{1}{2}, \tfrac{1}{2}, 1; \tfrac{1-u}{2}\right)$$

$$\boxed{a_D(u) = \frac{?}{2}(u-1)\, F\left(\tfrac{1}{2}, \tfrac{1}{2}, 2; \tfrac{1-u}{2}\right)}$$

To verify the log behavior use the identity

$$F(a, 1-a; c; z) = (1-z)^{c-1}(1-2z)^{a-c} F\left(\frac{c-a}{2}, \frac{c-a+1}{2}, c; \frac{4z^2-4z}{(1-2z)^2}\right)$$

$$= (\text{in our case}) = u^{-3/2}\left(\frac{1+u}{2}\right) F\left(\tfrac{3}{4}, \tfrac{5}{4}, 2; 1 - \tfrac{1}{u^2}\right)$$

$a_D(u) = $ (A.S. page 559) $=$

$$= \frac{i}{4} u^{-3/2}(u^2-1) \frac{1}{\Gamma(3/4)\Gamma(5/4)} \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(n!)^2} \Big(2\psi(n+1) - \psi(a+n)$$

$$- \psi(b+n) - \ln \frac{1}{u^2} \Big) \frac{1}{u^{2n}}$$

The leading singularity is

$$\frac{i}{4\pi} \sqrt{u}\, 2\sqrt{2}\, 2 \ln u = \frac{i}{\pi} \sqrt{2u} \ln u$$

$$\boxed{a_D(u) = \frac{i}{4} u^{-3/2} F\left(3/4, 5/4, 2; 1 - 1/u^2\right)}$$

Use now the Kummer identity:

$$c(1-z)F(a,b,c;z) - c\,F(a, b-1, c; z) + (c-a) z F(a,b, c+1; z) = 0$$

with $c = 1$, $a = b = 1/2$, $z = (1-u)/2$. It leads to:

$$F(1/2,1/2,2;\tfrac{1-u}{2}) = \frac{4}{1-u}F(-\tfrac{1}{2},\tfrac{1}{2},1;\tfrac{1-u}{2}) - 2\,\frac{1+u}{1-u}F(\tfrac{1}{2},\tfrac{1}{2};1;\tfrac{1-u}{2})$$

Since $k^2 = \dfrac{2}{1+u}$ $\quad \dfrac{1-u}{2} = -\dfrac{k'^2}{k^2}$ and we conclude:

$$\left.\begin{aligned} a(u) &= \frac{4}{\pi k}E(k) \\ a_D(u) &= \frac{4}{i\pi}E\left(\frac{ik'}{k}\right) + \frac{4i}{\pi k^2}K\left(\frac{ik'}{k}\right) \end{aligned}\right\}$$

Using:

$$F(a,b,c;z) = (1-z)^{-a}F\left(a,c-b,c;\frac{z}{z-1}\right)$$
$$= (1-z)^{-b}F\left(b,c-a,c;\frac{z}{z-1}\right)$$

$$\left.\begin{aligned} K(ik'/k) &= kK(k') \\ E(ik'/k) &= k^{-1}E(k') \end{aligned}\right\} \quad \text{hence:}$$

$$\boxed{\begin{aligned} a_D(u) &= \frac{4}{\pi i}\,\frac{E'-K'}{k} \\ a(u) &= \frac{4}{\pi k}E(k) \end{aligned}}\ .$$

$$\left.\begin{aligned} u &= -1 & k^2 &\to \infty \\ u &= +1 & k^2 &\to 1 \\ u &\to \infty & k^2 &\to 0 \end{aligned}\right\}$$

Legendre relation:

$$EK' + E'K - KK' = \frac{\pi}{2}$$

$$a_D = a\,\frac{iK'}{K} - \frac{2i}{kK(k)}$$

To obtain a_D as a function of a we need to express $u = u(a)$. This we can do using Lagrange's method: Given the equation

$$x = y - \phi(y) \equiv F(y)$$

when $\phi = 0$ $y = x$, we want to construct $y = y(x) = x + \cdots$

$$y(x) = \oint \frac{dF(y)}{2\pi i}\,\frac{y}{F(y)-x} = \oint \frac{F'(y)\,y}{y-x-\phi(y)}$$

$$= \sum_{n\geq 0}\oint \frac{(1-\phi'(y))\,y}{(y-x)^{n+1}}\,\phi(y)^n = \text{(integrate by parts)}$$

$$= x + \sum_{n=1}^{\infty}\frac{1}{n!}\,\frac{d^{n-1}}{dy^{n-1}}\phi^n(y)\Big|_{y=x}$$

In our case write

$$\frac{a^2}{2} = u\, F^2(-1/4, 1/4, 1; 1/u^2)$$

$$x = \frac{a^2}{2} \qquad \phi = u\,(1 - F^2(-1/4, 1/4, 1; 1/u^2))$$

A useful formula can be obtained for the transformation of $\mathcal{F}(a)$.

$$\Gamma = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL_2(\mathbb{Z}). \qquad \left.\begin{array}{l} a_D^\Gamma = \alpha a_D + \beta a \\ a^\Gamma = \gamma a_D + \delta a \end{array}\right\}$$

If we want to know the functional change of $\mathcal{F}$, proceed as follows $\mathcal{F}_\Gamma(a_\Gamma)$ can be thought as a function of a:

$$\frac{\partial \mathcal{F}_\Gamma}{\partial a} = \frac{\partial \mathcal{F}_\Gamma}{\partial a_\Gamma}\frac{\partial a_\Gamma}{\partial a} = a_D^\Gamma \left(\gamma \frac{\partial a_D}{\partial a} + \delta\right) =$$

$$= (\alpha \mathcal{F}' + \beta a)(\gamma \mathcal{F}'' + \delta) \quad ; \text{ since } \alpha\delta - \beta\gamma = 1$$

$$a\mathcal{F}'' = (a\mathcal{F})'' - 2\mathcal{F}'$$

$$= \frac{\partial}{\partial a}\left(\frac{1}{2}\alpha\gamma\,\mathcal{F}'^2 + \frac{1}{2}\beta\delta a^2 + \mathcal{F} + \beta\gamma a \mathcal{F}'\right)$$

$$\boxed{\mathcal{F}_\Gamma(a) = \frac{1}{2}a^2\beta\delta + \frac{1}{2}a_D^2\,\alpha\gamma + \beta\gamma a\mathcal{F}' + \mathcal{F}}$$

1) $\Gamma = T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ $\qquad \mathcal{F}_T(a) = \frac{1}{2}a^2 + \mathcal{F}$

2) $\Gamma = S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ $\qquad \mathcal{F}_S(a) = \mathcal{F}(a) - a\mathcal{F}'$

The case with Matter $N=2$, $N_f \neq 0$

In this part there are many technical issues and properties depending on the value of N_f. To keep asymptotic freedom we require $N_f \leq 4$. For $N_f = 4$ the theory is ~~one loop~~ finite to all orders in perturbation theory and very likely it is conformally invariant non-perturbatively ~~Seiberg & Witten~~ (with $m_{quark} = 0$). Seiberg & Witten provide evidence that this theory is another example of $SL(2,\mathbb{Z})$ invariant theory. Together with pure $N=4$ they provide the only such examples in four dimensional gauge theories.

There are some issues with normalizations which are important. In pure $N=2$ (or when the particles involved transform in the adjoint representation) the charges of particles and monopoles are integers. Hence in the formula $M = \sqrt{2}|Z|$, $Z = a\, n_e + a_D n_m$ $n_e, n_m \in \mathbb{Z}$. When quarks are present n_e could be half-integral. To ensure that n_e is integral, we multiply n_e by two and divide a by two to compensate. In terms of the gauge invariant quantity $u = \operatorname{Tr}\phi^2$, as $|u| \to \infty$ the asymptotic behavior with this normalization becomes

$$a \simeq \frac{1}{2}\sqrt{2u} \qquad a_D \simeq i\frac{4}{\pi}\, a \log a\,,$$

and the effective coupling is also rescaled $\tau = \partial a_D/\partial a = \frac{\theta}{\pi} + \frac{8\pi i}{g^2}$. This rescaling affects the form of the curve which determines the solution of the model.

Since we are going to study the case of $SU(2)$, and the most interesting dynamics appear in the Coulomb branch where $SU(2)$ breaks down to $U(1)$, it is convenient to study first the case of $N=2$ QED.

$N=2$ QED

The photon multiplet is (W_α, A) in terms of $N=1$ chiral superfields. Each quark hypermultiplet consists of two $N=1$ chiral superfields M^i, $\tilde{M}_i$ with charges resp. $+1, -1$. The only renormalizable $N=2$ invariant superpotential is:

$$W = \sqrt{2}\, A\, M^i \tilde{M}_i + \sum_i m_i M^i \tilde{M}_i$$

If $m_i = 0$, the global symmetry is $SU(k) \times SU(2)_R \times U(1)_R$, $SU(2)_R$ is the symmetry rotating both SUSY's and $U(1)_R$ is the usual R-symmetry which is afflicted by ~~anom~~ an anomaly. As in the case without matter the name of the game is to first determine the classical moduli space (CMS) and to determine whether the quantum moduli space (QMS) is qualitatively different. In general, for the CMS we can have the Higgs branch ~~an~~ or the Coulomb branch. In the Higgs branch some of the M^i's acquire VEV's. In this case (both for QED and QCD) the gauge group is completely broken, and the light states live on a hyperkähler manifold. Often the symmetries of the problem constrain the system to the point that the CMS and QMS, this ~~occ~~ occurs when the hyperkähler structure is unique. In the Higgs branch there are no monopoles or dyons, and hence the ~~poss~~ dynamical possibilities are not as rich as in the Coulomb branch. In the case at ~~hand~~ hand, if all $m_i = 0$, the Coulomb branch is determined by $\langle A \rangle \neq 0$, then $\langle M^i \rangle = \langle \tilde{M}_i \rangle = 0$, ~~the~~ ~~ga~~ $U(1)$ remains unbroken and all the M's become massive. For $k = 1$ there is no Higgs phase. When $k \geq 2$, and $m_i = 0$ using the global symmetries together with the vanishing condition for the D-term it is possible to rotate M and $\tilde{M}$ to $M = (B, 0, \ldots)$ $\tilde{M} = (0, B, 0, \ldots)$, and we can read off the different patterns of symmetry breaking as a function of the number of flavors.

As mentioned in the Higgs phase there are no quantum mechanical corrections because the symmetries of the problem lead to a unique hyperkähler metric. On the Coulomb branch we can have corrections. The Kähler potential is $K = \mathrm{Im}\, a_D(a)\, \bar{a}$, and the gauge kinetic term:

$$\int d^2\theta \, \frac{\partial a_D}{\partial a} W^\alpha W_\alpha$$

The 1-loop approximation to K is exact. Using the 1-loop β-function we obtain:

$$a_D = -\frac{ik}{2\pi}\, a \log(A/\Lambda)$$

(to reproduce this formula one should be careful about the extra factors of 2 in the supersymmetric way of defining g and the normalization convention mentioned earlier). The singularity at $|a| = \Lambda/e$ is the Landau pole. The theory does not make sense in the UV-region. If masses are inserted, the singularities in the Coulomb branch can move. Since when $a = -\frac{1}{\sqrt{2}} m_i$ one electron becomes massless we have:

$$a_D = -\frac{i}{2\pi} \sum_i (a + m_i/\sqrt{2}) \ln\left(\frac{a + m_i/\sqrt{2}}{\Lambda}\right)$$

It is clear that depending on the possible equality of two or more masses one can have Higgs and Coulomb branches touching, leading to an intricate structure that the reader is welcome to explore.

We see here also a manifestation of the modified form of the central charge of the $N=2$ SUSY algebra. For pure $N=2$ $Z = n_e a + n_m a_D$, however now the electron masses are not just $\sqrt{2}|a|$, rather the i-th multiplet has mass $|\sqrt{2}a + m_i|$. As shown in the last section the $U(1)$-charge S_i of the massive hypermultiplets appear in Z:

$$Z = n_e a + n_m a_D + \sum_i S_i m_i/\sqrt{2}$$

QCD with Matter CMS.

The superpotential ~~are~~ takes the form:

$$W = \sqrt{2}\, \tilde{Q}_i \Phi Q^i + \sum_i m_i \tilde{Q}_i Q^i$$

If the gauge group is $SU(2)$ when $m_i = 0$ the global symmetry is $O(2N_f) \times SU(2)_R \times U(1)_R$. The full symmetry is $O(2N_f)$ because there is a parity symmetry $Z_2 \subset O(2N_f)$ ~~[illegible]~~ acting as

$$P: \quad Q_1 \longleftrightarrow \tilde{Q}_1$$

Note also that a $Z_2 \subset U(1)_R$ is $(-1)^F$ contained in the Lorentz group. Hence there are some non-trivial discrete quotients one should keep in mind when writing the global symmetry.

When $N_f = 0, 1$ it is easy to see that only the Coulomb branch exists ($\langle\phi\rangle \neq 0$), $SU(2) \to U(1)$, all quarks are massive, and

$U(1)_R$ is spontaneously broken because Φ has $U(1)_R$ charge 2. For $N_f \geq 2$ we can have Higgs branches. We will not analyse this branch in detail. $SU(2)$ is fully broken and the dynamics is not as rich.

Some Properties of the Quantum Theory.

We are interested only in those cases where $N_f = 0, 1, 2, 3, 4$. By counting zero modes in the presence of an instanton it is easy to see that $U(1)_R$ and ρ are anomalous. There is a discrete $\mathbb{Z}_{4(4-N_f)}$ anomaly free subgroup. The combined $U(1)_R$ and $\mathbb{Z}_2(\rho)$ symmetry action is:

$$
(*) \quad \left.\begin{aligned} W_\alpha &\to \omega W_\alpha(\omega^{-1}\theta) \\ \Phi &\to \omega^2 \Phi(\omega^{-1}\theta) \end{aligned}\right\} \quad \left.\begin{aligned} Q^1 &\to \tilde{Q}_1(\omega^{-1}\theta) \\ \tilde{Q}_1 &\to Q^1(\omega^{-1}\theta) \end{aligned}\right\} \quad \left.\begin{aligned} Q^i &\to Q^i(\omega^{-1}\theta) \\ \tilde{Q}_i &\to \tilde{Q}_i(\omega^{-1}\theta) \end{aligned}\right. \quad i \neq 1
$$

For $N_f = 0$ we do not have the quarks to cancel the anomaly. Take $N_f \neq 0$. Count zero modes in the presence of an instanton λ has 4 zero modes and so does ψ_Φ. Each fermion in $Q^i, \tilde{Q}_i$ has one zero mode, hence the simplest non-vanishing amplitude is

$$
\langle (\lambda\lambda)^2 (\psi\psi)^2 \, \psi_{\tilde{Q}_1}\psi_{Q^1} \, \psi_{\tilde{Q}_2}\psi_{Q^2} \cdots \psi_{\tilde{Q}_{N_f}}\psi_{Q^{N_f}} \rangle
$$

Since $\lambda \to \omega\lambda$, $\psi \to \omega\lambda$; $\psi_{\tilde{Q}_1}\psi_{Q^1}$ change sign under ρ and $\psi_{Q^i} \to \omega^{-1}\psi_{Q^i}$ $\psi_{\tilde{Q}_i} \to \omega^{-1}\psi_{\tilde{Q}_i}$, we have under $U(1)_R \times \mathbb{Z}_2$ that this ~~term~~ amplitude changes by a factor:

$$
\omega^4 \omega^4 \underset{\substack{\uparrow \\ \rho \text{ factor}}}{(-1)} \omega^{-2N_f} = \omega^{2(4-N_f)} \cdot (-1) = 1
$$

hence $\omega = \exp i\pi / 2(4-N_f)$. With this ω the transformation is non-anomalous. For $N_f = 0$ only the square of $(*)$ holds. Also a $\mathbb{Z}_2 \subset \mathbb{Z}_{4(4-N_f)}$ equals $(-1)^F$. Using $SU(2)_R$ transformations we find a $\mathbb{Z}_{4(4-N_f)}$ symmetry commuting with SUSY

$$
\left.\begin{aligned} &\Phi \to \omega^2 \Phi \qquad Q^1 \to \omega^{-1}\tilde{Q}_1 \qquad \tilde{Q}_1 \to \omega^{-1} Q^1 \\ &Q^i \to \omega^{-1} Q^i \qquad \tilde{Q}_i \to \omega^{-1}\tilde{Q}_i \qquad i \neq 1. \end{aligned}\right\} A
$$

Since $u = \mathrm{Tr}\,\phi^2$ is the gauge invariant order parameter, it transforms as $u \to \exp 2\pi i/(4-N_f)\, u$. The global symmetry ~~is~~ on the u-plane is $\mathbb{Z}_{4-N_f}$ $N_f > 0$, and $\mathbb{Z}_2$ for $N_f = 0$ as we saw previously. In the spirit of Seiberg's previous works we can imagine that $U(1)_R \times \mathbb{Z}_2(\rho)$ are not broken by giving appropriate charge to u and Λ_{N_f}. For instance giving charge 4 ~~to~~ and even parity to u and charge $2(4-N_f)$ ~~to~~ and odd parity to the instanton factor $\Lambda_{N_f}^{4-N_f}$ (coming from the instanton amplitude $\propto \exp -\frac{8\pi^2}{g^2}$, and the β-function to one loop). For $N_f = 4$ $U(1)_R$ is non-anomalous and u has charge 4. $\mathbb{Z}_2(\rho)$ in $O(8)$ is still anomalous, but it can be treated as unbroken by ~~giving~~ assigning odd parity to the instanton factor $\exp(-\frac{8\pi^2}{g^2} + i\theta)$.

As for $N_f = 0$, the one-loop β-function determines the ~~large~~ large u-behavior of $a(u)$, $a_D(u)$:

$$a \simeq \frac{1}{2}\sqrt{2u} + \cdots$$

$$a_D \simeq i\,\frac{4-N_f}{2\pi}\, a(u) \log \frac{u}{\Lambda_{N_f}^2} + \cdots$$

To ~~figure~~ figure out the orders at which instantons make contributions it suffices to use the previous $U(1)_R \times \mathbb{Z}_2$ assignments. Since the metric depends on u which is even under ρ-parity, odd ~~instan~~ numbers of instantons do not contribute, but there is no problem with even numbers, we thus expect:

$$a = \frac{1}{2}\sqrt{2u}\left(1 + \sum_{n\geq 1} a_n(N_f)\left(\frac{\Lambda_{N_f}^2}{u}\right)^{n(4-N_f)}\right)$$

$$a_D = i\,\frac{4-N_f}{2\pi}\, a(u)\log\frac{u}{\Lambda_{N_f}^2} + \sqrt{u}\sum_{n\geq 0} a_D^{(n)}(N_f)\left(\frac{\Lambda_{N_f}^2}{u}\right)^{n(4-N_f)}$$

The difficulty is to compute the a_n, $a_D^{(n)}$ coefficients.

One more remark has to do with the way the global symmetries act. $SU(2)$ breaks only to $U(1)$ and therefore there will be massive charge states, and we want to know how the unbroken global symmetry acts on them. If we raise the symmetry A (page 83) to the power $4-N_f$, it leads to a

$\mathbb{Z}_4$ symmetry. This symmetry changes the sign of ϕ and therefore acts as charge conjugation on the charged fields. When $N_f = 1, 3$ the transformation acts as parity on $O(2N_f)$ $\not{\mathcal{E}}$, hence parity is realized on the spectrum but reverse electric and magnetic charges; states of given charge belong to $SO(2)$ $(N_f = 1)$ or $SO(6)$ $(N_f = 3)$ multiplets. For $N_f = 2, 4$ the parity symmetry is broken and the states are only in $SO(4)$ $(N_f = 2)$ or $SO(8)$ $(N_f = 4)$ representations.

BPS Saturated States.

We need not worry about the Higgs branch because $SU(2)$ is completely broken and there are no electric or magnetic charges. On the Coulomb branch the simplest BPS saturated states ~~are~~ providing reduced hypermultiplet representations of $N = 2$ (with bare masses $= 0$) are the quarks with masses $M = \sqrt{2}|a|$. They are in the vector of $SO(2N_f)$. There are also monopole configurations in the spectrum. For $N_f \neq 0$ each quark field has a ~~zero mode~~ fermion zero mode in the monopole background. With N_f hypermultiplets there are $2N_f$ zero modes, ~~and although the~~ transforming in the vector of $SO(2N_f)$. As in the quantization of the Ramond sector of the string these zero modes turn the monopoles into spinors of $SO(2N_f)$. At the quantum level the global symmetry is $Spin(2N_f)$. One of the monopole's collective coordinates is charge rotation, upon quantization one obtains dyonic states. A 2π-rotation by the electric charge operator however is not the identity. Using the Witten effect such a rotation ~~a~~ gives a topologically non-trivial gauge transformation with eigenvalue $e^{i\theta}(-1)^H$ for a monopole with $n_m = 1$. $(-1)^H$ is the center of $SU(2)$. Since we are normalizing the charge operator so that elementary quarks have charge ± 1, we have.

$$e^{i\pi Q} = e^{i\theta n_m}(-1)^H$$

Taking into account the Witten effect and writing $Q = n_e + n_m\theta/\pi$, $n_e \in \mathbb{Z}$, this relation implies that states of even n_e have $(-1)^H = 1$ and with odd n_e $(-1)^H = -1$. $(-1)^H$ can be identified with the chirality operator for the spinor representation of $SO(2N_f)$. For $N_f = 1, 3$ the parity transformation acts on the spectrum and it guarantees that a dyon transforming as a ~~positive~~ positive chirality spinor of $SO(2N_f)$ is degenerate with a particle of opposite e- and m. charge and opposite $SO(2N_f)$ chirality. No such relation exists for $N_f = 2, 4$. For $N_f > 0$ we will see that the spectrum contains states with $n_m > 1$. A convenient way of labelling states is in terms of their behavior under the center of $Spin(2N_f)$.

a). $N_f = 2$, $Spin(4) = SU(2) \times SU(2)$ with center $\mathbb{Z}_2 \times \mathbb{Z}_2 \ni (\epsilon, \epsilon')$. $\epsilon = 0$ for vector-like ~~and~~ irreps $\epsilon = 1$ for spinor-like irreps. An elementary quark transforms as $(\epsilon, \epsilon') = (1,1)$. Multiple quark states then transform as $(n_e \bmod 2, n_e \bmod 2)$. Since monopoles ~~to~~ behave like spinors we have

$$(\epsilon, \epsilon') = ((n_e + n_m) \bmod 2 \,,\ n_e \bmod 2) \qquad \underline{N_f = 2}$$

b) $N_f = 3$ $Spin(6) = SU(4)$, ~~Et~~ with center $\mathbb{Z}_4$. The elementary quarks are in the 6 of $Spin(6)$ Thus they have charge 2 w.r.t. the center. Since monopoles behave like spinors we conclude that $\mathbb{Z}_4$ acts as:

$$\exp \frac{i\pi}{4} (n_m + 2n_e) \qquad \underline{N_f = 3}$$

c). $N_f = 4$ $Spin(8)$ has center $\mathbb{Z}_2 \times \mathbb{Z}_2$. Hence using similar arguments $(\epsilon, \epsilon') = (n_m \bmod 2, n_e \bmod 2)$. $(0,1) \equiv v$, $(1,0) \equiv s$ $(1,1) \equiv c$, $(0,0) \equiv$ adj.

If an $N = 2$ invariant mass is turned on, say $m_{N_f} \neq 0$, $SO(2N_f) \to SO(2N_f - 2) \times SO(2)$, and the global abelian charge $SO(2)$ appears in the central charge

$$M = \sqrt{2}|Z| \,; \qquad Z = n_e a + n_m a_D + S \frac{m_{N_f}}{\sqrt{2}} \,,$$

hence for $a = \pm m_{N_f}/\sqrt{2}$ one of the elementary quarks becomes massless. This will be crucial later in determining the global structure of QMS.

<u>Duality</u> As in the $N_f = 0$ case we can compute the monodromy matrices for (a_D, a) around the singular points on the QMS. The simplest case corresponds to having a single massive quark, and ~~see~~ ~~instead~~ investigate what happens

(*)

as $a \simeq a_0 \equiv m_{N_f}/\sqrt{2}$. From the QED analysis we know that there is a log singularity. Near a_0:

$$\left.\begin{aligned} a &\simeq a_0 \\ a_D &\simeq \text{const.} - \frac{i}{2\pi}(a-a_0)\ln(a-a_0) \end{aligned}\right\}$$

with monodromy:

$$\left.\begin{aligned} a &\to a \\ a_D &\to a_D + a - \frac{m_{N_f}}{\sqrt{2}} \end{aligned}\right\}$$

unlike the $N_f = 0$ case we have an inhomogeneous transformation; $\begin{pmatrix} a_D \\ a \end{pmatrix}$ picks up a shift. Including $\begin{pmatrix} a_D \\ a \end{pmatrix}$ into a three-dimensional vector $(m/\sqrt{2}, a_D, a)$ we have

$$\begin{pmatrix} m/\sqrt{2} \\ a_D \\ a \end{pmatrix} \to \mathcal{M} \begin{pmatrix} m/\sqrt{2} \\ a_D \\ a \end{pmatrix} \; ; \quad \mathcal{M} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\mathcal{M}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & -1 \\ 0 & 1 & 1 \end{pmatrix} .$$

Arranging the charges as a row vector $W = (S, n_m, n_e)$, invariance of Z (of M) implies $W \to W\mathcal{M}^{-1}$. In general:

$$\mathcal{M} = \begin{pmatrix} 1 & 0 & 0 \\ r & k & l \\ q & n & p \end{pmatrix} \qquad \mathcal{M}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ lq-pr & p & -l \\ nr-kq & -n & k \end{pmatrix}$$

$$\det \mathcal{M} = 1$$

The e- and m-charges mix under monodromy but not with S (S is related to a global symmetry); however the value of S does change. For the simple case above:

$$(S, n_m, n_e) \to (S + n_m, n_m, -n_m + n_e)$$

This leads once again to issues of marginal stability. In the regime of large m_{N_f} we are in the semiclassical region, and there the values of S are bounded. Hence although by acting with $\mathcal{M}^{-1}$ we can increase the value of S at will, but what might have started as a one-particle state comes as a multiparticle state. Thus although the formalism is $SL(2,\mathbb{Z})$ covariant the spectrum is not

Seiberg and Witten provide many compelling arguments to suggest that the theory with $N_f = 4$, like the theory with $N=4$ is $SL_2(\mathbb{Z})$ invariant, but the details are quite different. The states $(n_m, n_e) = (0,1)$ are the elementary hypermultiplets in the vector v of $Spin(8)$. The states $(n_m, n_e) = (1,0)$ are in the s and $(n_m, n_e) = (1,1)$ in the c of $Spin(8)$. They are permuted by the group of outer automorphisms S_3 of $Spin(8)$. If one is willing to assume that there are monopole bound states for every pair of relatively prime integers (p,q), then the spectrum may be $SL_2(\mathbb{Z})$ invariant. For each such pair there are eight states transforming in a representation $8_v, 8_s, 8_c$ of $Spin(8)$ depending on the mod 2 reduction of (p,q). $SL_2(\mathbb{Z})$ would be mixed with Triality. The solution S&W provide for this model gives strong support to this possibility. The global symmetry group is a semi-direct product $Spin(8) \ltimes SL(2,\mathbb{Z})$

First Look at Singularities

As in the pure $N=2$ we first try to locate the singularities in moduli space before computing their monodromy. We recall that with the standard normalization of the gauge coupling constant the one loop β-function is given by:

$$\mu \frac{d}{d\mu} g = -\frac{g^3}{16\pi^2}(4-N_f) \Rightarrow \frac{1}{\alpha_{N_f}(\mu)} = \frac{4-N_f}{2\pi} \ln \frac{\mu}{\Lambda_{N_f}}$$

This allows us to relate Λ_{N_f} for different flavors at decoupling. Imagine we decouple ~~a quark of mass~~ $N_f - N_f'$ quarks of mass m. Imposing the matching condition

$$\frac{1}{\alpha_{N_f}(m)} = \frac{1}{\alpha_{N_f'}(m)} \Rightarrow \boxed{\Lambda_{N_f'}^{4-N_f'} = m^{N_f - N_f'} \Lambda_{N_f}^{4-N_f}}$$

For instance

$$N_f = 3 \quad N_f' = 0 \qquad \Lambda_0^4 = m^3 \Lambda_3$$

$$N_f = 1 \quad N_f' = 0 \qquad \Lambda_0^4 = m \Lambda_1^3 \qquad \text{etc.}$$

If we consider $N_f \leq 3$, and bare masses very large compared to Λ the theory obtained at low energies has $N_f = 0$ and the vacuum structure is already known, then we study what happens as the masses are decreased.

$\underset{\sim}{N_f = 3}$. Start with equal masses $m_i = m \gg \Lambda$. Thus the global symmetry $SU(4) \to SU(3) \times U(1)$. Classically there is a singularity at $a = m/\sqrt{2}$. The electrically charged massless fields form a $\underline{3}$ of $SU(3)$. For $m \gg \Lambda$ the singularity is in the semiclassical region $u \sim 2a^2 \gg \Lambda^2$. For $u \ll m^2$ the three massive quarks can be integrated out, $\Lambda_0^4 = m^3 \Lambda_3$, hence for small u the moduli space is that of a pure $N=2$ with scale Λ_0 and two singularities with $(n_m, n_e) = (1,0), (1,1)$ massless monopoles. Clearly these states are $SU(3)$ invariant.

As m is decreased the singularity at large u moves, and although the charges of the associated states can change (through monodromy matrices), their non-abelian charge cannot change. Hence with respect to $SU(3)$ the massless states at the various singularities transform as $\underline{3}, \underline{1}, \underline{1}$. In the $m=0$ limit these states must combine into representations of $SU(4)$. The only possibility is to have two singularities combining into a $\underline{4}$ of $SU(4)$ and the other somewhere else remains a singlet. ~~By~~ We conclude that the massless $N_f = 3$ has two singularities with massless particles in the $\underline{4}$ and $\underline{1}$ of $SU(4)$. ~~The~~ From our study of ~~the~~ how the different states transform under the center, the ~~minimal~~ smallest charge assignments for the singularities are $(n_m, n_e) = (1,0)$ for the $\underline{4}$, and $(n_m, n_e) = (2,1)$ for the $\underline{1}$. Semiclassically the first state exists, but it is not known whether the monopole-monopole bound state implied by the second does exist.

$\underset{\sim}{N_f = 2}$. With equal masses $m_i = m \gg \Lambda$, $Spin(4) \to SO(2) \times SO(2)$. There is a singularity at $a = m/\sqrt{2}$, the massless states transform under one or the other of the two $SO(2)$'s. In the region $u \ll m^2$ again we integrate out the quarks leading to a $N_f = 0$ theory with $\Lambda_0^4 = m^2 \Lambda_2^2$, and two singularities with $(n_m, n_e) = (1,0), (1,1)$, singlets under $SO(2) \times SO(2)$. In total we have four massless states. As $m \to 0$ we recover the full ~~SO~~ $Spin(4) \simeq SU(2) \times SU(2)$ global symmetry. The states move, and since each $SO(2)$ is contained in a different $SU(2)$, the only way to combine the singularities into representation of $SU(2)\times SU(2)$ is as $(\underline{2},\underline{1})$ and $(\underline{1},\underline{2})$; hence for $m=0$ there are two singularities, and the simplest charge assignments are $(n_m, n_e)=(1,0)$ in one spinor of $SO(4)$, and $(n_m,n_e)=(1,1)$ in the other spinor. Recall that the transformation under the center is $(\epsilon,\epsilon') = ((n_m+n_e) \bmod 2,\ n_e \bmod 2)$.

$\underset{\sim}{N_f=1}$. Now the massive theory has the same $SO(2)$ symmetry as the massless theory; however the same arguments as before imply the presence of three singularities. Since in this case we have a $\mathbb{Z}_3$ symmetry acting on the u-plane, from $a_D = i\frac{4-N_f}{2\pi} a(u) \ln \frac{u}{\Lambda_{N_f}} + \dots$ we know that a transforms homogeneously, but a_D is shifted by a: $a \to \omega^{1/2} a$, $\omega = e^{-2\pi i/3}$, $a_D \to \omega^{1/2}(a_D + a)$, hence if one of the states in the $m=0$ limit has $(n_m,n_e)=(1,0)$ the $\mathbb{Z}_3$-symmetry implies that the other states have $(n_m,n_e) = (1,1),(1,2)$. Hence there are three singularities with massless states $(n_m,n_e) = (1,0),(1,1),(1,2)$.

Determining the Metric.

For $N_f=0$ the solution was given by a family of elliptic curves:

$$y^2 = (x-\Lambda^2)(x+\Lambda^2)(x-u) \qquad (f1)$$

with the monodromy in $\Gamma(2)$. In this case (n_m,n_e) are both integer. If we want to have flavors, $Z = a_D n_m + a n_e$, and we multiply n_e by 2 and divide a by two

$$\begin{pmatrix} a_D \\ a \end{pmatrix} \to \begin{pmatrix} a_D \\ a' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix}\begin{pmatrix} a_D \\ a \end{pmatrix}, \text{ the monodromy changes:}$$

$$\begin{pmatrix} m & n \\ p & q \end{pmatrix} \to \begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix}\begin{pmatrix} m & n \\ p & q \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} m & 2n \\ p/2 & q \end{pmatrix}$$

Since $p, n \equiv 0 \pmod 2$, we obtain matrices in $\Gamma_0(4)$, the subgroup of $SL_2(\mathbb{Z})$ of the form $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $b \equiv 0 \pmod 4$. In these conventions the monodromy at ∞ is $\begin{pmatrix} -1 & 4 \\ 0 & -1 \end{pmatrix} = PT^{-4}$, and the monodromy around the massless monopole $(1,0)$ is $\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$. The family of curves as will be shown is:

$$y^2 = x^3 - ux^2 + \frac{1}{4}\Lambda^4 x \qquad (f2)$$

The family (f1) is convenient if we study $N=4$ susy while (f2) is convenient for $N_f \neq 0$.

Any genus one curve can be represented as a cubic:

$$y^2 = F(x) = (x-e_1)(x-e_2)(x-e_3),$$

~~base~~ i.e. a double cover of the plane branched over e_1, e_2, e_3, ∞. The curve become singular when two branch points coincide (e.g. $e_1 = e_2 \neq e_3$ or $e_3 \to \infty$). In this case the singularity is called stable. If ~~all~~ more than two branch points coincide the singularity is not stable, but a change of variables (reparametrization of x, y) can make the situation stable again. For (f2) the branch points are

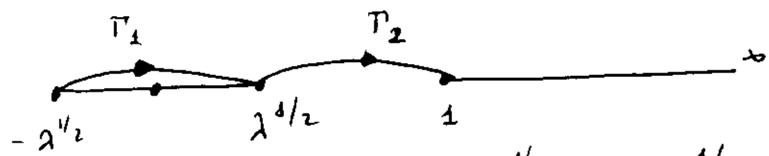
$$0, \ \frac{1}{2}\left(u \pm \sqrt{u^2 - \Lambda^4}\right), \infty$$

for $u = \pm\Lambda^2$ we have a stable singularity, but the $u \to \infty$ singularity is not stable. For ~~a~~ a stable singularity at for instance $u=0$, the family of curves ~~near~~ near $u=0$ can be written in the form

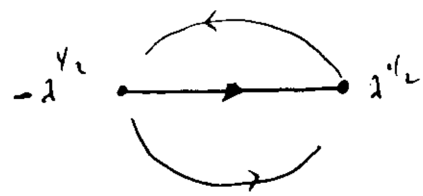
$$y^2 = (x-1)(x^2 - u^n) \qquad \text{for some } n$$

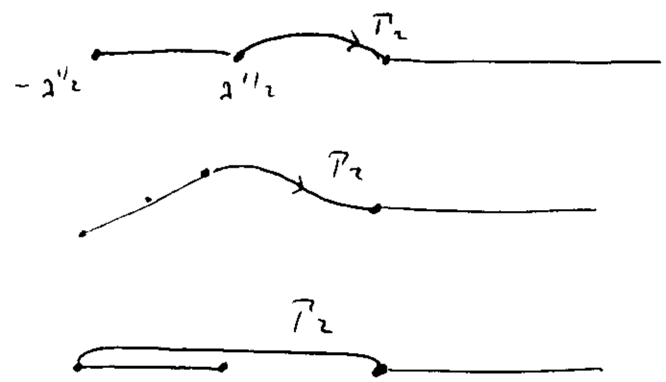
in this case the monodromy is conjugate to T^n, $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. The ~~abelian~~ holomorphic abelian differential is $\omega = dx/y$. Consider $y^2 = (x-1)(x^2 - \lambda)$ $\quad (\lambda = u^n)$

$$\omega_1 = \underset{(\Gamma_1)}{\int_{-\lambda^{1/2}}^{\lambda^{1/2}}} \frac{dx}{y} \qquad \omega_2 = \underset{(\Gamma_2)}{\int_{\lambda^{1/2}}^{1}} \frac{dx}{y} \qquad \text{are the periods}$$



$|\lambda| \ll 1$. As $\lambda \to e^{2\pi i}\lambda$ $\quad \lambda^{1/2} \to -\lambda^{1/2}$. The cut moves as in the figure, we exchange the branches of the integrand and therefore $\omega_1 \to \omega_1$. As for ω_2, the path Γ_2 is transformed as:





Thus $\omega_2 \to \omega_2 + \omega_1$. This is in terms of λ. Since $\lambda = u^n$, when $u \to e^{2\pi i} u$, λ makes n turns and we obtain the n-th power of the monodromy. In terms of λ the monodromy is just $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \Rightarrow u \to e^{2\pi i} u$ we have T^n. Given any polynomial, a useful quantity is its discriminant:

$$\Delta = \prod_{i<j} (e_i - e_j)^2 ,$$

where the e_i's are the roots. Δ can be expressed in terms of the coefficients of the polynomial. $\Delta = 0$ in our case when two branch points coincide (apart from ∞). From the example $y^2 = (x-1)(x^2 - u^n)$, $\Delta \sim u^n$ near $u = 0$ and the monodromy is conjugate to T^n. Hence in the stable case the exponent of the monodromy is the order of zero of Δ. For instance in the family (f2) Δ has a first order zero at $u = \Lambda^2$, and the monodromy is conjugate to T. Similarly for the one at $u = -\Lambda^2$. Look now at $u = \infty$. For large u the branch points are approximately at $0, \Lambda^4/4u, u, \infty$. Change variables

$$x = x' u \qquad y = y' u^{3/2} ,$$

then the branch points become $0, \Lambda^4/4u^2, 1, \infty$, and the discriminant behaves like $\Delta \sim u^{-4}$. Due to the square root of u in $y \to y'$, dx/y has monodromy PT^{-4}, because we pick and extra minus sign from the square root. As with $N_f = 0$ we can use the β-function to work out the monodromy at ∞. It is

$$M_\infty = P T^{N_f - 4}$$

The singularities at finite points in the u-plane correspond to massless magnetic monopoles with ~~charge~~ one unit of magnetic charge. The monodromy around such a point is determined by the one-loop QED ~~formula~~ β-function with k hypermultiplets. Since we have already worked out this case (we know a_D as a function of a), in this case the monodromy is conjugate to T^k, hence the monodromy around a point with k magnetic monopoles with $(n_m = 1, -n_e)$ is $(T^{n_e} S)\, T^k\, (T^{n_e} S)^{-1}$. Similarly for the $(2,1)$ state in $N_f = 3$. Altogether the monodromies are:

$$S T S^{-1} \qquad T^2 S\, T\, (T^2 S^{-1}) \quad \rightarrow \quad P T^{-4} \qquad N_f = 0$$

$$S T S^{-1} \quad T S T (T S)^{-1} \quad (T^2 S) T (T^2 S)^{-1} \rightarrow P T^{-3} \qquad N_f = 1$$

$$S T^2 S^{-1} \qquad (T S)\, T^2 (T S)^{-1} \quad \rightarrow \quad P T^{-2} \qquad N_f = 2$$

$$S T^2 S\, T\, (S T^2 S)^{-1} \qquad S T^4 S^{-1} \quad \rightarrow \quad P T^{-1} \qquad N_f = 3$$

Note that in the first three cases ($N_f = 0,1,2$), the ~~charges~~ electric charges assign to the singularities ~~...~~ differ in sign from those determined by our previous ~~or~~ arguments. This is consistent with the inherent ambiguities in relating the universal charges with the transformation properties under the center of the group. Using $S^2 = -1$, $(ST)^3 = 1$, it is easy to check that the product of the finite monodromy matrices yields M_∞.

<u>Properties of the $N_f = 0$ curve</u>.

1). The family describing the theory is $y^2 = F(x, u, \Lambda)$; F ~~cubic~~ at most cubic in x, u.

2). The cubic part in (x,u) in F is $F_0 = x^2 (x - u)$.

3). Assigning $U(1)_R$ charges $4, 2$ to u, x resp., from the instanton arguments we assign charge 2 to Λ, then F has charge 12 and y charge 6.

4). F can be written as $F = F_0 + \Lambda^4 F_1 \qquad F_1 = x/4$

Property one remains an ansatz for $N_f > 0$, it will lead to the correct monodromy matrices, and it can also be justified in part when considering the $N_f=4$ theory. Property 2) is a consequence of the fact that as $u\to\infty$ we must obtain the logarithm in a_D coming from the one loop β-function. The cubic part can be written as $(x-e_1u)(x-e_2u)(x-e_3u)$. When $u\to\infty$, ~~and~~ $|u|\gg\Lambda$ the only way to get a logarithm is if two e_i's coincide and the other is different, say $e_1=e_2\neq e_3$; then by a redefinition of x, we can bring F_0 to the form $x^2(x-u)$. Property 4) in the $N_f=0$ is a consequence of the ~~charge assignments~~ $U(1)_R$ charge assignments. Note that F has only a classical contribution plus a one instanton term: $\Lambda^4 x/4$. Now we use these properties to determine the curves for $N_f=1,2,3$.

$N_f=1$ $(m=0)$

The instanton amplitude is $\propto\Lambda_1^3$; however for $N_f\geq 1$ only even numbers of instantons contribute (Λ_1^3 is odd under ρ). Since Λ_1 has $U(1)_R$ charge 2, and F has charge 12, the only possibility is:

$$y^2=x^2(x-u)+t\Lambda_1^6 \qquad t\Lambda_1^6\equiv\tilde\Lambda_1^6$$

The constant t can be absorbed in a redefinition of Λ_1. The discriminant of this curve is

$$\Delta=\tilde\Lambda_1^6\,(4u^3-27\tilde\Lambda_1^6),$$

hence it has three zeroes whose monodromy is conjugate to T as expected from the general arguments.

$N_f=2$ $(m=0)$

Since the instanton factor is $\Lambda_{N_f}^{4-N_f}$, now it is Λ_2^2. Again without bare masses only even instantons contribute. Since Λ_2 has $U(1)_R$ charge 2, the general form of the curve is:

$$y^2=x^2(x-u)+(ax+bu)\Lambda_2^4$$

In our discussion of the singularities we expect two singularities each with two massless monopoles. Hence the monodromy at each singularity is conjugate to T^2; thus the discriminant at each finite singularity ~~sh~~ should have a double zero. This condition determines a and b. After some rescaling of Λ_2 the family of curves becomes

$$y^2 = (x^2 - \tilde{\Lambda}_2^4)(x-u)$$

with the expected $\mathbb{Z}_2$-symmetry.

$N_f = 3$ $(m=0)$

There are two singularities ~~are with ma~~ with monodromies conjugate to T^4 and T. Take the T^4 singularity to be at $u=0$ (there is no symmetry acting on the u-plane). The discriminant should have a 4-th order zero at $u=0$. This together with the instanton count (two-instanton contribute now with Λ_3^2) leads to:

$$F = a\Lambda_3^2 x^2 + bu^2 x + cux^2 + x^3$$

$b \neq 0$ (otherwise the curve is singular for all u). Requiring that the cubic part of ~~the~~ ~~or~~ F have the expected classical behavior, and after some rescaling leads to

$$y^2 = x^2(x-u) + \tilde{\Lambda}_3^2 (x-u)^2$$

This is probably a good point to stop. Using similar arguments one can determine the curves in the massive cases and also the $N_f = 4$ and $N=4$ curves, as well as the renormalization group flows from $N_f = 4$ to smaller N_f. The interested reader should now go to the papers of Seiberg and Witten to find the details. Hopefully these notes, which were just an introduction to their work will be useful. ~~a chatter~~