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SMR.858 - 3

Lecture I

SUMMER SCHOOL IN HIGH ENERGY PHYSICS AND COSMOLOGY

12 June - 28 July 1995

BASICS OF CFT, STRING COMPACTIFICATIONS, MIRROR SYMMETRY AND SPACETIME TOPOLOGY CHANGE

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Please note: These are preliminary notes intended for internal distribution only.

QUANTUM GEOMETRY

LECTURE 0:
- INTRODUCTION / OVERVIEW
- BASICS OF CFT

LECTURE 1:
- $N=2$ SUPERCONFORMAL ALGEBRA
GENERAL / PROPERTIES / EXAMPLES

LECTURE 2:
- FAMILIES of $N=2$ THEORIES, MARGINAL OPERATORS, MODULI SPACE I
- INTERRELATIONS BETWEEN $N=2$ SCFT's (LG/CY/MM), MODULI SPACE II

LECTURE 3:
- MIRROR SYMMETRY, MODULI SPACE III

LECTURE 4:
- "MILD" TOPOLOGY CHANGE (CFT)
MODULI SPACE IV

LECTURE 5:
- "DRASTIC" TOPOLOGY CHANGE (Type II STRINGS)
MODULI SPACE V

I.2) Quantum Geometry in String Theory

- Spacetime = $M_4 \times K_6$

Physical Observables "="

$$\int_{K_6} (\text{integrand}) \quad + \quad \text{additional terms} \quad \left(\text{which vanish as } \frac{\alpha'}{R_{K_6}^2} \to 0\right)$$

$$\begin{array}{ccc}
\int_{K_6}(\text{integrand}) & & \text{additional terms} \\
\uparrow & & \uparrow \\
\text{homotopically trivial configurations} & & \text{homotopically non-trivial} \\
\uparrow & & \uparrow \\
\text{point particle contribution} & & \text{"stringy" corrections} \\
\uparrow & & \uparrow \\
\text{classical geometry} & & \text{stringy corrections} \\
& \searrow \quad \swarrow & \\
& \text{quantum (stringy) geometry.} &
\end{array}$$

MAIN RESULTS TO BE COVERED

i) MIRROR MANIFOLDS :

- $M_4 \times K_6$
- $M_4 \times \tilde{K}_6$

$\longrightarrow$ IDENTICAL STRING PHYSICS

- NLσM ON K_6
- NLσM ON $\tilde{K}_6$

$\longrightarrow$ IDENTICAL N=2 SCFT's

"QUANTUM GEOMETRY"

ii) SPACETIME TOPOLOGY CHANGE :

$$M_4 \times K_6(\lambda) \quad \begin{matrix} \nearrow K_6(0) \\ \searrow K_6(1) \end{matrix}$$

DIFFERENT TOPOLOGY YET PHYSICALLY SMOOTH TRANSITION

TRULY "STRINGY" PHENOMENA.

- "MILD" TOPOLOGY CHANGE: (WITTEN; Aspinwall, BRG, Morrison)

  * $K_6(0)$ & $K_6(1)$
    - HAVE SAME # "holes" (HOMOLOGY)
    - DIFFER IN MORE SUBTLE TOPOLOGICAL INVARIANTS
    - CFT PHENOMENON (STRING TREE LEVEL)

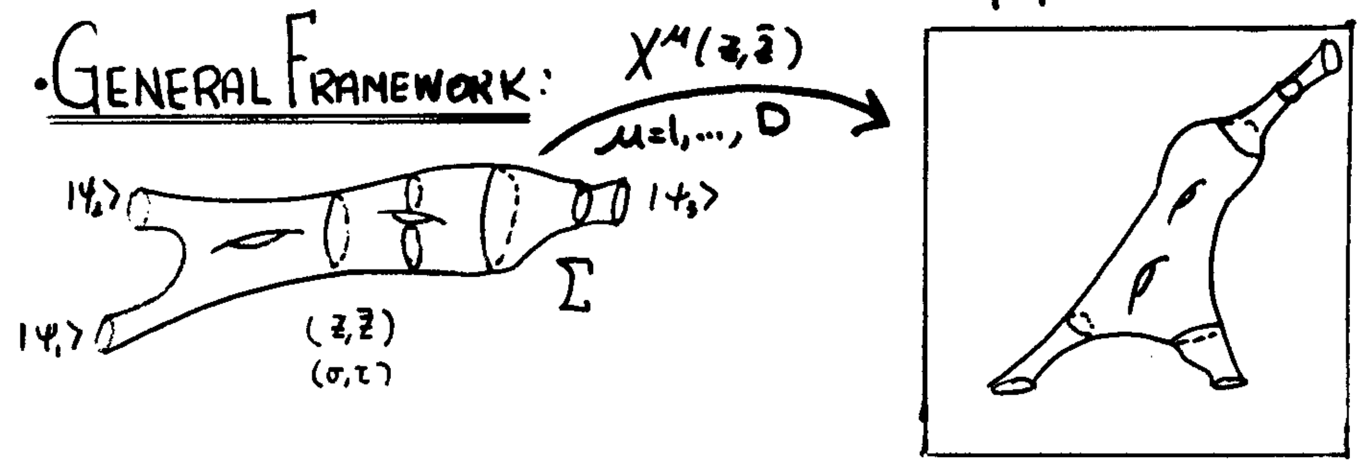
- "DRASTIC" TOPOLOGY CHANGE: (BRG, Morrison, Strominger)

  * $K_6(0)$ & $K_6(1)$
    - HAVE DIFFERENT # "holes" (HOMOLOGY DIFFERS)
    - REQUIRES NON-PERT. QUANTUM EFFECTS IN FORM OF BLACK HOLE "CONDENSATION"

0.1) String Theory

• **Motivation**: Quantum gravity ; Unified theory; fully predictive.

• **General Framework**:



• **Action**:

$$S = \frac{1}{4\pi\alpha'}\int_\Sigma d\sigma\, d\tau \sqrt{h}\, h^{ab}\,\partial_a X^\mu \partial_b X^\nu G^M_{\mu\nu}(X) + \cdots$$

DIMENSION OF AREA

• **Observables** (string scattering amplitudes):

$$A_k(\psi_1,\dots,\psi_k) = \int \mathcal{D}h \int_{\{\psi_i\}} \mathcal{D}X\, e^{-S}$$

- NOTE: TWO INDEPENDENT EXPANSIONS
  - GENUS : "QUANTUM CONTRIBUTIONS"
  - α' : "σ-MODEL EXPANSION"

- **CRUCIAL SYMMETRIES OF S**

  - TWO-DIMENSIONAL REPARAMETRIZATIONS:

$$\xi^a \to \xi^a + V^a(\xi) \qquad \xi^a = (\tau, \sigma)$$

$$\delta h_{ab} = \nabla_a V_b + \nabla_b V_a .$$

$$\delta X^\mu = X^\mu(\xi^a + V^a) - X^\mu(\xi^a) = V^a \nabla_a X^\mu$$

$g \to \delta_{z\bar{z}}$.

  - WEYL TRANSFORMATIONS :

$$\delta h_{ab} = -\,\delta\phi(\xi)\, h_{ab} \quad , \text{i.e.}$$

$$h_{ab} \mapsto e^{\phi(\xi)} h_{ab}$$

- COMBINING: IF $V^a(\xi)$ IS S.T. $\delta h_{ab} = \delta\phi(\xi) h_{ab} \Rightarrow$

* S INVARIANT UNDER $\delta\xi^a = V^a$; $\delta X^\mu = V^a \nabla_a X^\mu$, $\delta h_{ab} = 0$.

- SUCH $V^a(\xi)$ CALLED: CONFORMAL REPARAMETRIZATION
- COMBINED ACTION: CONFORMAL TRANSFORMATION

- EXPLICITLY: • $(\xi^a \; {}_{a=1,2}) \longmapsto (z, \bar{z})$
  - THEN: CONF. REPARAM. ARE:

$$\left.\begin{array}{l} z \longmapsto z + V(z) \\ \bar{z} \longmapsto \bar{z} + \bar{V}(\bar{z}) \end{array}\right\} \text{INFINITE DIMENSIONAL GROUP OF SYMMETRIES}$$

$$ds^2 = g_{z\bar{z}}\, dz\, d\bar{z} \longmapsto \left(\frac{\partial f}{\partial z}\right)\left(\frac{\partial \bar{f}}{\partial \bar{z}}\right)\left(g_{z\bar{z}}\, dz\, d\bar{z}\right)$$

$$f = z + V(z)$$

- S IS AN EXAMPLE OF A 2-D CONFORMAL FIELD THEORY

REMARKS: 1) IMPORTANCE OF SYMMETRY FOR STRING THEORY:
- ELIMINATE NEGATIVE NORM STATES

2) MORE GENERAL: YIELDS POWERFUL TOOL FOR ANALYZING THESE FIELD THEORIES

3) QUANTUM VIOLATION OF SYMMETRY OF CONFORMAL TYPE

0.2) Conformal Field Theory

A) Quantum Field Theory: (on flat base space)

Goal: $\langle \mathcal{O}_{i_1}(x_1) \cdots \mathcal{O}_{i_n}(x_n) \rangle = \int \mathcal{D}[\text{fields}]\, \mathcal{O}_{i_1}(x_1) \cdots \mathcal{O}_{i_n}(x_n)\, e^{-S}$

Say: $S = S(h_{\alpha\beta}, \phi_i)$

$\mathcal{O}_{i_j} = \mathcal{O}_{i_j}(\phi, \partial\phi, \ldots)$

B) Symmetries $\longmapsto$ Ward Identities

- Coordinate Invariance:

$$\langle \mathcal{O}_{i_1}(x_1) \cdots \mathcal{O}_{i_n}(x_n) \rangle = \int \mathcal{D}[\phi]\, \mathcal{O}_{i_1} \cdots \mathcal{O}_{i_n}\, e^{-S[\phi, h]}$$

$$= \int \mathcal{D}[\phi']\, \mathcal{O}'_{i_1} \cdots \mathcal{O}'_{i_n}\, e^{-S[\phi', h]}$$

Now: Take $x \to x' = x + \delta x$; $h \to h' = h + \delta h$, $\phi \to \phi' = \phi + \delta\phi$

associated to coordinate transform:

$$= \int \mathcal{D}[\phi]\, \mathcal{O}'_{i_1} \cdots \mathcal{O}'_{i_n}\, e^{-S[\phi', h]}$$

$$= \int \mathcal{D}[\phi]\, (\mathcal{O}_{i_1} + \delta\mathcal{O}_{i_1}) \cdots (\mathcal{O}_{i_n} + \delta\mathcal{O}_{i_n})\, e^{-S[\phi', h' - \delta h]}$$

NOW: COORDINATE INVARIANCE of $S \Rightarrow$

$$S[\phi', h'] = S[\phi, h] \Rightarrow$$

$$S[\phi', h' - \delta h] = S[\phi', h'] - \int \delta h_{ab} \frac{\delta S}{\delta h_{ab}}$$

$$= S[\phi, h] - \int \delta h \, \delta S / \delta h$$

THUS:

$$\langle \theta_{i_1}(x_1) \cdots \theta_{i_n}(x_n) \rangle = \int \mathcal{D}[\phi] \, \theta_{i_1} \cdots \theta_{i_n} e^{-S[\phi, h]}$$

$$= \int \mathcal{D}[\phi] \, (\theta_{i_1} + \delta\theta_{i_1}) \cdots (\theta_{i_n} + \delta\theta_{i_n}) \, e^{-S[\phi, h] + \int \delta h_{ab} \frac{\delta S}{\delta h_{ab}}}$$

EXPANDING TO LOWEST ORDER $\Rightarrow$

$$0 = \sum_j \int \mathcal{D}[\phi] \, \theta_{i_1} \cdots \delta\theta_{i_j} \cdots \theta_{i_n} e^{-S[\phi, h]} +$$

$$+ \int \mathcal{D}[\phi] \, \theta_{i_1} \cdots \theta_{i_n} \left(\int \delta h_{ab} \frac{\delta S}{\delta h_{ab}} \right) e^{-S[\phi, h]}$$

DEF'N: $T^{\alpha\beta}(x) = 2 \, \delta S / \delta h_{\alpha\beta}(x)$: ENERGY-MOMENTUM TENSOR

$$\Rightarrow \sum_j \langle \theta_{i_1} \cdots \delta\theta_{i_j} \cdots \theta_{i_n} \rangle = -\tfrac{1}{2} \int dx \, \delta h_{ab}(x) \langle T_{ab}(x) \, \theta_{i_1} \cdots \theta_{i_n} \rangle$$

- If $h_{ab}(x) = \delta_{ab} \overset{\text{PLANE}}{\Rightarrow} \delta h_{ab} = \partial_a \varepsilon_b(x) + \partial_b \varepsilon_a(x) \Rightarrow$

$$\sum_j \langle \theta_{i_1}(x_1) \cdots \delta\theta_{i_j} \cdots \theta_{i_n} \rangle = -\int d^2x \; \partial_a \varepsilon_b(x) \langle T^{ab}(x)\, \theta_{i_1}(x_1) \cdots \theta_{i_n}(x_n) \rangle$$

(NB: IBP's $\Rightarrow$ T CONSERVED AWAY FROM OPERATOR INSERTIONS)

- SPECIAL CASE:

$$\varepsilon^\alpha = \delta x^\alpha \equiv \lambda x^\alpha \Rightarrow$$

$$\sum \langle \theta_{i_1}(x_1) \cdots \delta\theta_{i_j}(x_j) \cdots \theta_{i_n}(x_n) \rangle = -\lambda \int d^2x \left\langle T^a{}_a(x)\, \theta_{i_1} \cdots \theta_{i_n} \right\rangle$$

- SO MUCH FOR COORD. INVARIANCE. NOW: ADD ASSUMPTION OF WEYL INVARIANCE:

INVARIANCE UNDER $\delta h_{ab} = -(\delta\gamma) h_{ab}$, i.e. $\frac{\delta S}{\delta h_{ab}}(-\delta\gamma h_{ab}) = 0$

i.e. $T^{ab} h_{ab} = 0$, i.e. $T^a{}_a(x) = 0$. $\Rightarrow$

$$\sum_j \langle \theta_{i_1}(x_1) \cdots \delta\theta_{i_j}(x_j) \cdots \theta_{i_n}(x_n) \rangle = 0 \qquad \text{i.e.}$$

CORRELATION FCNS SCALE-INVARIANT.

INF. VERSION OF $\langle \theta_{i_1}(x_1) \cdots \theta_{i_n}(x_n) \rangle = \langle \theta'_{i_1}(x'_1) \cdots \theta'_{i_n}(x'_n) \rangle$ FOR CONF. [illegible]

- CAN GET STRONGER CONSTRAINTS:

  - WORK IN COMPLEX COORDINATES $\begin{cases} z = x^1 + i x^2 \\ \bar{z} = x^1 - i x^2 \end{cases}$

$$\Rightarrow \{T_{ab}\} = \begin{cases} T_{zz} = T_{11} - T_{22} + 2i\, T_{12} \\ T_{\bar{z}\bar{z}} = T_{11} - T_{22} - 2i\, T_{12} \\ T_{z\bar{z}} = T_{11} + T_{22} \end{cases}$$

(WEYL INVARIANCE) • T TRACELESS $\Rightarrow$ $T_{z\bar{z}} = 0$

- CONSERVATION $\Rightarrow \partial^a T_{ab} = 0 \Rightarrow \partial_{\bar{z}} T_{zz} + \partial_z T_{\bar{z}z} = 0$

$\Rightarrow \partial_{\bar{z}} T_{zz} = 0$

$\Rightarrow$ T_{zz} IS ANALYTIC (INSIDE CORRELATION FCNS AWAY FROM COINCIDENT PTS)

($\bar{T}_{\bar{z}\bar{z}}$: ANTIANALYTIC)

- CONSIDER WARD IDENTITY FOR: $z \to z' = z + \varepsilon(z) = f(z)$

$\bar{z} \to \bar{z}' = \bar{z} + \bar{\varepsilon}(\bar{z}) = \bar{f}(\bar{z})$

& ASSUMPTION of WEYL INVARIANCE $T_{z\bar{z}} = 0$.

WHY? $ds^2 = dz\, d\bar{z} \longrightarrow |\partial f/\partial z|^2\, dz\, d\bar{z}$

VARIATION: $z_j \to z_j + \varepsilon(z_j)$

$$\Rightarrow \sum_{j=1}^{n} \langle \mathcal{O}_{i_1}(z_1,\bar{z}_1) \cdots \hat{\delta}\mathcal{O}_{i_j}(z_j,\bar{z}_j) \cdots \mathcal{O}_{i_n}(z_n,\bar{z}_n) \rangle$$

$$= -\int d^2z\, \partial_{\bar{z}} \varepsilon(z) \langle T_{zz} \mathcal{O}_{i_1} \cdots \mathcal{O}_{i_n} \rangle$$

- TAKE $\varepsilon(z) = 1/(z-w)$. THEN $\partial_{\bar{z}}(1/(z-w)) = \delta^{(2)}(z-w) \Rightarrow$

$$\boxed{\sum_{j=1}^{n} \langle \mathcal{O}_{i_1}(z_1,\bar{z}_1) \cdots \hat{\delta}\mathcal{O}_{i_j}(z_j,\bar{z}_j) \cdots \mathcal{O}_{i_n}(z_n,\bar{z}_n) \rangle = \langle T_{ww}(w)\, \mathcal{O}_{i_1} \cdots \mathcal{O}_{i_n} \rangle}$$

VARIATION $z_j \to z_j + 1/(z_j - w)$.

- THESE ARE POWERFUL CONSTRAINTS ON CORR. FCNS.
- ALSO ALLOWS USE OF COMPLEX ANALYSIS TO DEAL WITH CFT:

- Example: CAUCHY - $f(z) = \oint_{z} \frac{dw}{2\pi i} f(w) \frac{1}{w-z}$ (No poles of f)

$$\Rightarrow \langle \mathcal{O}_{i_1}(z_1,\bar{z}_1) \cdots \delta\mathcal{O}_{i_j}(z_j,\bar{z}_j) \cdots \mathcal{O}_{i_n}(z_n,\bar{z}_n) \rangle = \oint_{z_j} \frac{dw}{2\pi i} \varepsilon(w) \langle T(w) \mathcal{O}_{i_1} \cdots \mathcal{O}_{i_n} \rangle$$

WITH $\delta\mathcal{O}_{i_j}(z_j,\bar{z}_j) = \oint_{z_j} \frac{dw}{2\pi i} \varepsilon(w) \hat{\delta}\, \mathcal{O}_{i_j} \underset{\text{FROM ABOVE}}{=} \oint_{z_j} \frac{dw}{2\pi i} \varepsilon(w)\, T(w)\, \mathcal{O}_{i_j}(z_j,\bar{z}_j)$

- OPERATOR PRODUCT EXPANSION

GENERAL: $A(x)B(y) \sim \sum_i C_i(x-y)\, \mathcal{O}_i(y)$

COMPLETE SET OF LOCAL OPERATORS

(SINGULAR AS $x \to y$)

- PRIMARY FIELDS:

$\Phi(w,\bar{w})$ IS A PRIMARY FIELD WITH CONFORMAL WEIGHTS $(h, \bar{h})$ IF:

$$T(z)\,\Phi(w,\bar{w}) \sim \frac{h}{(z-w)^2}\Phi(w,\bar{w}) + \frac{1}{(z-w)}\partial_w \Phi(w,\bar{w}) + \cdots$$

$$\bar{T}(\bar{z})\,\Phi(w,\bar{w}) \sim \frac{\bar{h}}{(\bar{z}-\bar{w})^2}\Phi(w,\bar{w}) + \frac{1}{(\bar{z}-\bar{w})}\partial_{\bar{w}} \Phi(w,\bar{w}) + \cdots$$

- If Φ is primary $\Rightarrow$ Under $z \to z + \epsilon(z)$ we have

$$\delta\Phi(z,\bar{z}) = \oint_{z} \epsilon(\omega)\frac{d\omega}{2\pi i}\left[\frac{h}{(\omega-z)^2}\Phi(\omega,\bar{\omega}) + \frac{1}{(\omega-z)}\partial_\omega\Phi(\omega,\bar{\omega}) + \cdots\right]$$

Write: $\epsilon(\omega) = \epsilon(z) + (\omega - z)\partial_z\epsilon$

$$\Rightarrow \delta\Phi(z,\bar{z}) = \int_{z}\frac{d\omega}{2\pi i}\left[\frac{\partial_z\epsilon\cdot h}{\omega-z}\Phi + \frac{\epsilon(z)\partial_\omega\Phi}{\omega-z}\right]$$

$$= (h\,\partial_z\epsilon)\Phi(z,\bar{z}) + \epsilon(z)\partial_z\Phi(z,\bar{z})$$

- This is an infinitesimal version of

$$\Phi(z,\bar{z}) \longmapsto \left(\frac{\partial f}{\partial z}\right)^h\left(\frac{\partial\bar{f}}{\partial\bar{z}}\right)^{\bar{h}}\Phi(f(z),\bar{f}(\bar{z}))$$

under $z \longmapsto f(z)$; $\bar{z} \to \bar{f}(\bar{z})$.

- Conformal Ward Identities

$$\langle\Phi_1(z_1,\bar{z}_1)\cdots\Phi_n(z_n,\bar{z}_n)\rangle = \langle\Phi_1'(z_1',\bar{z}_1')\cdots\Phi_n'(z_n',\bar{z}_n')\rangle$$

$$= \left(\frac{\partial z_1'}{\partial z_1}\right)^{h_1}\left(\frac{\partial\bar{z}_1'}{\partial\bar{z}_1}\right)^{\bar{h}_1}\cdots\left(\frac{\partial z_n'}{\partial z_n}\right)^{h_n}\left(\frac{\partial\bar{z}_n'}{\partial\bar{z}_n}\right)^{\bar{h}_n}\cdot\langle\Phi_1(z_1',\bar{z}_1')\cdots\Phi_n(z_n',\bar{z}_n')\rangle$$

- Eg: $n=2$ $\langle\Phi_1\Phi_2\rangle = C_{12}/(z_1-z_2)^{2h}(\bar{z}_1-\bar{z}_2)^{2\bar{h}}$
  - $n=3$ $\langle\Phi_1\Phi_2\Phi_3\rangle = C_{123}\cdot 1/z_{12}^{h_1+h_2-h_3}\,z_{23}^{h_2+h_3-h_1}\,z_{13}^{h_3+h_1-h_2}\cdot 1/\text{BARRED}$

12.25
13.1
13.2
13.3

• Descendent Fields, Virasoro Algebra, Representation Theory

• Have seen: $T(\omega)\,\Phi(z,\bar z) = \frac{h}{(\omega-z)^2}\Phi(z,\bar z) + \frac{1}{(\omega-z)}\partial_z\Phi + \cdots$

• What's in ...?

• Nonsingular Terms

• $T(\omega)\,\Phi(z,\bar z) = \frac{h}{(\omega-z)^2}\Phi(z,\bar z) + \frac{1}{(\omega-z)}\partial_z\Phi +$

$$\mathcal{L}_{-2}\Phi(z,\bar z) + (\omega-z)\mathcal{L}_{-3}\Phi + (\omega-z)^2\mathcal{L}_{4}\Phi + \cdots$$

• $\mathcal{L}_{-n}\Phi(z,\bar z)$ = Descendent Fields

• $\mathcal{L}_{-n}\Phi(z,\bar z) = \oint \frac{d\omega}{2\pi i}(\omega-z)^{-n+1}\,T(\omega)\Phi(z,\bar z)$

$\Rightarrow\ \mathcal{L}_0\Phi(z,\bar z) = h\Phi(z,\bar z)$; $\mathcal{L}_{-1}\Phi(z,\bar z) = \partial_z\Phi(z,\bar z)$

$\mathcal{L}_{-2}\Phi(z,\bar z)$ = new field, ...

• Dynamics of Descendents:

$$\langle \Phi_{i_1}(z_1,\bar z_1)\cdots\ (\mathcal{L}_{-n}\Phi_i(z_i,\bar z_i))\cdots\Phi_{i_n}(z_n\bar z_n)\rangle$$

$$= \langle \Phi_{i_1}(z_1\bar z_1)\cdots\ \oint\frac{d\omega}{2\pi i}(\omega-z_i)^{-n+1}\,T(\omega)\Phi_i(z_i,\bar z_i))\cdots\Phi_{i_n}(z_n\bar z_n)\rangle$$

$$= \oint\frac{d\omega}{2\pi i}(\omega-z_i)^{-n+1}\left\langle T(\omega)\,\Phi_{i_1}(z_1\bar z_1)\cdots\Phi_i(z_i\bar z_i)\cdots\Phi_{i_n}(z_n\bar z_n)\right\rangle$$

$$= \oint\frac{d\omega}{2\pi i}(\omega-z_i)^{-n+1}\sum_j\langle\Phi_{i_1}\cdots\,\delta\Phi_{i_j}\cdots\Phi_{i_n}\rangle$$

$$\hookrightarrow = \left[h_{i_j}/(\omega-z_j)^2 + 1/(\omega-z_j)\,\partial_{z_j}\right]$$

EXPLICITLY:

EXPAND IN POWERS OF $(\omega - z_i)$; KEEP simple poles:

$$\bullet\,(\omega - z_i)^{-n+1}\left[\frac{h_{i_j}}{(\omega - z_j)^2} + \frac{1}{(\omega - z_j)}\partial_{z_j}\right]$$

USE $$1/(\omega - z_j) = \frac{1}{\omega - z_i + z_i - z_j} = \frac{1}{(z_j - z_i)}\sum_{m=0}\left(\frac{\omega - z_i}{z_j - z_i}\right)^m \Rightarrow$$

$$\bullet\,(\omega - z_i)^{-n+1}\left[(n-1)\,h_{i_j}\,\frac{(\omega - z_i)^{n-2}}{(z_j - z_i)^n} + \frac{1}{(z_j - z_i)^{n-1}}(\omega - z_i)^{n-2}\partial_{z_j}\right]$$

$$\Rightarrow \left\langle \Phi_{i_1}(z_1\bar{z}_1)\cdots(\mathcal{L}_{-n}\Phi_i(z_i,\bar{z}_i))\cdots\Phi_{i_n}(z_n\bar{z}_n)\right\rangle =$$

7.2) $$\left(\sum_{j\neq i}\frac{(n-1)h_{i_j}}{(z_j - z_i)^n} + \frac{1}{(z_j - z_i)^{n-1}}\cdot\partial_{z_j}\right)\left\langle\Phi_{i_1}(z_1\bar{z}_1)\cdots\Phi_i(z_i\bar{z}_i)\cdots\Phi_{i_n}(z_n\bar{z}_n)\right\rangle$$

DESCENDENTS OF DESCENDENTS..

$$\mathcal{L}_{-m}\mathcal{L}_{-n}\Phi(z,\bar{z}) = \oint\frac{d\omega}{2\pi i}\oint\frac{d\omega'}{2\pi i}(\omega - z)^{-m+1}(\omega' - z)^{-n+1}\,T(\omega)T(\omega')\Phi(z,\bar{z}),$$

$$\bullet\bullet\bullet$$

* <u>CONFORMAL FAMILY</u>: $\{\underset{\text{PRIMARY}}{\Phi(z,\bar{z})}, \mathcal{L}_{-n}\Phi(z,\bar{z}), \mathcal{L}_{-m}\mathcal{L}_{-n}\Phi, \mathcal{L}_{-n_1}\cdots\mathcal{L}_{-n_j}\Phi, \ldots$

* ¿WHAT IS $T(\omega)\cdot T(z)$?

• IF T WERE PRIMARY:

$$T(\omega)\cdot T(z) = \frac{c/2}{(\omega-z)^4} + \frac{2}{(\omega-z)^2}T(z) + \frac{1}{(\omega-z)}\partial_z T(z) + \mathcal{L}_2 T(z)\cdots$$

• BUT: CONSIDER $\mathcal{L}_{-2}(\mathbb{I}) = \oint \frac{d\omega}{2\pi i}(\omega-z)^{-1}\, T(\omega)\cdot I = T(z)$

• T IS NOT PRIMARY ⇒ ADDITIONAL TERM
SYMMETRIES ⇒ ABOVE FORM

• C = "CENTRAL CHARGE". VALUE DEPENDS ON THEORY.
ROUGHLY: C COUNTS DEGREES OF FREEDOM

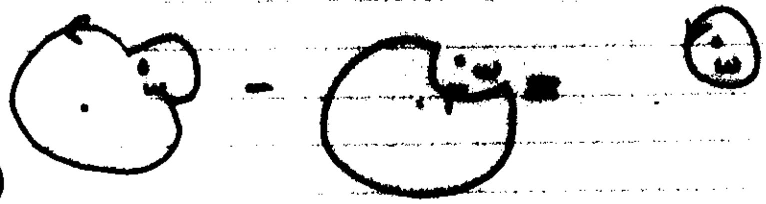
• MODE EXPANSIONS

CONVENIENT: $T(z) = \sum_{n\in\mathbb{Z}} z^{-n-2} L_n$ $\left(\Rightarrow L_n = \oint \frac{dz}{2\pi i} z^{n+1}\, T(z)\right)$

OPE of $T(\omega)\cdot T(z)$ ⇒ COMMUTATION RELATIONS ON $\{L_n\}$:

$$[L_n, L_m] = \left(\oint\frac{dz}{2\pi i}\oint\frac{d\omega}{2\pi i} - \oint\frac{d\omega}{2\pi i}\oint\frac{dz}{2\pi i}\right) z^{n+1}\,T(z)\,\omega^{m+1}\,T(\omega)$$

z- CONTOUR FOR EACH ω:



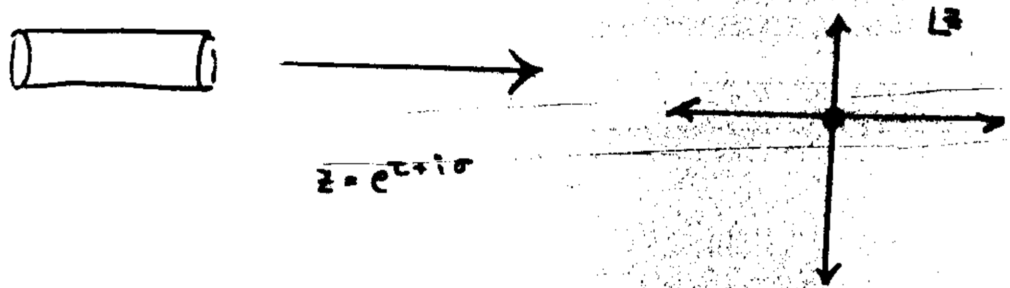
WHY?
TOTAL CONTOUR:

$$\Rightarrow [L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{m+n,0} \quad \left(\begin{matrix} T\to\bar T \\ L\to\bar L\end{matrix}\right)$$

VIRASORO ALGEBRA

FIELDS / STATES & HIGHEST WEIGHT REPRESENTATIONS

FROM



→ CONFORMAL FIELDS AT $z=0$ CREATE ASYMPTOTIC "IN" STATES

$$\Rightarrow |\phi\rangle = \phi(0)|0\rangle$$

$$\Rightarrow \underline{\underline{\mathcal{L}_{-n}\phi(0)|0\rangle}} = \oint \frac{d\omega}{2\pi i}\, \omega^{-n+1}\, T(\omega)\,\phi(0)|0\rangle$$

$$= \oint \frac{d\omega}{2\pi i}\, \omega^{-n+1} \sum \omega^{-m-2} L_m\, \phi(0)|0\rangle$$

$$= \oint \frac{d\omega}{2\pi i}\, \frac{1}{\omega} L_{-n}\, |\phi\rangle = \underline{\underline{L_{-n}|\phi\rangle}}$$

$n \geq 0$ • ϕ HAS WT $h \Rightarrow \oint \frac{d\omega}{2\pi i}\,\omega^{n+1}\, T(\omega)\phi(z) = \oint \frac{d\omega}{2\pi i}\,\omega^{n+1}\left[\frac{h}{(\omega-z)^2} + \frac{1}{(\omega-z)}\partial_z\right]\phi$

WRITING $\omega^{n+1} = z^{n+1} + (n+1)z^n(\omega - z) + \cdots \Rightarrow$

$$\mathcal{L}_n\,\phi(0)|0\rangle = L_n|\phi\rangle = \lim_{z\to 0}\left[\left(h(n+1)z^n\phi(z) + z^{n+1}\partial\phi(z)\right)|0\rangle\right]$$

FOR $\left.\begin{array}{l} n=0 : L_0|\phi\rangle = h|\phi\rangle \\ n>0 : L_n|\phi\rangle = 0 \end{array}\right\}$ $|\phi\rangle$ IS "HIGHEST WT STATE"

<u>DESCENDENT STATES</u>: $\{\underbrace{|\phi\rangle = \phi(0)|0\rangle}_{\text{HIGHEST WT}},\ \underbrace{\overbrace{L_{-n_1}\cdots L_{-n_j}}^{\text{CREATION OPERATORS}}|\phi\rangle}_{\text{DESCENDENTS}}\}$

← HIGHEST WT REPRESENTATION →

- **IN FACT**: IF $L_0|\phi\rangle = h|\phi\rangle \Rightarrow$

$$\underline{L_0[L_{-n}|\phi\rangle]} = L_{-n}L_0|\phi\rangle + [L_0, L_{-n}]|\phi\rangle$$
$$= hL_{-n}|\phi\rangle + nL_{-n}|\phi\rangle$$
$$= \underline{(h+n)(L_{-n}|\phi\rangle)}$$

- HENCE: $L_n \begin{cases} n<0 & \text{RAISING OPS} \\ n>0 & \text{LOWERING OPS} \\ n=0 & \text{CARTAN ELEMENT: LABEL REPRESENTATIONS} \end{cases}$

THUS: 1) HILBERT SPACE IS ORGANIZED INTO HIGHEST WT REPS OF VIRASORO CHARACTERIZED BY c & h_i, h_i = VARIOUS L_0 EIGENVALUES of HW STATES.

2) HW STATES ASSOCIATED TO PRIMARY FIELDS OF WT h_i

$$|\phi\rangle = \phi(0)|0\rangle$$

3) CORR FCNS ALL DETERMINED BY THOSE of PRIMARIES

4) INCLUDING $\bar{z}$ PART OF THEORY:

$$\mathcal{H} = \bigoplus_{i,j}\left(\mathcal{H}_i \otimes \bar{\mathcal{H}}_j\right), \quad \begin{aligned} \mathcal{H}_i &= \text{HW REP BASED ON } \phi_i \\ \bar{\mathcal{H}}_j &= \quad \text{''} \qquad\qquad \text{''} \quad \bar{\phi}_j \end{aligned}$$

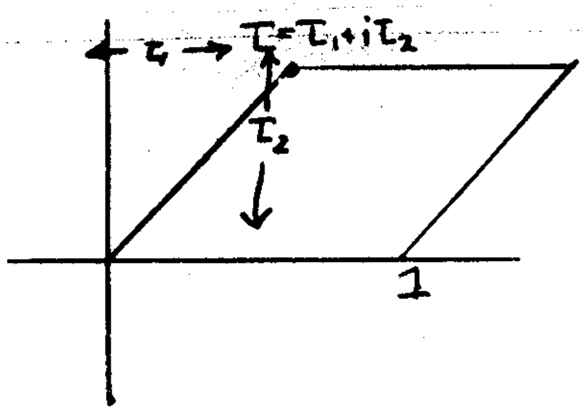
CHARACTERS & MODULAR INVARIANCE

- ENCODE SPECTRUM OF A THEORY:

$$\underline{\underline{Z}} = \sum_{\text{ALL STATES}} q^{L_0 \text{ of STATE}} \, \bar{q}^{\bar{L}_0 \text{ of state}}$$

$$= \sum_{(h_i, \bar{h}_j)} \chi_{h_i} \chi_{\bar{h}_j} \; ; \quad \chi_{h_i} = \text{"CHARACTER"} = \sum_{\substack{\text{conformal} \\ \text{family } \Phi_{h_i}}} q^{L_0}$$

- CONSIDER CFT ON TORUS:



THEN: $\int e^{-S} = \mathrm{Tr}\, e^{2\pi i \tau_1 P} e^{-2\pi \tau_2 H}$

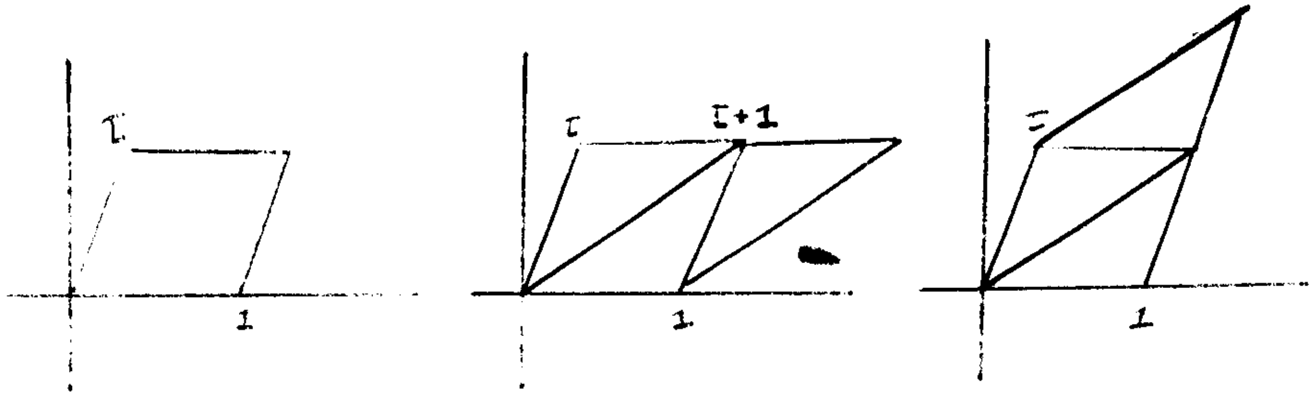
$$= \mathrm{tr}\, e^{2\pi i \tau_1 (L_0 - \bar{L}_0)} e^{-2\pi \tau_2 (L_0 + \bar{L}_0)}$$

$$= \mathrm{tr}\, e^{2\pi i \tau L_0} e^{-2\pi i \bar{\tau} \bar{L}_0}$$

So, IDENTIFYING $q = e^{2\pi i \tau}$ $\Rightarrow$

$$\underline{\underline{Z = \int e^{-S} = \mathrm{tr}\, q^{L_0} \bar{q}^{\bar{L}_0}}}$$

• Modular Invariance



Modular Parameters:

$$\tau/1 = \tau \quad ; \quad (\tau+1)/1 = \tau+1 \quad ; \quad \tau/(\tau+1)$$

• More Generally: Identify $\tau \leftrightarrow \dfrac{a\tau+b}{c\tau+d}$; $a,b,c,d \in \mathbb{Z}$ $ad-bc=1$ i.e. $SL(2,\mathbb{Z})$.

• Generators: $\tau \to \tau+1$; $\tau \to -1/\tau$. $(*)$

• Modular Invariance:

$$Z(\tau,\bar{\tau}) = \int e^{-S} = \sum_i \chi_{h_i}(\tau)\, \chi_{\bar{h}_j}(\bar{\tau})$$

invariant under $(*)$