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REVIEW OF 2D SUSY SIGMA MODELS AND RELATION TO 4D GAUGE THEORIES

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Please note: These are preliminary notes intended for internal distribution only.

Topological reduction of 4D SYM to 2d σ-model

4D Gauge Theories & 2d σ-models have much in common:

- ASYMPTOTIC FREEDOM
- INSTANTONS
- MASS GAP
- $1/N$ EXPANSION

...

For **SUSY** theories this analogy can be made more precise.

Consider 4D SYM on a manifold

$$M_4 = C_\epsilon \times \Sigma$$

- In the limit when size of C_ϵ shrinks to zero one recovers 2d theory on Σ
- Under certain conditions this procedure results in 2d <u>σ-model</u>

List of the results

1. Reduction of $N=2$ SYM

- relates Donaldson invariants to quantum cohomology ring of the moduli space of flat connections

2. Reduction of $N=4$ SYM

- relates S-duality to T-duality

3. Non topological information

- relation between β-functions in $D=4$ & $d=2$ implies

— $G_{ij} \sim R_{ij}$ $\left(\begin{array}{l} G_{ij} \text{ is W.P. metric on the} \\ \text{moduli space of flat connections} \end{array} \right)$

— Other non trivial relations on G_{ij}

4D $N=2$ SYM β-function is 1-loop exact

$\Rightarrow$ 2d β-function is 1-loop exact

$\Rightarrow \sum RRR = \text{const}$; .....

Reduction:

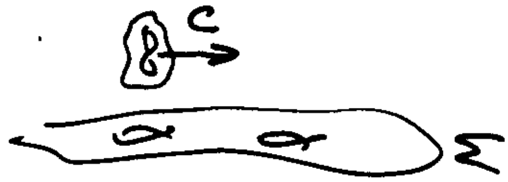
- In general the resulting theory is very complicated
  Different regimes of field configurations
  results in different limits

<u>To avoid this complications</u>

- assume that $G = SU(N)/\mathbb{Z}_N$ (we usually consider $SO(3)$)

- restrict path integral to field configurations
  with nontrivial $\mathbb{Z}_N$ magnetic flux $z \in H^2(M, \mathbb{Z})$

$$z(C) \neq 0$$



- This implies that $\Delta_C = D_C^2$ is **INVERTIBLE**

- To establish precise correspondence consider
  reduction of 4D topologically twisted
  (or partially twisted) SYM
  $\Rightarrow$ 4D theory is $\|C\|$ independent

- $N=1$ SYM $\longrightarrow$ $(2,0)$ σ-model
- $N=2$ SYM $\longrightarrow$ $(2,2)$ σ-model ✓
- $N=4$ SYM $\longrightarrow$ $(4,4)$ σ-model ✓

Reduction (pure YM)

metric on $M_4 = C \times \Sigma$ $\qquad g = \begin{pmatrix} \epsilon g_C & 0 \\ 0 & g_\Sigma \end{pmatrix}$

YM action

$$S = \int \Big[\frac{1}{\epsilon} |F_{w\bar{w}}|^2 + |F_{zw}|^2 + |F_{z\bar{w}}|^2 + \epsilon |F_{z\bar{z}}|^2 \Big]$$

$(w, \bar{w}) \leftarrow$ coordinates on C

$(z, \bar{z}) \leftarrow$ coordinates on Σ

In the limit $\epsilon \to 0$ only <u>flat connections</u>

$\underline{F_{w\bar{w}} = 0}$ survive

$\mathcal{M}(C) \leftarrow$ Moduli space of flat connections on C

$$A_C(w, \bar{w}, z, \bar{z}) = A_C(w, \bar{w} \,|\, X(z, \bar{z}))$$

where $X(z, \bar{z}) : \Sigma \to \mathcal{M}(C)$

dynamical field of 2d σ-model

5)

- Space, tangent to $\mathcal{M}(C)$ is spanned by $\{\alpha_I\}$

$$D_A \alpha = 0 \quad (\text{variations}) \; ; \quad D^\mu \alpha_\mu = 0 \quad (\text{gauge cond.})$$

$$\frac{\partial A_w}{\partial X^i} = \alpha_{iw} + D_w E_i \qquad \left(E_i \text{ defines a connection on } \mathcal{M}(C) \right)$$

- $\mathcal{M}(C)$: Kahler manifold of $\dim_{\mathbb{C}} = 3(h-1)$ genus C

$$\omega_{IJ} = \int_C \alpha_I \wedge \alpha_J \quad ; \quad G_{IJ} = \int \alpha_I \wedge * \alpha_J$$

- $A_z, A_{\bar z}$ play the role of auxiliary fields (ignoring a kinetic term $\sim \epsilon$)

Integrating $A_z, A_{\bar z}$ out (Δ is **INVERTIBLE**)

$$A_z = E_I \partial_z X^I \, , \quad A_{\bar z} = E_I \bar\partial_z X^I$$

<u>**Action**</u> $\quad S = \int_M |F_{zw}|^2 + |F_{z\bar w}|^2 + o(\epsilon) \longrightarrow$

$$\longrightarrow \int_M \partial_z X^I \alpha_I \wedge * \bar\partial_z X^J \alpha_J \longrightarrow$$

$$\longrightarrow \int_\Sigma \partial_z X^I \bar\partial_z X^J \, G_{IJ} \quad \Big\} \; \sigma\text{-model action !}$$

6)

Comments

- turning on θ-angle in YM $\Longleftrightarrow$
  complexification of Kähler class

- $\dim H^2(\mathcal{M}(C)) = 1$; $\tau = \theta + \frac{i}{g^2}$

- YM instantons in 4D $\Longleftrightarrow$
  holomorphic instantons of 2d σ-model
- Additional contributions from def, interactions
- For SUSY theories one can be NAIVE
  CLASSICAL arguments **O.K.**

- limit $\epsilon \to 0$ is smooth **!**

N=2 SYM in 4D

Bosons: A (connection), φ (complex scalar)

Fermions: (Twisted theory)
- η (scalar)
- $\lambda,\ \lambda_{zw},\ \bar\lambda_{\bar z\bar w}$ (self-dual 2-form)
- $\chi_w, \chi_{\bar w}, \chi_z, \chi_{\bar z}$ (vector)

$$S_F = \eta\left(D_w \chi^{\sigma}_{\bar w} + \cancel{\epsilon D_z \chi_{\bar z}}\right) + \lambda\left(D_{\bar w}\chi^{\sigma}_{w} + \cancel{\epsilon D_{\bar z}\chi_z}\right) +$$

$$+ \underline{\lambda^{\sigma}_{zw}\left(D_{\bar z}\chi^{\sigma}_{\bar w}\right.} - D_{\bar w}\chi_{\bar z}\big) + \underline{\bar\lambda^{\sigma}_{\bar z\bar w}\left(D_z\chi^{\sigma}_{w}\right.} - D_w\chi_z\big) + \text{int}$$

contribute to σ-model action.

Integrating out the fields $\eta, \lambda, \chi_z, \chi_{\bar z}$ one imposes the constraints

$$D_w\chi_{\bar w} = D_{\bar w}\chi_w = D_w\bar\lambda_{\bar z\bar w} = D_{\bar w}\lambda_{zw} = 0$$

$$\Rightarrow \quad \chi_{\bar w} = \chi^{i}\alpha_{i\bar w}, \quad \chi_w = \chi^{\bar i}\alpha_{\bar i w}$$

$$\lambda_{zw} = \rho_z^{\bar i}\alpha_{\bar i w}, \quad \bar\lambda_{\bar z\bar w} = \rho_{\bar z}^{i}\alpha_{i\bar w}$$

where α_I span the tangent space to $\mathcal{M}(C)$

$\chi^i, \chi^{\bar{i}}, \rho^i_{\bar{z}}, \rho^{\bar{i}}_z$ are <u>σ-model</u> fields

Taking into account interaction one recovers the full (2,2) supersymmetric σ-model action

$$S = \frac{1}{e^2} \int G_{i\bar{j}} \left(\partial X^i \bar{\partial} X^{\bar{j}} + \bar{\partial} X^i \partial X^{\bar{j}} + \rho^i_{\bar{z}} D_z \chi^{\bar{j}} + \chi^i \bar{D}_{\bar{z}} \rho^{\bar{j}}_z \right) +$$
$$+ R_{i\bar{j}k\bar{l}}\, \chi^i \chi^{\bar{j}} \rho^k_{\bar{z}} \rho^{\bar{l}}_z$$

It is topological <u>A-model</u>

$$c_1(\mathcal{M}) = N \cdot h_2 \qquad \left(h_2 \in H^2(\mathcal{M}, \mathbb{Z}) \right)$$

Fermion # violation $= 2N$ (in 4D $N=1$ SYM)

$$\Downarrow$$

Fermion # violation $= 2N$ (in 2d σ-model)

Observables

- The only observable in topological 4D SYM is $\mathcal{O} = \mathrm{Tr}\, \varphi^2$ and its descendants.

- In 2d σ-model the observables are given by d-cohomology. The full cohomology space is generated by H^2, H^3 and H^4.

- Moreover :
$$\dim H^2 = 1$$
$$\dim H^3 = 2h \qquad (\,\mathrm{genus}(C) = h\,)$$
$$\dim H^4 = 2 - \delta_{h,2}$$

$$a \in H^2 \,, \quad \gamma_K \in H^3 \qquad ; \quad \{a^2, b\} \in H^4$$

Under Reduction:

$$\mathcal{O} = \mathrm{Tr}\, \varphi^2 \longrightarrow b = \chi^i \chi^{\bar{j}} \chi^k \chi^{\ell}\, b_{i\bar{j}k\bar{\ell}}$$

- by the equation of motion φ is <u>bilinear</u> in fermions

- Normal ordering : $\quad b \Longleftrightarrow \mathcal{O} - \mathrm{const}$

Kahler class "a" corresponds to non local operator

$$I(C) = \int_C \mathcal{O}^{(2)} \iff a = \omega_{IJ} \chi^I \chi^J ,$$

which becomes <u>local</u> after reduction.

Similarly:

$$I(\Gamma_K) = \int_{\Gamma_K} \mathcal{O}^{(1)} \iff \gamma_K = \ldots ,$$

where $\Gamma_K \subset C$ (to insure locality)

Elements of H^3 are labeled by $H_1(C)$

This promotes the action of modular group on H^*

The invariant part of cohomology H^*_{inv} is generated by

$$a, \; b, \; c = \sum \gamma_i \gamma_j \left(\Gamma^i \# \Gamma^j \right)$$

σ-Model TARGET SPACE

Let the gauge group be

$$SO(3) = SU(2)/\mathbb{Z}_2$$

Flux $z \in H^2(M, \mathbb{Z}_2)$

Magnetic flux has $4gh+2$ components :

$$H^2(C,\mathbb{Z}_2) + H^1(C,\mathbb{Z}_2) \otimes H^1(\Sigma,\mathbb{Z}_2) + H^2(\Sigma,\mathbb{Z}_2)$$

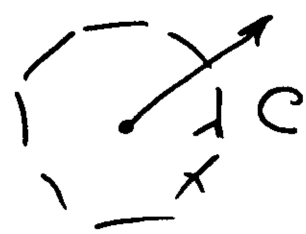
To avoid singular limit we will always choose $z(C) = 1$.

What is the meaning of remaining $4gh+1$ comp. ?

Let $\tilde{\mathcal{M}}$ be the space of representations

$$\tilde{\pi}_1(C) = \left\{ a_i, b_i \,\middle|\, \prod_1^h a_i b_i a_i^{-1} b_i^{-1} = -1 \right\} \quad \text{in } SU(2)$$

- $\tilde{\mathcal{M}}$ is smooth & compact



- Flipping sign of a_i, b_i defines the action of $G = H^1(C, \mathbb{Z})$, which is <u>NOT FREE</u>

The target space of 2d σ-model is an ORBIFOLD $\mathcal{M} = \tilde{\mathcal{M}}/_{G}$

For the worldsheet Σ_g, the sectors are classified by the boundary conditions on A_i, B_j cycles, i.e. elements of

$$H^1(\Sigma, G) \simeq H^1(\Sigma, \mathbb{Z}_2) \times H^1(C, \mathbb{Z}_2)$$

The choice of 4gh components of the flux z is equivalent to <u>boundary conditions</u>

The remaining component of the flux is $z(\Sigma)$

Relation between instanton numbers

4D : $\quad p \equiv -z^2/4 \pmod 1$

2d : $\quad k \equiv 2p \equiv - z(\Sigma) + \text{const}$

depends only on orbifold sector

$z(\Sigma)$ governs the PARITY of instanton number in 2d

Quantum Cohomology for $\mathcal{M}(C)$

classically : $H^*_{inv} = \mathbb{C}[a,b,c]/I^h$

where ideals $\ldots \supset I_h \supset I_{h+1} \supset \ldots$

and $I_h = \{ R_1^h, R_2^h, R_3^h \}$

1) A. King & P. Newstead
2) B. Seibert & G. Tian

$$\left.\begin{aligned} R_1^{h+1} &= a R_1^h + h^2 R_2^h \\ R_2^{h+1} &= b R_2^h + \frac{2h}{h+1} R_3^h \\ R_3^{h+1} &= c R_3^h \end{aligned}\right]$$

where $R_1^1 = a$, $R_2^1 = b$, $R_3^1 = c$

The Relation between Donaldson theory ($N=2$ SYM, E. Witten)

$$\boxed{\sum_{\substack{z(\Sigma) \\ z(C)=1}} \left\langle e^{\alpha \mathcal{O} + \beta I(C)} \right\rangle_z^{C\times\Sigma} = \left\langle e^{\alpha a + \beta b} \right\rangle_{\Sigma, \mathcal{M}(C)}}$$

Witten

The Relation between corr. functions implies that "b" has two eigenvalues $\underline{\pm 8q}$, and "a" has $(2h-1)$ eigenvalues $\underline{\{0, \pm 4\sqrt{q}, \pm 8i\sqrt{q}, \ldots}\}$

$c = \sum \gamma_i \gamma_i$ is nilpotent

$R \longrightarrow Q$

$\boxed{q = e^{2\pi i \tau}}$

$$Q_1^1 = a, \quad Q_2^1 = b - 8q, \quad Q_3^1 = c$$

$$\left[\begin{aligned} Q_1^{h+1} &= a\, Q_1^h + h^2 Q_2^h \\ Q_2^{h+1} &= (b + (-1)^{h-1} 8q)\, Q_2^h + \frac{2h}{h+1} Q_3^h \\ Q_3^{h+1} &= c\, Q^h \end{aligned}\right.$$

For $h=2$ these formulas reproduce the result of **Donaldson**.

This information is enough to reconstruct the full cohomology ring

Topologically Twisted $N=4$ SYM

Bosons : $(A, \varphi, \bar{\varphi}, f, \bar{f}, \phi_w, \phi_{\bar{w}})$

After Reduction:

$F=0$ for $N=2$ get replaced by

$$\begin{cases} F_{w\bar{w}} = \delta[\phi_w, \phi_{\bar{w}}] \\ D_w \phi_{\bar{w}} = D_{\bar{w}} \phi_w = 0 \end{cases} \quad (*) \qquad \textbf{Hitchin}$$

$\mathcal{M}^H \leftarrow$ moduli space of solutions of $(*)$

1) $\mathcal{M}^H$ is smooth, non compact

2) $\mathcal{M}^H$ – partial compactification of $T\mathcal{M}(C)$

3) $\mathcal{M}^H$ is hyperKähler

4) $\dim_{\mathbb{C}} \mathcal{M}^H = 6h-6$

A distinct complexified Kahler form

$\Leftrightarrow \quad \tau = \theta + \frac{i}{g^2}$ in 4D

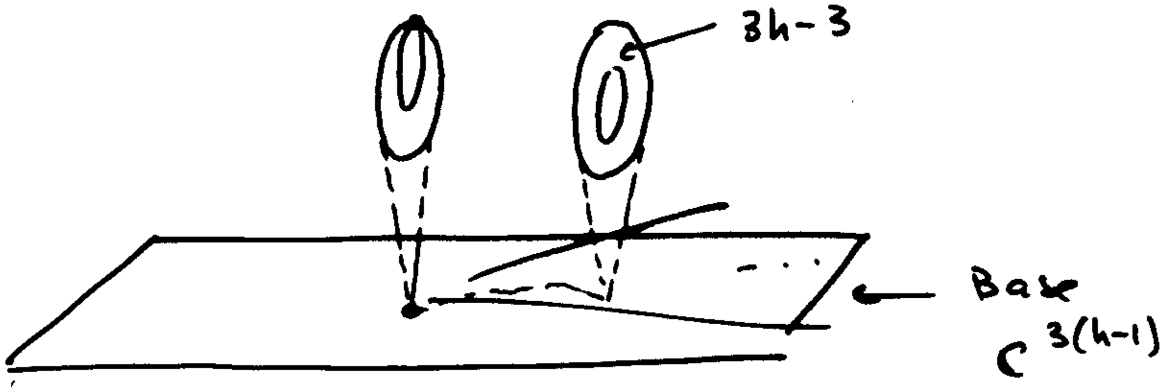
S-Duality $\Longleftrightarrow$ T duality

Description of Hitchin space

$K = \tau\omega$

For $su(2)$

$$\mathcal{M}^H = \left\{ \begin{array}{l} \text{Fibration of complex tori} \\ T^{3(h-1)} \text{ over } \mathbb{C}^{3(h-1)} \end{array} \right\} \begin{array}{l} \sigma\text{-model} \\ \text{metric } G \end{array}$$



At some points the torus might degenerate, but the full space is smooth.

Base: $b_{ww} = \det \Phi_w$ ← Holomorphic 2-diff.

Let Σ_c be a double cover of Σ, branched over the zeroes of 2-diff. b_{ww}

$$T^{3(h-1)} = \left\{ T^{3(h-1)} \subset \mathrm{Jac}(\Sigma_c) \,\middle|\, \text{odd w/r } \mathbb{Z}_2 \right\}$$

- Generalization to $SU(N)$ is straightforward

Generalization for $SU(N)$ is clear

Base $= \{$ Hol. j-diff $\mathrm{Tr}\,\Phi_w^j$, $j = 2, \ldots N \}$

Fiber $= \{$ Prime variety of N-fold cover $\}$

- We do not know how to solve σ-model directly.
- One can use <u>adiabatic approximation</u>, by viewing complex moduli of the fiber as slow varying (with <u>fixed</u> Kahler structure)
- Modular properties of the Kahler module (of $\mathcal{M}^H$) will be the same as for each individual fiber (as long as adiabatic approximation is valid)
- Adiabatic approximation breaks down near degenerated fibers, but the total space is <u>non singular</u>.

But we can still trust the modular properties

Cycles on Σ : a_i, b_i $\quad i=1\dots h$

Cycles on Σ_c : $a_i \pm \sigma(a_i)$, $b_j \pm \sigma(b_j)$

a_α, b_β $\quad \alpha, \beta = 1, \dots 2h-3$

(cycles around branch points)

$$a_\alpha \circ b_\beta = \delta_{\alpha\beta}$$
$$(a_i \pm \sigma(a_i)) \circ (b_j \pm \sigma(b_j)) = 2\,\delta_{ij}$$

$\Rightarrow$ Induces the Kahler class

$$K = \sum_{\alpha=1}^{2h-3} dx_\alpha \wedge dY^\alpha + 2 \sum_{i=1}^{h} dx_i \wedge dY^i$$

In general $\quad K = \sum n_i \, dx_i \wedge dY^i \quad (n \in \mathbb{Z}_+)$

For $su(N)$ $\quad n_i \in \{1, N\}$

$2h-3$ cycles have area 1

h cycles have area 2

For the later ones the instanton expansion is in powers of q^2 (instead of q)

This is the origin of $\Gamma_0(2) \subset SL(2,\mathbb{Z})$ invariace.

One can be a little more precise.

Moduli space of T^{2d} $\qquad d = 3(k-1)$

$$\mathrm{Mod} = \frac{SO(2d,2d)}{SO(2d)\times SO(2d)\times SO(2d,2d,\mathbb{Z})}$$

Let $D_{ij} = \delta_{ij} n_i$; $J = \begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix}$

Deformations of complex structure, that preserve Kahler form K

$$Sp(2d) \big/ U(d)\times Sp(2d,\mathbb{Z})$$

Now we can split the complex & Kahler directions

Complex: $(\sigma_x \otimes S,\ 1\otimes A)/(1\otimes A)$

Kahler: $(t = \sigma_x \otimes 1,\ b = i\sigma_y \otimes J,\ \sigma_z \otimes J)/(\sigma_z \otimes J)$

S, A — symmetric, antisymmetric generators of $Sp_J(2d)$ $\Bigg\}$ we also rescaled coordinates s.t. $D_{ij} = \delta_{ij}$

Generators of Kähler deformations <u>commute</u> with those of complex deformations and form $SL(2)$ group.

$SL(2) \subset SO(2d,2d)$ is the maximal subgroup which commutes with $Sp(2d)$.

<u>Modular Group.</u>

1) Restoring Normalization

$$J = \oplus J_i$$

2) $SL(2)$ sits diagonally in $\otimes SL_i(2)$ with common moduli τ is identified as $n_i \tau_i$ (which correspond to $q_i^{n_i}$)

Assume, that n_i's have no common divisor

$n \Leftarrow$ Least common multiple of n_i

Common intersection $\cap SL_i(2) = \underline{\Gamma_0(n)}$,

generated by T, ST^nS

Duality group for $SU(N)$ SYM in 4D.