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I.C.T.P., P.O. BOX 586, 34100 TRIESTE, ITALY, CABLE CENTRATOM TRIESTE

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NEW RESULTS IN TOPOLOGICAL FIELD THEORY AND ABELIAN GAUGE THEORY

G. THOMPSON
ICTP
Trieste, Italy

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George Thompson: thompson@ictp.trieste.it

Warning Very Preliminary

Contents

1 Introduction

String theory is claimed to be the theory of 'everything'. In principle, within the context of string theory, one can calculate any process one can dream of and also some processes, involving zero mass black holes which were previously quite unimaginable. This makes the study of string theory both fascinating and, in general, difficult. Topological field theory, on the otherhand, is the theory of 'nothing'. This means that one does not calculate any physical transitions at all (at least not directly). The reason for this being that there are no dynamical degrees of freedom in these theories. This makes the theory, in some instances, more tractable but, perhaps surprisingly, does not diminish its inherent interest.

Why is a theory of 'nothing' interesting? One aspect of the answer to this question is that, even though there are no physical degrees of freedom, these are fully interacting, and at first sight, very complicated field theories. It comes as a surprise that one can solve these models exactly. Nevertheless, this is so in many cases. Solvable gauge field theories are few and far between yet Yang-Mills theory and the G/G coset models on a general Riemann surface and Chern-Simons theory and various relatives on arbitrary three manifolds are examples. We can hope in this way to gain a better understanding of what a non- perturbative solution to a field theory could be like.

What topological field theories calculate are invariants. That is numbers that are robust: they are independent of couplings or of any dynamics. These numbers are usually the dimension of some space or an Euler character. These numbers may be of interest to phyicists or mathematicians or, indeed, to both. An example of interest to both parties is Chern-Simons theory [15]. The partition function of Chern-Simons theory on a manifold of the form $\Sigma \times S^1$, where Σ is a Riemann surface, calculates the dimension of the space of conformal blocks of the G WZW model. For physicists this is the dimension of the Hilbert space of the theory while for mathematicians this is the dimension of the space of sections of the determinant line over the space of flat connections on Σ (somewhat of a mouthful). One can calculate this directly in the conformal theory, as E. Verlinde did, but one only needs familiar gauge theory techniques from the Chern-Simons point of view and one by passes completely the conformal field theory technology.

While topological field theories are interesting in their own right some of the interest in them is also due to their, rather direct, relationship with conventional "physical" theories. For example, one can calculate Yukawa couplings in ($N = 2$ supersymmetric-) string theory with the standard sigma model or with either of the two possible topological theories[1]. The topological field theories can also act as "easier" testing grounds for ideas in physical theories. An example of this is the

[1]The calculation of Yukawa couplings in string theory was covered in Brian Greene's lectures. The relationship with the topological theory comes about as one is restricting ones attention to chiral primaries. On this restricted field set the supersymmetry charges act like BRST operators.

idea of duality in supersymmetric Yang-Mills theory in four dimensions. Suppose there is a correspondence between the weak (strong) coupling of one theory and the strong (weak) coupling of its dual theory. If one of these field theories has a topological field theory hidden within it, then so does the other as the topological field theory will exist in both phases of the original theory. Now the form of the topological field theory may well be different in the dual model. The equivalence of the two descriptions of the topological field theories is then a necessary condition for the duality of the starting models. In practice it may be easier to check for duality at the level of the topological field theories.

This is the situation studied by Vafa and Witten [9]. The twisted $N = 4$ super Yang-Mills theory on a four manifold calculates the Euler character of the moduli space of instantons. Fortuitously, the mathematicians have calculated these for certain compact Kähler manifolds as well as for ALE spaces. Vafa and Witten were able to confirm, using the results of the mathematicians, that indeed the partition function of the $N = 4$ $SU(2)$ gauge theory transforms in strong coupling to the $N = 4$ $SO(3)$ gauge theory at weak coupling. This provides a direct test of the duality hypothesis.

Another example where one can use the topological theory (counter historically) to test results in a physical theory is provided by the candidate exact solution of $N = 2$ $SU(2)$ super Yang-Mills theory of Seiberg and Witten. The $N = 2$ theory is related to some rather deep mathematics of four manifolds. As explained by L. Alvarez-Gaume in his lectures, when one is at weak coupling dominant contributions come from instantons. The bulk of the mathematical analysis is to come to grips with the moduli space of instantons. Rather than working at weak coupling $u \to \infty$ one can pass to the points $|u| = 1$ where the physics is given in terms of a massless monopole and the magnetic photon. This system is Abelian and easily analysed, and should allow one, if the picture is correct, to reproduce the Donoldson Polynomials. Witten [13] shows in fact that the theory at this point in the u plane gives non-trivial invariants of four manifolds (it is still a conjecture that one reproduces Donaldson theory in general).

These notes are very similar to sets of lectures presented earlier this year in Trieste. The first was by Braam and the second by Dijkgraaf. The reason for the overlap is easy to explain, our sources are almost identical. The papers by Taubes, Verlinde, Witten and others are very clear and hardly need elucidation -as for background the books by Freed and Uhlenbeck and especially that by Donaldson and Kronheimer are excellent for the elaboration of many of the notions presented in the cited publications. There are, however, also some differences in presentation. These stem from the different audiences that we were addressing and from the fact that I am not an expert in the field of four manifolds (whereas the Dutch gentlemen cited are).

2 Topological Field Theories

The type of topological field theories that will be of interest to us here are easy to construct. We will want a field theory that devolves to some moduli space, that is to say, that describes the space of solutions to a set of equations. Suppose that we have a set of fields, $\{\Phi_i\}$ and the equations of interest are

$$s^a(\Phi_i) = 0\,, \tag{2.1}$$

we denote the space of solutions by $\mathcal{M}(\Phi)$. A typical example is the space of flat connections on a manifold X. In this case the fields Φ_i are only the gauge field A and the equations s^a are

$$s = F_A = dA + A^2 = 0\,. \tag{2.2}$$

Usually one wishes to also factor out the action of the gauge group and this would correspond to yet more conditions on A (a choice of gauge). Another choice of interest is the space of self dual connections over a 4-manifold, the context in which Witten first introduced topological field theories of this kind and to which we will pay some attention later in these notes.

An important ingredient in the construction of a topological field theory is the topological symmetry. We denote generator of the symmetry by Q. Its action on the fields $\{\Phi_i\}$ is to give a new set of fields $\{\Psi_i\}$ which are in all ways identical to the original set except that the new fields have opposite Grassman character. If one starts with a scalar field then its 'superpartner' is also of spin zero, but it is an anticommuting field. If, on the other hand, the field Φ is a spinor field (anticommuting) then its partner Ψ will be a commuting spinor field, and so on. This property of the set of fields $\{\Psi_i\}$ makes them more like Fadeev-Popov ghost fields and this is insinuated when one says that the operator Q is a BRST operator. So we have

$$Q\Phi_i = \Psi_i\,. \tag{2.3}$$

In order to be able to impose the equations (2.1) we also need to introduce a set of Grassman odd fields $\{\overline{\Psi}_a\}$ and Grassman even fields $\{B_a\}$ with transformation properties

$$Q\overline{\Psi}_a = B_a\,. \tag{2.4}$$

Supose that the theory we are interested in is defined on some manifold (or more generally some space) X. In order to fully define the theory we may need also to use a metric on X, or some other coupling constants; denote these collectively by t_i. The action of interest is then, schematically,

$$\begin{aligned} S &= \int_X \{Q, \left(t_0\overline{\Psi}_a s^a(\Phi_i) + \sum_{i=1} t_i V^i\right)\} \\ &= \int_X \left(t_0 B_a s^a(\Phi) - t_0\overline{\Psi}_a \frac{\delta s^a(\Phi)}{\delta\Phi_i}\Psi_i + \dots\right). \end{aligned} \tag{2.5}$$

The associated path integral is

$$Z = \int DY \exp\left(iS(Y)\right) \tag{2.6}$$

where Y denotes all the fields.

Notice that if the multiplier fields B_a appear as in (2.5) then integrating over them yield a delta function constraint on $s^a = 0$. Hence the partition function will devolve to a (finite dimensional) integral over the moduli space. The integration over all of the fields $\{\overline{\psi}^a\}$ will yield some function $\mu(\Phi)$, so that

$$Z = \int_{\mathcal{M}} \mu(\Phi), \tag{2.7}$$

and $\mu(\Phi)$ may be interpreted as some measure on $\mathcal{M}$.

Remark: It is sometimes possible to write down a topological action without the need of an auxillary set of fields. Two dimensional Donaldson theory is an example of this.

Metric and Coupling Constant Independence

In order to write down the equations of interest or an action one may need to introduce extraneous parameters. For example, even if one can formulate the flatness equation without recourse to a metric, a metric is, nevertheless, required in order to gauge fix the gauge field. Another example is the instanton equation which requires a metric from the word go. Under suitable conditions one will be able to establish that nothing depends on these choices. When that is so- the name of the game is to choose the parameters to make life as simple as possible.

The variation of the partition function with respect to any of the parameters t_i (including the metric) is

$$\begin{aligned} \frac{\partial Z}{\partial t_j} &= \int DY \exp\left(iS(Y)\right).i\frac{\partial S(Y)}{\partial t_j} \\ &= \int DY \exp\left(iS(Y)\right).i\{Q, V(Y)\}, \end{aligned} \tag{2.8}$$

for some V. The righthand side vanishes by a Ward identity. Consider the obvious equality

$$\int DY \exp(iS(Y))V(Y) = \int D(Y+QY)\exp iS(Y+QY)V(Y+QY). \quad (2.9)$$

This leads to non-trivial information if both the measure DX and the action are Q invariant. In the topological field theory the action is Q exact, its Q variation will be Q^2 acting on something, but $Q^2 = 0$, so $S(Y+QY) = S(Y)$. Presume the measure is also invariant (one can check that at a formal level this is the case), so that $D(Y+QY) = DY$. We can now conclude that

$$\begin{aligned}\int DY \exp(iS(Y))V(Y) &= \int DY \exp(iS(Y))V(Y+QY) \\ &= \int DY \exp(iS(Y))\left(V(Y) + \{Q, V(Y)\}\right) \quad (2.10)\end{aligned}$$

which implies that

$$\int DY \exp(iS(Y))\{Q, V(Y)\} = 0. \quad (2.11)$$

This establishes that the partition function in a topological field theory is independent of both the metric and coupling constants, providing the theory remains well defined as we vary the parameters. Let the parameter space be $\mathcal{T}$, which, for simplicity we take to be connected. One can get from one set of parameters t_i to another t_i' along some path in $\mathcal{T}$. Pick such a path and suppose that for all points along the path the theory is well behaved. Now perturb the path. If only for very special choices of the perturbation does the path go through points that lead to an ill defined theory, one says that the partition function is independent under generic variations of the couplings. This is good enough.

Unfortunately, it is not always the case that a generic path misses the 'bad' points, as we will see later, there are situations in Donaldson and in Witten theory where the dimension of the moduli space jumps as one varies the metric and that one cannot avoid this.

No Physical Degrees of Freedom

When dealing with any field theory, there are certain restrictions on the field content. For example, one should have a good quadratic form. This means that, in the present situation, up to a finite number of zero modes, there should be an equal number of degrees of freedom in the set of fields $\{\Phi\}$ and $\{B\}$, even though the labels are different. For example, the four dimensional Yang-Mills action (plus a θ term), may be written as

$$\mathrm{Tr}\int_{\mathbb{R}^4}\left(B_+^{\mu\nu}F_{\mu\nu} - \frac{g^2}{2}B_+^{\mu\nu}B_{\mu\nu}^+\right), \quad (2.12)$$

where $B_{\mu\nu}^+$ is a self dual anti-symmetric tensor. The equation of motion for B_+ is algebraic, so that one can substitute this back to obtain the more usual form of the action[2]. Now $B_{\mu\nu}$, only has three independent components while A_μ has four. We need to gauge fix and we can do so by introducing a multiplier field b and Fadeev-Popov ghosts to the theory. Now one adds

$$\mathrm{Tr}\int_{\mathbb{R}^4}(b\partial^\mu A_\mu + \bar{c}\partial^\mu D_\mu c) \quad (2.13)$$

to the action. B and b together match have four degrees of freedom and so match the gauge field, while c has one degree of freedom and matches $\bar{c}$ so all is well.

This counting implies that we have a well defined action but does not tell us what the *physical* degrees of freedom are. We know that in d-dimensions a vector field has $d-2$ physical degrees of freedom. There is a simple way of getting this from the gauge fixed action. A gauge field has d degrees of freedom, and each of the two ghosts has -1 degrees of freedom, giving us a total of $d-2$. The B and b fields do not count as they are 'non-propagating', meaning one can eliminate them algebraically.

The topological field theory has, by construction, for every field an associated 'ghost' field identical in every respect except that it has opposite Grassmann parity. As every field is matched by a ghost, the total number of physical degrees of freedom is always zero in a topological field theory.

Index theory and the Dimension of $\mathcal{M}$

We have seen that there are no propagating degrees of freedom, we are down to the zero, or topological, modes and some of these make up the moduli space $\mathcal{M}_s$. What is the dimension of $\mathcal{M}_s$? To simplify life let us assume that $\mathcal{M}_s$ is connected and smooth about most points. One way to determine the dimenion of $\mathcal{M}_s$, is to fix a point $\phi \in \mathcal{M}_s$ and then see in how many directions you can go and stay in $\mathcal{M}_s$. We saw an example of this argument in the lectures of J. Harvey for the moduli space of monopoles we proceed in the same way. Hence, we want $s(\phi) = 0$ and if $\phi + \delta\phi$ is a nearby point we also require $s(\phi + \delta\phi) = 0$, or we look for solutions to

$$\left.\frac{\delta s^a(\Phi)}{\delta \Phi_i}\right|_\phi \delta\phi_i = 0. \quad (2.14)$$

If we are lucky the operator $D = \delta s^a(\phi)/\delta\phi$ has no co-kernel, as in the case of the monopoles, and the index is known. Then as $index(D) = Ker(D) - CoKer(D)$

[2]In the Path integral we are dealing with a Gaussian integral over B which amounts to the same thing.

we would have the dimension of the moduli space. There are many situations where we are not lucky. On a compact odd dimensional manifold the operator D that is associated with the space of flat connections (2.2) has index zero (as the kernel and cokernel are equal). In such situations one must look elsewhere for a handle on the moduli space. For flat connections the equivalent description in terms of homomorphisms of the fundamental group of the underlying manifold into the gauge group (modulo the adjoint action of the gauge group) contains the necessary information.

The index of D is sometimes called the virtual dimension of the moduli space.

2.1 The Euler Character

One of the classic invariants of a closed compact n-manifold X is its Euler character. It is defined to be

$$\chi(X) = \sum_{i=1}^{n}(-1)^i b_i \tag{2.15}$$

where the Betti numbers are $b_i = dim H^i(X, \mathbb{R})$. Now there are two well known formula for this invariant. The first, due to Gauss and Bonnet states that if $R_{\mu\nu\kappa\lambda}$ is the Riemann curvature tensor then

$$\chi(X) = \frac{1}{(2\pi)^{n/2}}\int_X R^{n/2}. \tag{2.16}$$

The second formula, due to Poincare and Hopf, counts the number of non-degenerate critical points x_P of a function f on X with sign,

$$\chi(X) = \sum_P sign \det\left(\partial_\mu\partial_\nu f\right). \tag{2.17}$$

These formulae arise naturally in the context of supersymmetric quantum mechanics. The aim there is to give a path integral representation of the index of certain differential operators. The index of the de-Rham operator offers a third representation of the Euler character. As Witten showed [11] there is a twisting of the de-Rham operator d that interpolates between the two formulae, $d \to e^{-tf} d e^{tf}$. In the supersymmetric quantum mechanics path integral one can take the limit $t \to 0$ to arrive at (2.16) or $t \to \infty$ to arrive at (2.17). By supersymmetry invariance the path integral is formally independent of t and the equality of the two formulae is thus established.

This situation prompted Quillen [?] and Matthai and Quillen [8] to develop a completetly classical formula that interpolates between (2.16) and (2.17). We will give a physicists 'derivation' of this shortly. Before doing that I would like to

explain the historical relationship of the Matthai-Quillen formalism to topological field theory. The supersymmetric quantum mechanics, alluded to above, was, perhaps, the first example of a Witten type topological field theory. After Wittens introduction of Donaldson theory it was shown, by Atiyah and Jeffery [1], that one could re-interpret the construction as an infinite dimensional version of the Matthai-Quillen formalism. These infinite dimensional Matthai-Quillen theories devolve to finite dimensional Matthai-Quillen models of the type we will presently discuss. In the quantum mechanics context, the supersymmetric theory considered by Witten can be viewed as an infinite dimensional Matthai-Quillen construction, while the finite dimensional formula that it gives rise to and that interpolates between (2.16) and (2.17) is the 'classical' Matthai-Quillen formula. An account of this construction in topological field theory is to be found in [6, 4, ?, 9]. Physicists will see that this construction is equivalent to the existance of a Nicolai map [2, 7, 3].

Now back to buisiness. Let X be a closed and compact manifold, and f a map, $f : X \to \mathbb{R}$, which has isolated critical points. The points at which

$$df = 0 \tag{2.18}$$

define our moduli space $\mathcal{M}_f$. If the critical points are isolated this means that the second derivative of f is not zero at those points. From our discussion of the dimension of the moduli space we should be looking for solutions to the variation of df,

$$\frac{D^2 f}{Dx^\mu x^\nu}\delta x^\mu = 0. \tag{2.19}$$

If the eigenvalues of $D^2f/Dx^\mu Dx^\nu$ are not zero then the only solution is $\delta x^\mu = 0$, that is the critical point is isolated.

The supersymmetry algebra is

$$\begin{aligned} Qx^\mu &= \psi^\mu, \quad Q\psi^\mu = 0, \\ Q\bar{\psi}_\mu &= B_\mu - \bar{\psi}_\nu \Gamma^\nu_{\mu\kappa}\psi^\kappa, \\ QB_\mu &= B_\nu \Gamma^\nu_{\mu\kappa} - \frac{1}{2}\bar{\psi}_\nu R^\nu_{\mu\lambda\kappa}\psi^\lambda\psi^\kappa. \end{aligned} \tag{2.20}$$

with $Q^2 = 0$. Now we may create a topological (field) theory with the action

$$\begin{aligned} S_f &= \{Q, \bar{\psi}_\mu g^{\mu\nu}(it\partial_\nu f + \frac{1}{2}B_\nu)\} \\ &= itB^\mu\partial_\mu f + \frac{1}{2}g_{\mu\nu}B^\mu B^\nu - it\bar{\psi}_\mu \frac{D^2 f}{Dx^\mu Dx^\nu}\psi^\nu - \frac{1}{4}R_{\mu\nu\kappa\lambda}\bar{\psi}^\mu\psi^\kappa\bar{\psi}^\nu\psi^\lambda \end{aligned} \tag{2.21}$$

The transformation rules have been chosen to give us a covariant action.

Notice that there is a second supersymmetry that one gets by exchanging $\psi \to \overline{\psi}$ and $\overline{\psi} \to -\psi$. This happens quite naturally whenever one wishes to write down a topological field theory for an Euler character. The correspondence comes because on the space of forms Q acts like d while the second supersymmetry charge $\widetilde{Q}$ behaves as d^*. The construction is most easily understood in terms of supersymmetric quantum mechanics, which, unfortunately, there is no time to go into here.

The partition function

$$Z_f = -\frac{1}{(2\pi i)^n}\int_X \int dB d\psi d\bar{\psi} e^{-S_f} \tag{2.22}$$

is independent of smooth deformations of the parameters, as we discussed previously, since the derivative of the action with respect to either the metric or t is of the form $\{Q, \ldots\}$ (we see this directly from the first line of (2.21)). In particular, it does not depend on t, so that we are free to take various limits.

$t \to \infty$

In this case the entire contribution to the 'path integral' is around the critical points of f. To see this send $B \to \frac{1}{t}B$ and $\bar{\psi} \to \frac{1}{t}\bar{\psi}$,

$$S_f(\infty) = ig^{\mu\nu}B_\mu\partial_\nu f - ig^{\mu\nu}\bar{\psi}_\mu \frac{D^2 f}{Dx^\nu Dx^\lambda}\psi^\lambda . \tag{2.23}$$

An important feature of this scaling is that the Jacobian of the transformation is unity.

We let $\{x_P\} = \mathcal{M}_f$ be the critical point set and expand around each point as $x = x_P + x_q$. Furthermore, around a critical point we can pick a flat metric $g_{\mu\nu} = \delta_{\mu\nu}$ for which the christoffel symbol vanishes. The integral over B, gives

$$\begin{aligned}(2\pi)^n \delta\left(\partial_\mu f(x)\right) &= (2\pi)^n \sum_P \delta\left(x_q^\nu \partial_\mu\partial_\nu f(x_P)\right) \\ &= (2\pi)^n \sum_P \frac{1}{|\det \partial_\nu\partial_\mu f|}\delta\left(x_q^\nu\right)\end{aligned} \tag{2.24}$$

On the other hand the integral over the fermionic fields,

$$\int d\psi d\bar{\psi} \exp\left(-i\bar{\psi}^\nu \frac{\partial^2 f}{\partial x^\nu \partial x^\mu}\psi^\mu\right) = -(i)^n \det \partial_\nu\partial_\mu f . \tag{2.25}$$

The path integral, therefore becomes

$$\int_X \sum_P \frac{\det \partial_\nu\partial_\mu f}{|\det \partial_\nu\partial_\mu f|}\delta\left(x_q^\nu\right) = \sum_P \frac{\det \partial_\nu\partial_\mu f}{|\det \partial_\nu\partial_\mu f|} . \tag{2.26}$$

We can write this in the form

$$Z_f = \sum_P \epsilon_P \tag{2.27}$$

with

$$\epsilon_P = sign \det H_P(f) \tag{2.28}$$

with $H_P f = \partial^2 f/\partial x^\mu \partial x^\nu$. $H_P f$ is called the Hessian of f at x_P.

Example: Riemann Surfaces

Consider the example of a height function of a genus g Riemann surface as given in the figure.

Figure

The critical points of the height function are marked. The bottom of the surface, $f = h_{min}$, is a minimum, and hence the sign of both eigenvalues of $\partial^2 f/\partial x^\mu \partial x^\nu$ are plus. The Hessian is therefore $+1$. At the top of the surface, $f = h_{max}$, is a maximum, both eigenvalues are negative, the determinant is positive, and the Hessian is $+1$. For each hole (there are g of them) one has two turning points. The turning points are at $f = h_1, \ldots, h_{2g}$. One of the eigenvalues is positive, the other negative so, at each of these points, the Hessian is -1.

All in all we obtain in this way,

$$Z = 2 - 2g \tag{2.29}$$

which we recognise as the Euler character of the Riemann surface.

$t \to 0$

In this limit we are left only with the curvature term, so that

$$Z = -\frac{1}{(2\pi i)^n}\int_X \int dB e^{-\frac{1}{2}g_{\mu\nu}B^\mu B^\nu} \int d\psi d\overline{\psi} e^{\frac{1}{4}R_{\mu\nu\kappa\lambda}\overline{\psi}^\mu \psi^\kappa \overline{\psi}^\nu \psi^\lambda} \tag{2.30}$$

The integrals over the fields B, $\bar{\psi}$ and ψ are now easy to perform, leaving us with the Gauss-Bonnet formula for the Euler character of a manifold X,

$$\chi(X) = \frac{1}{(2\pi)^{n/2}}\int_X R^{n/2} \tag{2.31}$$

Perturbations

The form of the partition function (2.27) appears to depend quite strongly on the function f that we started with. Yet, topological (BRST) invariance allowed us to equate this with the form of the partition function as given in (2.31), which does not depend at all on the function f that we started with. There is a nice way to see why this might be so. Consider a perturbed height function f' as displayed in the figure.

Figure

The difference between this and f of figure is the addition of a 'hill' and of a valley. Now, the apex of the new hill is a maximum, so the Hessian there is $+1$. On the otherhand the bottom of the valley is a turning point with Hessian -1. So we see that the addition of the Hessians for f' at these two critical points cancel out, and the sum reverts to that of the other critical points where f' agrees with f! One can easily convince oneself that whenever a valley is added then so is a hill (when you dig a hole you get a mound of dirt) as well as the converse, and the contributions of the Hessian always cancel out .

This takes care of the undulations but what happens when we hit a plateau? Such a situation is depicted in the figure for a height function g.

Figure

This situation means that the moduli space, i.e. the solution set to $dg = 0$, is not made up of just isolated points. In the current situation there are two ways out. The first, is simply to note that the general formula (2.22), works in this instance as well. Indeed there are two different limits that can be used. One can take the $t \to 0$ limit without fear. Alternatively, away from the plateau, one may use the the $t \to \infty$ limit, and as one approaches the plateau, revert to the $t \to 0$ limit. The second way to proceed, which will be of importance later, is to perturb the function g to a new function g'. The perurbation $g' - g$ need only be ever so slight and then the critical points are isolated again. The perturbed equation is

$$dg = \epsilon v \tag{2.32}$$

where v is a vector field (essentially $dg' - dg$). As long as $\epsilon > 0$ there are only isolated solutions to this equation. One can construct a new action which takes its values at (2.32),

$$\{Q, \overline{\psi}^{\mu}(dg - \epsilon v_{\mu})\}, \tag{2.33}$$

(all reference to the metric has been dropped, as it plays no role here, we have taken the $t \to \infty$ limit). The partition function function once more, gives the Euler character of the surface, and the $\epsilon \to 0^+$ may be taken with impunity.

Invariants and Zero Dimensional Moduli Space

Quite generally (up to certain compactness requirements) given a system of equations

$$s^a(\Phi) = 0 \tag{2.34}$$

with isolated solutions $\{\phi\}$, the signed sum of solutions

$$\sum_{\phi} \epsilon_{\phi} \tag{2.35}$$

where $\epsilon_{\phi} = sign \det \delta^2 s/\delta\phi\delta\phi$, is a topological invariant.

To be continued

3 S Duality In Maxwell Theory and Abelian Instantons

In this section we will study two different aspects of four-dimensional Abelian theories. The first is a study of S duality in the Abelian context. M. Bershadsky has described, in his lectures, a relationship between S duality in four dimensions and T duality in two dimensions, for non-Abelian theories. Why the interest in the Abelian case? From the lectures of Harvey and Alvarez-Gaume we have seen the important part role played by the breaking of $SU(2)$ down to $U(1)$ in the $N = 2$ supersymmetric Yang-Mills theory. The effective theory, at strong coupling is a $U(1)$ theory.

There have appeared in the last month two very interesting papers on S duality in Maxwell theory, which go in different directions. E. Verlinde [10] has studied the S duality of Maxwell theory and its relationship to T duality, string theory and higher dimensional (free) field theories. E. Witten [12] has used it to probe the modular properties of the partition function so as to fix some of the τ dependence of the $N = 2$ theory on arbitrary four manifolds.

In the following I will give a 'bare bones' description of S duality for Maxwell theory which I hope, though it does no justice to the above works, will nevertheless entice the reader to look into the references.

The second subject will be a quick tour of Abelian instantons and the construction of a topological field theory that describes the moduli space. That model should not be taken too seriously- I have included it so as to explain how a topological field theory may contain 'τ' dependence and to introduce some ideas that we will need later on.

3.1 Maxwell Theory on X and S Duality

The usual action for pure Maxwell theory with a theta term is

$$S = \frac{1}{g^2}\int_X F_A * F_A + i\frac{\theta}{8\pi^2}\int_X F_A F_A. \tag{3.1}$$

The partition function is a function of both τ and $\bar{\tau}$ where

$$\tau = \frac{\theta}{2\pi} + i\frac{4\pi}{g^2}. \tag{3.2}$$

What we would like to know how the partition function behaves under the action of $SL(2,\mathbb{Z})$ on τ. That action is described by

$$\tau \to \frac{a\tau + b}{c\tau + d} \tag{3.3}$$

where the constants a, b, c, d are integers and obey $ad - bc = 1$ (so that one may group them together into a matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{3.4}$$

which clearly defines the group $SL(2,\mathbb{Z})$). One can generate $SL(2,\mathbb{Z})$ by the transformations

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{3.5}$$

and

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \tag{3.6}$$

From these we see that $T(\tau) = \tau + 1$ (just substitute into (3.3) the values $a = b = d = 1$ and $c = 0$) or $\theta \to \theta + 2\pi$. If the partition function is invariant under T then we are saying the physics is periodic in θ with period 2π. Likewise the action of S on τ is $S(\tau) = -1/\tau$, or $g^2/4\pi \to 4\pi/g^2$. The label S is thus apt for it has the effect of exchanging weak and strong coupling.

It is strange to talk about weak and strong coupling for a free theory! The context in which strong-weak duality has been discussed in the school is in theories with monopoles. Now one can mimic the presence of monopoles by allowing for manifolds with non trivial two-cycles then the 'Dirac quantization' condition applies or, put another way, the integral of F_A over such a surface must be 2π times an integer. A small surprise is that for some four manifolds the partition function transforms well under T and S^2 but not under S. We will find that the partition function is a modular form of particular weight (see below), of $SL(2,\mathbb{Z})$ or a (finite index) subgroup thereof.

We can now check for the properties of the partition function under both $\tau \to \tau + \alpha$ and $\tau \to -1/\tau$.

<u>$\tau \to \tau + 1$ or $\tau \to \tau + 2$</u>

$\tau \to \tau + \alpha$ is easy to check as it corresponds to $\theta \to \theta + 2\pi\alpha$. The action shifts by

$$i\frac{\alpha}{4\pi}\int_X F_A F_A. \tag{3.7}$$

Now we know that $F_A = da + 2\pi n^i \gamma_i$ where $a \in \Omega^1(X, \mathbb{R})$ and γ_i is a basis of $H^2(X, \mathbb{R})$ dual to a basis of $H_2(X, \mathbb{R})$, i.e. $\int_{\gamma_j} \gamma_i = \delta_{ij}$, and that the n^i are integers. This decomposition is due to the, by now, familiar magnetic flux condition

$$\int_{\gamma_i} F_A = 2\pi n^i. \tag{3.8}$$

With these conventions (3.7) becomes

$$i\pi\alpha n^i Q_{ij} n^j \tag{3.9}$$

where $Q_{ij} = \int_X [\Sigma_i][\Sigma_j]$. More on the matrix Q later. If $n.Q.n$ is even then $\exp(-S)$ is unchanged for $\alpha = 1$, $\theta \to \theta + 2\pi$, while if $n.Q.n$ is odd one needs to take $\alpha = 2$, $\theta \to \theta + 4\pi$, to have an invariance. There are general results that tell us that for manifolds on which fermions are defined n^2 must be even. One says that there is a spin structure. For manifolds which do not admit a spin structure there is no condition on n^2. Lets turn to some examples.

- $X = \Sigma_1 \times \Sigma_2$

  We know that we can have spinors on Riemann surfaces so in this case we would expect that $n.Q.n$ is even. First we have to decide what the available harmonic two-forms are. We can use the Kunneth formula $H^p(X_1 \times X_2) = \sum_{q=0}^{q=p} \oplus H^{(q)}(X_1) \otimes H^{(p-q)}(X_2)$ and set $p = 2$. To simplify matters I will do the case of $X = S^2 \times S^2$ and leave the genral case as an exercise. Let us denote the basic two forms by ω_1 and ω_2 of the first and the second S^2 respectively. Kunneth tells us that these two span the second cohomology group of $S^2 \times S^2$. The ω_i are normalised by

$$\int_{S^2_i} \omega_j = \delta_{ij} \tag{3.10}$$

and the only non-zero components of Q are

$$Q_{12} = Q_{21} = \int_{S^2 \times S^2} \omega_1 \omega_2 = 1. \tag{3.11}$$

Then we have

$$n.Q.n = 2n^1 n^2 \tag{3.12}$$

which is even, as promised.

- $X = \mathbb{CP}^2$

The second cohomology group of $\mathbb{CP}^2$ has only one generator which we denote by ω. The harmonic part of F_A is therefore proportional to ω

$$F_A = 2\pi n \omega \tag{3.13}$$

and consequently $n.Q.n = n^2$ which can be even or odd. When discussing Witten theory we will see that indeed $\mathbb{CP}^2$ does not admit a spin structure.

$\tau \to -1/\tau$

In order to check the modular properties of the partition function under $\tau \to -1/\tau$ we need to re-write the action. Consider instead of (3.1) the action

$$S(F, V) = \frac{1}{4\pi} \int_X F(iRe\tau + Im\tau *)F - \frac{i}{2\pi} \int_X dVF. \tag{3.14}$$

Here V is to be understood as a connection on a non-trivial bundle, in defining the path integral we should sum over all such lines, and F is an arbitrary two-form. We can write

$$F_V = dV = dv + 2\pi m^i [\Sigma_i]. \tag{3.15}$$

Firstly lets establish that the theory defined in this way is equivalent to Maxwell theory, also summed over all non-trivial line bundles. The partition function is

$$Z(\tau, \overline{\tau}) = \sum_{m^i} \int DFDv\, e^{-S(F,V)}. \tag{3.16}$$

F may be decomposed as $F = *dB + da + c^i[\Sigma_i]$ where the b^i are real numbers. The integral over v gives a delta function constraint setting $dF = 0$, which implies $B = 0$. We are left with the sum

$$\sum_{m^i} \exp\left(im^i Q_{ij} c^j\right) \tag{3.17}$$

which is a periodic delta function which forces $c^j = 2\pi n^j$. So the requirement that F, after integrating out V, is F_A fixes the coefficient of the dVF term in the action (up to a sign). This establishes that the partition function agrees with the Maxwell partition function.

Now we integrate out F instead of V. This is a simple Gaussian integral (and I leave it as an exercise) giving

$$Z(\tau, \overline{\tau}) = \sum_{m^i} \int Dv \exp\left(\frac{-1}{g'^2} \int_X F_V * F_V - i\frac{\theta'}{8\pi^2} \int_X F_V F_V\right) \tag{3.18}$$

where

$$\frac{4\pi}{g'^2} = \frac{Im\tau}{\tau\overline{\tau}}, \quad \frac{\theta'}{2\pi} = -\frac{Re\tau}{\tau\overline{\tau}}. \tag{3.19}$$

These equalities correspond to $\tau \to -1/\tau$. Notice that we have not shown invariance of the partition function but rather described its 'covariance'.

In checking the duality transformations we have not at all worried about the normalisation of the path integral measure. We will certainly have to worry about this if we want to be sure of the complete dependence on τ of the partition function.

'Semi-Classical Expansion'

In order to perform the path integral over gauge fields with non-trivial first Chern class (monopole number) we write the field strength as

$$F_A = 2\pi n^i [\Sigma_i] + da \tag{3.20}$$

and let the path integral be an integral over the globally defined vectors a, though they may have non-trivial harmonic 1-form pieces, as well as a summation over the n^i. It is worth remarking that the split (3.20) is a standard one we often employ in field theory, namely a split into a classical configuration plus a quantum part. A classical configuration in Maxwell theory is one that satisfies

$$d * F_A = 0 \tag{3.21}$$

while all F_A satisfy $dF_A = 0$. This tells us that $F_A \in H^2(X, \mathbb{R})$. The monopole quantization condition tells us that indeed $F_A/2\pi \in H^2(X, \mathbb{Z})$, so that classical configurations take the form $F_A = 2\pi n^i [\Sigma_i]$.

Substituting (3.20) into the action (3.1) we see that the classical and quantum configurations do not talk

$$S = \frac{1}{g^2} \int_X da * da + \frac{4\pi^2}{g^2} n^i G_{ij} n^j + i\frac{\theta}{2} n^i Q_{ij} n^j \tag{3.22}$$

where

$$G_{ij} = \int_X [\Sigma_i] * [\Sigma_j] \tag{3.23}$$

is the metric on the space of harmonic two-forms. Consequenttly we may split the path integral into a product of the τ dependent classical part Z_c and the $\mathrm{Im}\tau$ dependent quantum part Z_q,

$$Z(\tau) = Z_c(\tau) Z_q(\tau) \tag{3.24}$$

with

$$Z_c(\tau) = \sum_{n^i} \exp - \left(\frac{4\pi^2}{g^2} n^i G_{ij} n^j + i\frac{\theta}{2} n^i Q_{ij} n^j \right), \tag{3.25}$$

and

$$Z_q(\tau) = \int Da \exp\left(-\frac{1}{g^2} \int_X da * da \right). \tag{3.26}$$

We see that, with some definition, the τ dependence of Z_q is only in the form of $\mathrm{Im}\tau$. Witten gives us a way to fix this dependence. The dependence is of the form

$$Z_q(\tau) \sim \left(\sqrt{\mathrm{Im}\tau}\right)^{b_1 - 1}. \tag{3.27}$$

Lets have a look at some simple examples.

- $X = \mathbb{CP}^2$

  We know that the classical part of $F_A = 2\pi n\omega$. Furthermore, the metric is such that $*\omega = 1$. Hence

  $$Z_c(\tau)_{\mathbb{CP}^2} = \sum_n \exp\left(-i\pi\overline{\tau} n^2\right). \tag{3.28}$$

  Notice that this is only invariant under $\tau \to \tau + 2$. To discover the behaviour of $Z_c(\tau)_{\mathbb{CP}}$ under $\tau \to \tau$ we use a small trick. We rewrite $Z_c(\tau)_{\mathbb{CP}}$ as

  $$\sum_{n\in\mathbb{Z}} \exp\left(-i\pi\overline{\tau}n^2\right) = \sum_{m\in\mathbb{Z}} \int_{-\infty}^{\infty} dy \exp\left(-i\pi y^2 \overline{\tau} + 2\pi i y m\right) \tag{3.29}$$

  and the equality holds because the sum over m will give zero unless y is an integer. Now we can integrate out y to obtain

  $$Z_c(\tau)_{\mathbb{CP}} = \frac{1}{\sqrt{i\overline{\tau}}} Z_c(-\frac{1}{\tau})_{\mathbb{CP}} \tag{3.30}$$

  which tells us that $Z_c(\tau)_{\mathbb{CP}}$ is a modular form of degree $(0, 1/2)$.

- $X = \overline{\mathbb{CP}}^2$

  The only difference between $\mathbb{CP}^2$ and $\overline{\mathbb{CP}}^2$ is an reversal of orientation, so that in this case $*\omega = -\omega$, and

  $$Z_c(\tau)_{\overline{\mathbb{CP}}^2} = \sum_n \exp\left(-i\pi\overline{\tau}n^2\right). \tag{3.31}$$

  This is a modular form of degree $(1/2, 0)$.

- $X = S^2 \times S^2$

  From our previous discussion we know that $F_A = 2\pi n^1 \omega_1 + 2\pi n^2 \omega_2$. The only thing we need to specify is the metric $*$. We do this by setting

  $$*\,\omega_1 = R^2 \omega_2, \quad *\omega_2 = \frac{1}{R^2}\omega_1, \tag{3.32}$$

  which satisfies $*^2 = 1$. Z_c for such configurations is

  $$Z_c(\tau)_{S^2\times S^2} = \sum_{n_1,n_2} \exp\left(-i\frac{\pi}{2}\tau(n_1 R - \frac{n_2}{R})^2 + i\frac{\pi}{2}\overline{\tau}(n_1 R + \frac{n_2}{R})^2 \right) \tag{3.33}$$

  which has a form familiar from the study of the $R \to 1/R$ symmetry in string theory for a boson compactified on a circle of radius R (what the partition function lacks are the propagating modes).

At this point there are various lines of investigation available. One is to establish the τ dependence of Z_q, see [12]. On the otherhand the similiraity between the partition functions obtained here and those of string theory lead one in another direction [10]. Instead we turn to a topological field theory.

3.2 Abelian Instantons

One may wonder what the space of solutions to the Abelian instanton equation is. We would like to solve

$$F^+_{\mu\nu} = 0. \tag{3.34}$$

We know that, by the Bianchi identity, $dF = 0$, and as $F = *F$ this also implies $d^*F = 0$, or that $F \in H^2(X, \mathbb{R})$. Actually, as the flux of the gauge field through any two-surface is quantised, $(F/2\pi) \in H^2(X, \mathbb{Z})$. If F is a solution to (3.34) then $(F/2\pi) \in H^2_-(X, \mathbb{R})$.

One important observation is that the space of solutions *is* metric dependent. Indeed on any 4-manifold X with $b_2^+ > 0$, after a small perturbation of the metric, there are no Abelian instantons except flat ones (so that if X is simply connected as well the only solution to (3.34) is the trivial gauge field $A_\mu = 0$). The reason for this is that, as we have seen, F must lie on a lattice, however it also lives in $H^2_-(X,\mathbb{R})$, which will generically lie off the lattice.

To see how this could be so consider $S^2 \times S^2$ with the two generators of $H^2(X,\mathbb{Z})$ described previously, $b_2^+ = b_2^- = 1$. The metric was taken to satisfy

$$* \omega_1 = \omega_2 , \quad * \omega_2 = \omega_1 , \tag{3.35}$$

so that $*^2 = 1$. In this case any $\omega \in H^2(S^2 \times S^2, \mathbb{Z})$ can be expressed as

$$\begin{aligned} \omega &= n_1\omega_1 + n_2\omega_2 \\ &= \frac{n_1+n_2}{2}\omega_+ + \frac{n_1-n_2}{2}\omega_- , \end{aligned} \tag{3.36}$$

where $\omega_\pm = (\omega_1 \pm \omega_2)$ are the generators of $H^2_\pm(S^2 \times S^2, \mathbb{R})$. From these expressions one sees that H^2_- intersects the lattice at $n_1 = -n_2$.

Figure

Now we are going to perturb the metric a little. Let

$$* \omega_1 = \lambda \omega_2 , \quad * \omega_2 = \frac{1}{\lambda}\omega_1 , \tag{3.37}$$

where λ can be close to unity. With this metric the self-dual anti-self dual basis is

$$\omega_\pm = \omega_1 \pm \lambda\omega_2 , \tag{3.38}$$

and evidently $H^2(\mathbb{Z}) \cap H^2_-(\mathbb{R}) = (0,0)$. The situation is summarised in the diagram.

Figure

Consequently, for a generic metric, the only Abelian instantons are flat connections. On $S^2 \times S^2$ there are no flat connections so generically the moduli space is empty. We are also furnished with an example of how one cannot avoid a jump in the dimension of the moduli space along a one parameter family of metrics on X when $b_2^+ = 1$. λ parameterises the relative volume of the two spheres. If one follows a path in the space of metrics with $\lambda > 1$ to $1 > \lambda$ then one cannot avoid passing through $\lambda = 1$ at which point there are solutions to self dual equations. We summarise this as follows

Figure

This is a persistent problem for both Donaldson theory and Witten theory.

The argument above breaks down if $b_2^+ = 0$ as then $H^2_-(X,\mathbb{R})$ is the entire vector space (so, in particular, the lattice lives there). For simply connected X, this can only happen if $\chi = 2 - \sigma$. One manifold for which this equality holds is $\overline{\mathbb{CP}}^2$. In this case $b_2 = b_2^- = 1$.

Moduli Space

The moduli space of Abelian instantons is taken to be the space of solutions to the instanton equation modulo gauge transformations. When Abelian instantons exist it is not difficult to describe the moduli space. We may as well demand that $b_2^+(X) = 0$, then any Abelian instanton is described by

$$F_A = 2\pi n_i [\Sigma_i] + da \tag{3.39}$$

where $d * da = 0$ (as $d * F_A = 0$). We also must fix the gauge, so we do this by demanding $d * a = 0$. Now the restriction that $d * da = 0$ implies

$$\int_X a d * da = 0 \Rightarrow \int_X da * da = 0 \Rightarrow da = 0. \tag{3.40}$$

We now have the conditions that $da = d*a = 0$ or that $a \in H^1(X,\mathbb{R})$. The gauge invariant description of the points in the moduli space is to consider $\int_{\gamma_i} a$.

Actually this is not quite the end of the story as there are large gauge transformations to take into account. Roughly these arise as follows: a gauge transformation can be thought of as a map $g : X \to U(1)$, these maps fall into different classes because we can map the different 1-cycles non-trivially into $U(1)$. For example, consider a non-trivial one cycle γ with coordinates $0 < \sigma \leq 2\pi$, then one has non-trivial maps $g = \exp(in\sigma)$. Under such a gauge transformation $a \to a + g^{-1}idg$ one finds that $\int_\gamma a \to \int_\gamma a + 2\pi m$ so that these points of the moduli space are defined up to periodicity, they lie on a torus. In general one has that the moduli space is indeed the torus

$$H^1(X,\mathbb{R})/H^1(X,\mathbb{Z}). \tag{3.41}$$

This is the moduli space for fixed first Chern number.

3.3 Topological Field Theory for Abelian Instantons

We will now construct a topological model for Abelian instantons. Even though in general they do not exist we will take the topology to be on our side, namely, we will work with manifolds for which $b_2^+ = 0$.

The fields that we will need are; the gauge field A_μ, its super partner ψ_μ (Grassmann odd), and a scalar field ϕ on the one hand (these encode the geometry) while on the otherhand one needs a self dual tensor field $B^+_{\mu\nu}$, its super partner $\chi_{\mu\nu}$ (also self dual but Grassmann odd) and a pair of scalar super partners $\bar{\phi}$ and η (the first Grassmann even the second odd). All the fields are matched except ϕ (but its superpartner is the ghost field that one gets on gauge fixing A_μ, and which I have supressed).

The transformation rules are

$$\begin{aligned} QA_\mu = \psi_\mu\,, \quad Q\psi_\mu = \partial_\mu\phi\,, \quad Q\phi = 0\,, \\ Q\chi^+_{\mu\nu} = B^+_{\mu\nu}\,, \quad QB^+_{\mu\nu} = 0\,, \\ Q\bar{\phi} = \eta\,, \quad Q\eta = 0\,, \end{aligned} \tag{3.42}$$

Notice that $Q^2 = \mathcal{L}_\phi$, where $\mathcal{L}_\phi$ acts on the fields as a gauge transformation. This means that, even though $Q^2 \neq 0$ acting on a gauge invariant function it will give zero. With this in mind we take the following as our action on any four-manifold X

$$\begin{aligned} S &= \{Q, \int_X \chi^+ F_A + \bar{\phi} d * \psi\} \\ &= \int_X \left(B^+ F_A - \chi^+ d\psi + \eta d * \psi + \bar{\phi} d * d\phi\right). \end{aligned} \tag{3.43}$$

This action is inadequate because of the presence of zero modes. Clearly there is one zero mode for each of ϕ, $\bar{\phi}$, and η (the constant mode; $b_0 = 1$ on any manifold). There are also b_1 zero modes each for A and ψ, while there are b_2^+ zero modes for B^+ and for χ^+. We fix on a manifold with $b_2^+ = 0$, so that we need not worry about zero for either B_+ or χ_+. We can, by hand, simply declare the zero modes of $\bar{\phi}$ and η to be zero. One may also declare that the zero modes of ϕ and the Faddeev-Popv ghost field are also zero (though this is less natural). One can do this in a BRST invariant manner [?]. We also do not have to worry about the zero modes of the gauge field since they lie naturally on a torus and integrating over them will give some finite factor. We are left with the zero modes of ψ to worry about. However, we can soak these up by inserting operators of the form $\int_\gamma \psi$ into the path integral, where $\gamma \in H_1(X, \mathbb{R})$. Notice that these are BRST invariant operators. Everything is now more or less under control-but what does it mean?

Interpretation

Firstly the B_+ integral tells us that we are on the moduli space

$$F_A^+ = 0, \tag{3.44}$$

(providing we also gauge fix). The χ_+ integral enforces

$$(1 + *)d\psi = 0. \tag{3.45}$$

This equation tells us that ψ is tangent to the moduli space. To see this we note that if a is an abelian instanton and $a + \delta a$ is also an abelian instanton then $(F_A - F_{A+\delta A})^+ = 0$ and consequently $(1 + *)\delta a = 0$, which is the equation satisfied by ψ. From a geometrical point of view such a δA would correspond to a tangent to the moduli space providing we also impose that $d * \delta A = 0$. One really only wants tangents which do not lie in the gauge directions and the integral over η imposes that ψ has no components in that direction, i.e.

$$d * \psi = 0. \tag{3.46}$$

Indeed as ψ is Grassmann odd it is most naturally thought of as a one-form on the moduli space, and Q then has the interpretation of being the exterior derivative on $\mathcal{M}$. How many such ψ are there? We can answer this with a simple 'squaring' argument. From (3.45), using the same argument as above for the gauge field, we can conclude that

$$d\psi = 0. \tag{3.47}$$

Taken together with (3.46) this equation implies that ψ is harmonic. We can expand $\psi = \lambda^i\gamma_i$ for λ^i Grassmann parameters and γ_i a basis for $H^1(X, \mathbb{R})$. The ψ probe the tangent space to the torus and there are as many of them as the dimension of the torus.

Relationship with Maxwell Theory

To make some contact with Maxwell theory we note that we can write the action (3.1) as

$$S = \frac{2}{g^2}\int_X F_A^+ F_A^+ + i\frac{\tau}{4\pi}\int_X F_A F_A. \tag{3.48}$$

An equivalent theory is thus obtained on using

$$i\frac{2}{g^2}\int_X B_+ F_A + \frac{1}{2}\int_X B_+ B_+ + i\frac{\tau}{4\pi}\int_X F_A F_A \tag{3.49}$$

as the action. There is a relationship between topological theories and physical theories that comes about by 'twisting'. This will be decribed in the next section, but one part of the relationship that we need is that the actions of the topological and physical theories are, at first, equivalent. This means that what we really wish to consider as the action of the topological theory is not just (3.43), but rather (3.49). This differs from the bosonic part of (3.43) by a topological piece, the theta term, and by a BRST trivial piece B_+^2,

$$\int_X \{Q, \chi_+ B_+\} = B_+ B_+. \tag{3.50}$$

One can, therefore, use (3.49) as the bosonic part of the topological action and maintain topological invariance. Our general arguments tell us that, in principle, the addition of B_+^2 does not change the results.

The upshot of this is that one calculates, within the topological theory

$$\exp\left(-i\frac{\tau}{4\pi}\int_X F_A F_A\right)\left(\prod_i \oint_{\gamma_i}\psi\right) \tag{3.51}$$

This automatically gives us

$$Z(\tau)_{top} \sim \sum_{n_i} \exp\left(-i\pi n^i Q_{ij} n^j\right). \tag{3.52}$$

To be continued

4 A Digression on 4-Manifolds

The basic invariants of any manifold are the homotopy and homology groups.

The Hodge star operator $*$ squares to unity in four dimensions; $*^2 = 1$.

If a two-form $\alpha \in \Omega^2(X, \mathbb{R})$ is self dual then we may refine the Hodge decomposition somewhat. Let

$$\alpha = dA + *dB + \gamma \tag{4.1}$$

where A and B are one forms and γ is a harmonic form. Anti-self duality, or $(1+*)\alpha = 0$ means that

$$\alpha = - * \alpha = -dB - *dA - *\gamma \tag{4.2}$$

or, as the Hodge decomposition is unique, that

$$A = -B\,, \quad *\gamma = -\gamma. \tag{4.3}$$

This means that we can write for any (anti) self dual two-form

$$\alpha = (1-*)dA + \gamma \tag{4.4}$$

with $\gamma = -\gamma$.

4.1 Intersection Form

Just as in two dimensions, there are surfaces in our 4-manifold that cannot be contracted to a point. A simple example is that of $X = S^2 \times S^2$, it is clear that neither of the S^2 factors is contractible to a point. The homology groups $H_p(X, \mathbb{R})$ are a measure of noncontractible surfaces of dimension p in X. Let X be a compact, oriented simply connected 4-manifold. If α represents a class in $H_2(X, \mathbb{Z})$ then, by Poincare duality, we can identify it with a class in $H^2(X, \mathbb{Z})$, which I will also denote by α. Given $\alpha, \beta \in H_2(X, \mathbb{Z})$ we can define a quadratic form, (a $b_2 \times b_2$ matrix),

$$Q : H_2(X;\mathbb{Z}) \times H_2(X, \mathbb{Z}) \to \mathbb{Z}, \tag{4.5}$$

by

$$\int_X \alpha\beta \equiv \alpha.\beta\,. \tag{4.6}$$

Q enjoys the following properties.

- It is unimodular ($\det Q = \pm 1$). This follows from the fact that it provides the Poincare duality isomorphism between $H_2(X)$ and $H^2(X)$.
- It is symmetric. This is a trivial consequence of the fact that two form commute, i.e. $\alpha\beta = \beta\alpha$ whenever either of α or $\beta \in H^2(X, \mathbb{R})$.

Examples:

1. The four sphere S^4 has trivial H_2 so all the intersection numbers vanish.
2. Let $X = S^2 \times S^2$. There are two basic two forms, ω_1 and ω_2 dual to the second and first S^2 respectively. We have

$$Q(\omega_1, \omega_2) = \int_{S^2\times S^2} \omega_1\omega_2 = \int_{S^2\times\{p\}} \omega_1 = \int_{\{p\}\times S^2} \omega_2 = 1. \tag{4.7}$$

With this we see that

$$Q = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{4.8}$$

3. Consider a product 4-manifold, $X = \Sigma_1 \times \Sigma_2$, where the Σ_i are Riemann surfaces of genus g_i. In this case

$$Q = \sum_{2g_1g_2+1} \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{4.9}$$

4. The manifold $\mathbb{CP}^2$, has $b_2 = 1$, so $Q = 1$.

5. The K_3 surface has $b_2 = 22$! This means that Q is a 22×22 matrix. Indeed,

$$Q = 3\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus 2(-E_8), \tag{4.10}$$

where E_8 is the Cartan matrix of the exceptional Lie algebra e_8,

$$E_8 = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 2 \end{pmatrix} \tag{4.11}$$

To be continued

5 Donaldson Theory

Donaldsons original motivation for studying the moduli space of instantons over a compact closed and simply connected four manifold X was to get a handle on the possible differentiable structures that one could place on X. The dimension of the moduli space of instantons, for $SU(2)$, is

$$\begin{aligned} dim\mathcal{M} &= 8c_2 - \frac{3}{2}(\chi + \sigma) \\ &= 8c_2 - 3(1 - b_1 + b_2^+). \end{aligned} \tag{5.1}$$

For $c_2 = 1$ and $b_1 = b_2^+ = 0$ the formal dimension is $dim\mathcal{M} = 5$. The space looks like this,

Figure

The sharp ends are the reducible connections. These are self-dual connections for which the gauge group does not act freely. That is there are non-trivial solutions to $d_A\phi = 0$. This happens precisely when A is an Abelian connection living, say, in the 3 direction of $su(2)$ and ϕ constant also lying in the 3 direction. We saw before that we could avoid Abelian instantons *except* when $b_2^+ = 0$. Of course not all self dual Abelian gauge fields are allowed, they must satisfy

$$\begin{aligned} 1 &= \frac{-1}{8\pi^2}\int_X \mathrm{tr}\, F_A F_A \\ &= -\int_X \frac{F_A}{2\pi}\frac{F_A}{2\pi} \end{aligned} \tag{5.2}$$

the notation being that in the first line one is dealing with the $su(2)$ matrix

$$\begin{pmatrix} F_A & 0 \\ 0 & -F_A \end{pmatrix}. \tag{5.3}$$

How many solutions are there? The answer is $2b_2$. As $H^2 = H^2_-$ we have for a basis $[\Sigma_i]$ of $H^2(X, \mathbb{Z})$

$$\int_X [\Sigma_i] * [\Sigma_j] = \delta_{ij} \tag{5.4}$$

but as they are all anti self-dual $*[\Sigma_i] = -[\Sigma_i]$ we obtain

$$\int_X [\Sigma_i][\Sigma_j] = -\delta_{ij}. \tag{5.5}$$

One expands $F_A = 2\pi n_i[\Sigma^i]$ and so the constraint becomes $n_i n^i = 1$. All vectors of the form $(0, \ldots, \pm 1, \ldots, 0)$ satisfy this and in a vector space of dimension b_2 there are $2b_2$ such vectors.

The other end of the moduli space is a copy of the original manifold X. Recall that $SU(2)$ instantons on $\mathbb{R}^4$ (or S^4) are parameterised by their position and their scale. This is also true on a compact manifold (thanks to some work of Taubes). When one shrinks the scale down to zero (almost) they are parameterised only by their position on X. Saying that backwards: X parameterises the zero size instantons and hence appears at the end of the moduli space.

All of this was the original motivation. Later Donaldson realised that one could work with all sorts of instanton moduli spaces (of various dimensions). One could then define cohomology classes on those spaces which, under good conditions, would be 'topological'invariants that one could associate with the underlying manifold X. It turns out that Wittens topological field theory gives a ready description of these classes and so we turn to that.

5.1 Topological Field Theory of $\mathcal{M}$

In order to write down an action that devolves to an integral over the moduli space of instantons we adopt the same field content as in the $U(1)$ case, except the fields all take values in the adjoint representation. In a nutshell

$$\begin{array}{ll} QA = \psi & Q\psi = d_A\phi \\ Q\chi_+ = B_+ & QB_+ = [\chi_+, \phi] \\ Q\overline{\phi} = \eta & Q\eta = [\overline{\phi}, \phi] \end{array} \tag{5.6}$$

with $Q^2 = \mathcal{L}_\phi$. We can now write down the action

$$S = \int_X \{Q, \left(\chi_+ F_A + \frac{s}{2}\chi_+ B_+ + t\overline{\phi} d_A * \psi\right)\}$$

$$= \int_X \Big(B_+F_+ - \chi_+ d_A\psi + \frac{s}{2}B_+B_+ - \frac{s}{2}\chi_+[\chi_+,\phi] + t\eta d_A * \psi$$
$$+ t\overline{\phi} d_A * d_A\phi + t\overline{\phi}\{\psi, *\psi\}\Big). \qquad (5.7)$$

The observables of the theory can be obtained by the descent equation

$$(Q-d)\,\mathrm{Tr}\,(F_A + \psi + \phi)^n = 0. \qquad (5.8)$$

To be continued

5.2 Relationship to $N=2$ Super Yang-Mills Theory

To establish the relationship between the topological theory and a physical theory one needs the notion of twisting. It is easiest to start with the $N=2$ theory. From various lectures we have seen that the theory is described by one $N=2$ chiral superfield with components

$$\begin{matrix} & A_\mu & \\ \lambda^1 & & \lambda^2 \\ & \phi & \end{matrix} \qquad (5.9)$$

The $N=2$ theory has for global symmetry the Lorentz group $SU(2)_L \times SU(2)_R$ as well as an internal $SU(2)_I$ which acts on $\lambda^I = (\lambda^1, \lambda^2)$, thus exchanging the two supersymmetries. The quantum numbers of the fields under $SU(2)_L \times SU(2)_R \times SU(2)_I$ are

$$\begin{aligned} A_\mu &: \left(\frac{1}{2}, \frac{1}{2}, 0\right) \\ \lambda^1 &: \left(\frac{1}{2}, 0, \frac{1}{2}\right) \\ \lambda^2 &: \left(0, \frac{1}{2}, \frac{1}{2}\right) \\ \varphi &: (0,0,0) \end{aligned} \qquad (5.10)$$

Twisting amounts to redefining the Lorentz group to be $SU(2)_L \times SU(2)_{R'}$ where $SU(2)_{R'}$ is the diagonal sum of $SU(2)_R$ and $SU(2)_I$. The transformation of the fields under $SU(2)_L \times SU(2)_{R'}$ are

$$\begin{aligned} A_\mu &: \left(\frac{1}{2}, \frac{1}{2}\right) \\ \lambda^1 &: \left(\frac{1}{2}, \frac{1}{2}\right) \\ \lambda^2 &: (0,1) \oplus (0,0) \\ \varphi &: (0,0). \end{aligned} \qquad (5.11)$$

We are now in a position to match these fields with those in the topological theory. The gauge field is of course the same in either theory as is $\varphi = (\phi, \overline{\phi})$. λ^1 is now a vector and is what we called ψ_μ in the topological theory. λ^2 is now a sum of a scalar and a self dual two-form and hence corresponds to (χ_+, η). Notice that nothing in sight can be identified with B_+; this is no cause for alarm as B_+ is a multiplier field and may be eliminated algebraically.

One can also determine the new weights of the supersymmetry charges. Originally these were

$$\begin{aligned} Q^I_\alpha &: \left(\frac{1}{2}, 0, \frac{1}{2}\right) \\ \overline{Q}^I_{\dot\alpha} &: \left(0, \frac{1}{2}, \frac{1}{2}\right), \end{aligned} \qquad (5.12)$$

and after the twisting become $(1/2, 1/2)$ and $(0,0) \oplus (0,1)$ respectively. The $(0,0)$ component is a scalar supersymmetry charge and is what we have been calling Q. The charges may be denoted by Q_μ, $Q^+_{\mu\nu}$ and Q.

To be continued

Remarks

- The topological action is guaranteed only to be invariant under Q for arbitary parameters s and t.
- One can twist the $N=2$ action and this will correspond to (??), for certain values of s and t, up to theta terms.
- The $N=2$ action in twisted from will be invariant under Q_μ, $Q^+_{\mu\nu}$ and Q.

5.3 Relationship with the Monopole Equations

Donaldson theory, as described above, is related to $N=2$ super Yang-Mills theory. Indeed the usual way to relate the correlation functions in the field theory to the Donaldson invariants is at weak coupling in the ultraviolet. This corresponds to $u \sim \infty$ in the quantum moduli space, as explained in the lectures of Alvarez-Gaume. The beauty of the topological theory is that it is coupling constant independent and so if one could also evaluate the physical theory at strong coupling (in the infrared) one would get a different, though, equivalent, description of the Donaldson invariants.

Seiberg and Witten found that the infrared limit of the $N = 2$ theory in the infrared is equivalent to a weakly coupled theory of Abelian gauge fields coupled to 'monopoles'. In the u plane this region corresponds to the vicinity of the points $|u| = 1$. Away from these points only the Abelian gauge field is massless while at the degeneration points $u = \pm 1$, the monopoles also become massless.

As the theory is weakly coupled at $|u| = 1$, one is tempted to twist the physical theory at those points to get a different description of the Donaldson polynomials. At those points there is the photon plus an $N = 2$ hypermultiplet (also called a scalar multiplet) of two Weyl fermions, q and $\tilde{q}^\dagger$ and complex bosons, massless 'monopoles', B and $\tilde{B}^\dagger$. We put them into a diamond:

$$\begin{array}{ccc} & q & \\ B & & \tilde{B}^\dagger \\ & \tilde{q}^\dagger & \end{array} \quad . \tag{5.13}$$

Once more the $SU(2)_I$ symmetry acts on the rows and thus non-trivially only on $(B, \tilde{B}^\dagger)$. If we twist, q and $\tilde{q}^\dagger$ remain Weyl spinors, however, B and $\tilde{B}^\dagger$ transform as $(0, 1/2)$, that is, as spinors.

To be continued

6 Witten Theory

The equations that we will be studying in this section are

$$\begin{aligned} F^+_{\mu\nu} &= -\frac{i}{2}\overline{M}\sigma_{\mu\nu}M\,, \\ \gamma^\mu D_\mu M &= 0 \end{aligned} \tag{6.1}$$

where M^α is a Weyl spinor, satisfying $\gamma_5 M = M$, $\overline{M}_\alpha$ its complex conjugate. The gamma matrices satisfy $\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}$ and $\sigma_{\mu\nu} = \frac{1}{2}[\gamma_\mu, \gamma_\nu]$. In order to define the covariant derivative of a spinor on an arbitrary 4-manifold we need to introduce a spin connection ω^{ab}_μ with this in hand one has

$$D_\mu = \partial_\mu + iA_\mu + \frac{1}{4}\omega^{ab}_\mu\sigma_{ab}\,. \tag{6.2}$$

A complete set of conventions for γ-matrices, spinors, the spin connection and vierbeins is given in the appendix.

There are a few points that we ought to check about these equations. Firstly the i factor on the right hand side of the first equation in (6.1) is needed as the gauge field A_μ is taken to be real. Secondly, as the self dual part of $F_{\mu\nu}$ appears on the left hand side, only the self dual part should appear on the right hand side. This is indeed the case. With our definitions

$$\sigma_{\mu\nu}\gamma_5 = \frac{1}{2}\epsilon_{\mu\nu}{}^{\kappa\lambda}\sigma_{\kappa\lambda} \tag{6.3}$$

so that,

$$\begin{aligned} \overline{M}\sigma_{\mu\nu}M &= \frac{1}{2}\epsilon_{\mu\nu}{}^{\kappa\lambda}\overline{M}\sigma_{\kappa\lambda}M \\ &= \frac{1}{2}\left(\delta^\kappa_\mu\delta^\lambda_\nu + \frac{1}{2}\epsilon_{\mu\nu}{}^{\kappa\lambda}\right)\overline{M}\sigma_{\kappa\lambda}M \end{aligned} \tag{6.4}$$

which is self dual[3].

The sign on the righthand side of the first equation in (6.1) is important as well. We will see, shortly, that with this choice of sign strong restrictions can be placed on the solution set[4].

Our next objective is to analyse these equations in more detail and to see what they imply for 4-manifolds, but first a digression.

6.1 Spin and Spin$_\mathbb{C}$ Structures

There is a (would be) catch to writing down these equations. The bad news is that on many manifolds there are topological obstructions to defining spinors. The good news is that spinors can be defined on any smooth compact orientable 4-manifold if they are coupled to gauge fields that satisfy a certain restriction. We will firstly review the obstructions and then proceed to the coupling to gauge fields.

To be continued

6.2 $U(1)$ Bundles

To be continued

6.3 The Invariant

The Dimension of $\mathcal{M}$

[3] Had we changed the sign of γ_5 we would have found this combination to be anti-self dual

[4] One can flip the sign if one takes M to have opposite charge, ie. $M \in \Gamma(S_+ \otimes L^{-1})$.

Let (A, M) be a solution to the monopole equations. For $(A+\delta A, M+\delta M)$ to be an (infinitesimally) nearby solution, we require

$$(\delta A)^{+} = \tag{6.5}$$

The number of linearly independent $(\delta A, \delta M)$ tells us the dimension of the moduli space.

The virtual dimension is, therefore

$$d = -\frac{2\chi + 3\sigma}{4} + c_1(L)^2 . \tag{6.6}$$

When $0 > d$ there are generically no solutions to the monopole equations. More interesting for us is when $d = 0$. Let $x = -c_1(L^2) = -2c_1(L)$, then W vanishes precicely when

$$x^2 = 2\chi + 3\sigma . \tag{6.7}$$

Generically, when x satisfies $x^2 = 2\chi + 3\sigma$, there will be a set of t_x isolated solutions to the monopole equations (up to gauge transformations). Label these points by $P_{i,x}$, $i = 1, \ldots t_x$. We can now define

The Witten Invariant

Fix an x that satisfies (6.7) and to each $P_{i,x}$ associate a sign $\epsilon_{i,x} = \pm 1$- the sign of the determinant of T. Our discussion of the Euler character of a Riemann surface has prepared us for the following definition. The Witten invariant, n_x, is the integer

$$n_x = \sum_i \epsilon_{i,x} . \tag{6.8}$$

Under certain conditions

Perturbations

We have seen that the formal dimension of the moduli space is

$$d = \frac{1}{4}\left(2\chi(X) + 3\sigma(X)\right) + \frac{1}{4} c_1(L).c_1(L) . \tag{6.9}$$

In these notes we will be interested in the case where $d = 0$. However, this is the vanishing of the formal dimension. What we would really like is to have that the formal and true dimension of $\mathcal{M}$ coincide. To achieve this one may have to perturb the equations. We do this by passing to

$$\begin{aligned} F^{+}_{\mu\nu} &= -\frac{i}{2}\overline{M}\sigma_{\mu\nu}M + p_{\mu\nu} , \\ \not{D} M &= 0 , \end{aligned} \tag{6.10}$$

with p some real self-dual two form. Here is a nice fact due to Taubes.

Fact: Let X be a compact, oriented, 4-manifold with $b_2^{+} \geq 1$ and with a symplectic two form ω, then the space of solutions to (6.10), $\mathcal{M}(p)$, will be a smooth manifold for a generic choice of p with dimension (6.9).

In this situation we are in the same position as we were for the calculation of the Euler character previously. When $d = 0$, for a judicicious, though generic, p, and with $b_2^{+} > 1$, $\mathcal{M}(p)$ is a finite union of signed points and the Witten invariant is the sum over these points of the corresponding ± 1's.

When we come to considering Kähler surfaces one can be very explicit about the perturbation. Indeed, following Witten, we will give a thorough description of the perturbed moduli space.

6.4 Bochner-Lichnerowicz-Weitzenboch Formula

Given a set of first order equations, like the monopole equations, there is a technique for extracting some very useful information. The idea goes back to Weitzenboch, but was used most effectively by Bochner. In the context of the Dirac equation, it was Lichnerowicz who first derived In the complex domain, Kodaira has also put this idea to good use. For reasons that will become apparent shortly, I will simply refer to this as the squaring argument.

$$\begin{aligned} \not{D}^2_A &= \gamma^{\mu}\gamma^{\nu}D_{\mu}D_{\nu} \\ &= \left(\frac{1}{2}\{\gamma^{\mu},\gamma^{\nu}\} + \frac{1}{2}[\gamma^{\mu},\gamma^{\nu}]\right) D_{\mu}D_{\nu} \\ &= D^{\mu}D_{\mu} + \frac{1}{2}\sigma^{\mu\nu}[D_{\mu}, D_{\nu}] \\ &= D^{\mu}D_{\mu} + \frac{1}{2}\sigma^{\mu\nu}F_{\mu\nu} - \frac{R}{4} \end{aligned} \tag{6.11}$$

This formula holds irrespective of the dimension of the space that we are working on.

Let

$$s_{\mu\nu} = F^{+}_{\mu\nu} + \frac{i}{2}\overline{M}\sigma_{\mu\nu}M \quad k^{\alpha} = (\not{D}M)^{\alpha} , \tag{6.12}$$

and using the above formula we have

$$\int_X d^4x\sqrt{g}\left(\frac{1}{2}|s|^2 + |k|^2\right) \equiv \int_X d^4x\sqrt{g}\left(\frac{1}{2}g^{\mu\kappa}g^{\nu\lambda}s_{\mu\nu}s_{\kappa\lambda} + \overline{k}_{\alpha}k^{\alpha}\right) = 0 . \tag{6.13}$$

The aim is to cast this identity into a useful form. In order to do this we look at the seperate parts that appear and simplify them.

Notice that

$$\begin{aligned}
\int_X d^4x\sqrt{g}\overline{\not{D}M}\not{D}M &= -\int_X d^4x\sqrt{g}\overline{M}\not{D}\not{D}M \\
&= -\int_X d^4x\sqrt{g}\overline{M}\left(D^\mu D_\mu + \frac{i}{2}\sigma^{\mu\nu}F_{\mu\nu} - \frac{R}{4}\right)M \\
&= -\int_X d^4x\sqrt{g}\overline{M}\left(D^\mu D_\mu + \frac{i}{2}\sigma^{\mu\nu}F^+_{\mu\nu} - \frac{R}{4}\right)M\,, \quad (6.14)
\end{aligned}$$

the last line follows as we know the combination $\overline{M}\sigma M$ is self dual.

One more relationship that we need is by way of a Fierz identity,

$$\begin{aligned}
\delta^\alpha_\beta\delta^\gamma_\epsilon &= \frac{1}{4}\Big(\delta^\alpha_\epsilon\delta^\gamma_\beta + (\gamma_\mu)^\alpha_\epsilon(\gamma^\mu)^\gamma_\beta + (\gamma_5)^\alpha_\epsilon(\gamma_5)^\gamma_\beta \\
&\quad -\frac{1}{2}(\sigma_{\mu\nu})^\alpha_\epsilon(\sigma^{\mu\nu})^\gamma_\beta - (\gamma_\mu\gamma_5)^\alpha_\epsilon(\gamma^\mu\gamma_5)^\gamma_\beta\Big)\,. \quad (6.15)
\end{aligned}$$

Multipy this equation with $M_\alpha M_\gamma\overline{M}^\beta\overline{M}^\epsilon$ and recall that the spinor M is Weyl to obtain

$$-\frac{1}{8}\overline{M}\sigma_{\mu\nu}M\overline{M}\sigma^{\mu\nu}M = \frac{1}{2}\left(\overline{M}M\right)^2 . \quad (6.16)$$

Putting all the pieces together, one arrives at

$$\begin{aligned}
&\int_X d^4x\sqrt{g}\left(\frac{1}{2}|s|^2 + |k|^2\right) \\
&= \int_X d^4x\sqrt{g}\left(\frac{1}{2}|F^+|^2 + g^{\mu\nu}\overline{D_\mu M}.D_\nu M + \frac{1}{2}|M|^4 + \frac{1}{4}R|M|^2\right) . \quad (6.17)
\end{aligned}$$

Notice that the cross terms, $F^+\overline{M}\sigma M$, which are present in both $|s|^2$ and $|k|^2$ cancel in the sum. This is why the sign in the first of (6.1) is important.

6.5 Vanishing Theorems

The vanishing of (6.17) puts some constraints on the solution set of (6.1). For example, if there is a metric on X for which $R > 0$ then all the terms in (6.17) are positive and so they must individually vanish. In particular this implies that $M = 0$ and $F^+_{\mu\nu} = 0$. This is a 'vanishing' theorem.

Even when the scalar curvature is positive the squaring argument puts strong constraints on the solution set. As

$$\int_X d^4x\sqrt{g}\frac{1}{2}\left(|M|^2 + \frac{1}{4}R\right)^2 \geq 0 \quad (6.18)$$

we conclude that

$$\int_X d^4x\sqrt{g}\left(\frac{1}{2}|M|^4 + \frac{1}{4}R|M|^2\right) \geq -\frac{1}{32}\int_X d^4x\sqrt{g}R^2 . \quad (6.19)$$

Now, re-write (6.17) as

$$\int_X d^4x\sqrt{g}\left(\frac{1}{2}|F^+|^2 + |DM|^2\right) = -\int_X d^4x\sqrt{g}\left(\frac{1}{2}|M|^4 + \frac{1}{4}R|M|^2\right) , \quad (6.20)$$

which yields an inequality

$$\int_X d^4x\sqrt{g}|F^+|^2 \leq -\int_X d^4x\sqrt{g}\left(\frac{1}{2}|M|^4 + \frac{1}{4}R|M|^2\right) . \quad (6.21)$$

Combining the two inequalities yields

$$\int_X d^4x\sqrt{g}\frac{1}{2}|F^+|^2 \leq \frac{1}{16}\int_X d^4x\sqrt{g}R^2 . \quad (6.22)$$

The line bundle in question, L, satisfies $c_1(L)^2 = (2\chi + 3\sigma)/4$, but we can also express this as

$$c_1(L)^2 = \frac{1}{(2\pi)^2}\int_X F^2 = \frac{1}{(2\pi)^2}\int_X \sqrt{g}\left(|F^+|^2 - |F^-|^2\right) . \quad (6.23)$$

When the dimension of the moduli space vanishes the bound on F^+ also places a bound on F^-,

$$\int_X d^4x\sqrt{g}|F^-|^2 \leq \frac{1}{16}\int_X d^4x\sqrt{g}R^2 - \pi^2(2\chi + 3\sigma) . \quad (6.24)$$

This places a bound on the number of x's that will lead to a zero dimensional moduli space. Hence, associated to every four manifold, there will only be a finite number of invariants n_x. One can also read the inequality as a condition on the underlying four manifold, namely there will be a zero dimensional moduli space only if

$$\frac{1}{(4\pi)^2}\int_X d^4\sqrt{g}R^2 \geq 2\chi + 3\sigma . \quad (6.25)$$

6.6 Kähler Manifolds

When X is a Kähler manifold we may decompose the components of M according to

$$\Omega_+ = \omega.\Omega^0 \oplus \Omega^{(2,0)} \oplus \Omega^{(0,2)}$$

Let us choose our complex co-ordinates on $\mathbb{R}^4$ to be $z^1 = x^1 + ix^2$, $z^2 = x^3 + ix^4$. The $(2,0)$ and the $(0,2)$ forms are spanned by

$$\begin{aligned}
dz^1dz^2 &= \left(dx^1dx^3 - dx^2dx^4\right) + i\left(dx^1dx^4 + dx^2dx^3\right) \\
d\bar{z}^1d\bar{z}^2 &= \left(dx^1dx^3 - dx^2dx^4\right) - i\left(dx^1dx^4 + dx^2dx^3\right) \quad (6.26)
\end{aligned}$$

The symplectic two form, ω can be taken to be

$$\omega = \frac{i}{2}dz^1 d\bar{z}^1 + \frac{i}{2}dz^2 d\bar{z}^2 = dx^1 dx^2 + dx^3 dx^4 . \qquad (6.27)$$

Self dual two forms Φ satisfy

$$\Phi_{\alpha\beta} = \frac{1}{2}\epsilon_{\alpha\beta\mu\nu}\Phi^{\mu\nu}, \qquad (6.28)$$

so that

$$\begin{aligned}\Phi &= 2\Phi_{12}\left(dx^1dx^2 + dx^3dx^4\right) + 2\Phi_{13}\left(dx^1dx^3 - dx^2dx^4\right) + 2\Phi_{14}\left(dx^1dx^4 - dx^2dx^3\right) \\ &= 2\Phi_{12}\omega + (\Phi_{13} - i\Phi_{14})\, dz^1dz^2 + (\Phi_{13} + i\Phi_{14})\, d\bar{z}^1 d\bar{z}^2 . \qquad (6.29)\end{aligned}$$

From this we see that we may decompose the space of self dual two forms, Ω_+, as $i\omega\Omega^{(0,0)} \oplus \Omega^{(2,0)} \oplus \Omega^{(0,2)}$. This decomposition holds on any Kähler surface.

$K^{1/2} \otimes L \oplus K^{-1/2} \otimes L$

To be continued

Computations

After these preliminaries we find that the 'monopole' equations take on the following simple form

$$\begin{aligned}F^{(2,0)} &= \alpha\beta \\ F^{(1,1)}_{\omega} &= -\frac{\omega}{2}\left(|\alpha|^2 - |\beta|^2\right) \\ F^{(0,2)} &= \overline{\alpha\beta} . \qquad (6.30)\end{aligned}$$

In this notation (6.17) can be rewritten

$$\begin{aligned}\int_X d^4x\sqrt{g}\left(\frac{1}{2}|s|^2 + |k|^2\right) &= \int_X d^4x\sqrt{g}\left(\frac{1}{2}|F^+|^2 + g^{\mu\nu}\overline{D_\mu\alpha}D_\nu\alpha + g^{\mu\nu}\overline{D_\mu\beta}D_\nu\overline{\beta}\right. \\ &\left. \frac{1}{2}(|\alpha|^2 + |\beta|^2)^2 + \frac{1}{4}R(|\alpha|^2 + |\beta|^2)\right) . \qquad (6.31)\end{aligned}$$

We notice that the right hand side has a symmetry

$$A \to A, \quad \alpha \to \alpha, \quad \beta \to -\beta . \qquad (6.32)$$

Even though this is not a symmetry of the equations, it does have strong implications. Firstly, the right hand side of (6.31) has a zero only at a solution of the monopole equations (as from the left hand side we would require $s = k = 0$). If for some β the right hand side is zero then, by the symmetry, it must be also zero for $-\beta$. Thus if (A, α, β) is a solution to the monopole equation so is (A, α, β).

The situation just described implies that

$$F^{(2,0)} = \alpha\beta = -\alpha\beta . \qquad (6.33)$$

We have thus learnt that

$$0 = F^{(2,0)} = F^{(0,2)} = \alpha\beta = \overline{\alpha\beta} . \qquad (6.34)$$

These equations imply that either $\alpha = 0$ or $\beta = 0$. We can deduce which of the two is zero by integrating the $(1,1)$ part of (6.30),

$$\frac{1}{2\pi}\int_X \omega F = -\frac{1}{4\pi}\int_X \omega\omega\left(|\alpha|^2 - |\beta|^2\right) . \qquad (6.35)$$

The left hand side is the degree of the holomorphic line bundle L, sometimes denoted by $\deg(L)$, which is a topological invariant. When $\deg(L) = 0$, there is the possibility of having trivial instantons (in this case both α and β must be zero) and we consider metrics for which this is not possible. Now

Hence, the topological data all but fixes the solutions.

The equations for the spinors are

$$\begin{aligned}\overline{D_{z_1}}\alpha - iD_{z_1}\overline{\beta} &= 0 , \\ iD_{z_2}\overline{\beta} + \overline{D_{z_1}}\alpha &= 0 . \qquad (6.36)\end{aligned}$$

6.7 Perturbation

For a Kähler manifold the condition $b_2^+ > 1$ is equivalent to $H^{(2,0)}(X) \neq 0$. In this case we can take $p = \eta + \overline{\eta}$, where η is a non-zero holomorphic two-form. Before perturbing, the first Chern class of the line bundle L was given completely by the $(1,1)$ component of F. The perturbation is chosen so that this remains the case, namely

$$\int_X F^{(2,0)}\overline{\eta} = \int_X F^{(0,2)}\eta = 0 . \qquad (6.37)$$

An argument similar to the one that lead to (6.34) yields

$$0 = F^{(2,0)} = \alpha\beta - \eta . \qquad (6.38)$$

The vanishing of $F^{(0,2)}$ means that we are still in the realm of holomorphic bundles. The important equation is

$$\alpha\beta = \eta . \tag{6.39}$$

Now $\eta \in H^{(2,0)}(X)$ $(\equiv H^0(X,K))$, and α and β are holomorphic sections of $K^{1/2} \otimes L^{\pm 1}$. Let the divisor of η be a union of irreducible components Σ_i with multiplicity r_i. Then

$$c_1(K) = \sum_i r_i[\Sigma_i], \tag{6.40}$$

where $[\Sigma_i]$ denotes the cohomology class that is Poincare dual to the Riemann surface Σ_i. We take the Σ_i to span $H_2(X,\mathbb{Z})$, and consequently $[\Sigma_i]$ to span $H^2(X,\mathbb{Z})$. The integers $r_i \geq 0$ as the sections are holomorphic. Likewise as α is a holomorphic section of $K^{1/2} \otimes L$ and β a holomorphic section of $K^{1/2} \otimes L^{-1}$

$$\begin{aligned} c_1(K^{1/2} \otimes L) &= \sum_i s_i[\Sigma_i] \\ c_1(K^{1/2} \otimes L^{-1}) &= \sum_i t_i[\Sigma_i], \end{aligned} \tag{6.41}$$

with $0 \leq s_i$, and $0 \leq t_i$. For line bundles, E and F, $c_1(E \otimes F) = c_1(E) + c_1(F)$. Set

$$c_1(L) = \sum_i u_i[\Sigma_i], \tag{6.42}$$

where there is no apriori constraint on the sign of the integers u_i. Even though we do not know the sign of the u_i we do know that

$$t_i = \frac{1}{2}r_i - u_i \geq 0 \tag{6.43}$$

so that $r_i \geq 2u_i$. We also know that

$$s_i = \frac{1}{2}r_i + u_i \tag{6.44}$$

but, because of the bound on the u_i, we obtain $0 \leq s_i \leq r_i$. As we could have run through the argument with s_i and t_i interchanged we conclude that $0 \leq t_i \leq r_i$.

The basic class is of the form $x = -2c_1(L)$ or

$$x = -\sum_i (2s_i - r_i)[\Sigma_i]. \tag{6.45}$$

To be continued

Appendix 1: Vierbeins and the Spin Connection

We will recall here some of the basic formula for the coupling of spinors to a gravitational field. One introduces a vierbein e^a_μ, where the label a is an internal Lorentz label. The gauge field for the internal gauge group $SO(4)$ is called the spin connection and is denoted by ω^{ab}_μ.

The covariant derivative is

$$D_\mu M = \left(\partial_\mu + \frac{1}{4}\omega^{ab}_\mu \sigma_{ab}\right) M . \tag{.46}$$

The matrices $\sigma_{ab} = \frac{1}{2}[\gamma_a, \gamma_b]$, satisfy the algebra of the Lorentz group, namely

$$[\sigma_{ab}, \sigma_{cd}] = \delta_{ad}\sigma_{bc} - \delta_{bd}\sigma_{ac} + \delta_{bc}\sigma_{ad} - \delta_{ac}\sigma_{bd} . \tag{.47}$$

The spin connection is determined from the fact that one requires that the covariant derivative of the vierbein vanishes

$$D_\mu e^a_\nu = \partial_\mu e^a_\nu - \Gamma^\lambda_{\mu\nu} e^a_\lambda + \omega^a_{\mu b} e^b_\nu = 0 \tag{.48}$$

With this one calculates that

$$\begin{aligned} [D_\mu, D_\nu] &= \left(\partial_\mu \omega^{ab}_\nu - \partial_\nu \omega^{ab}_\mu + [\omega_\mu, \omega_\nu]^{ab}\right) \frac{1}{4}\sigma_{ab} \\ &= R^{ab}_{\mu\nu} \frac{1}{4}\sigma_{ab} . \end{aligned} \tag{.49}$$

This gives back the definition of the Riemann curvature tensor

Some properties of the Riemann curvature tensor that will be useful are

$$R_{\kappa\lambda\mu\nu} + R_{\nu\lambda\kappa\mu} + R_{\mu\lambda\nu\kappa} = 0 . \tag{.50}$$

Consider

$$\begin{aligned} \frac{1}{2}\gamma^\mu\gamma^\nu[D_\mu, D_\nu] &= \frac{1}{8}\left(\partial_\mu\omega_\nu - \partial_\nu\omega_\mu + [\omega_\mu, \omega_\nu]\right)^{ab} \gamma^\mu\gamma^\nu\sigma_{ab} \\ &= \frac{1}{8}\left(\partial_\mu\omega_\nu - \partial_\nu\omega_\mu + [\omega_\mu, \omega_\nu]\right)_{\kappa\lambda} \gamma^\mu\gamma^\nu\gamma^\kappa\gamma^\lambda \\ &= \frac{1}{8}R_{\mu\nu\kappa\lambda}\gamma^\mu\gamma^\nu\gamma^\kappa\gamma^\lambda \end{aligned} \tag{.51}$$

We can use the identity (.50)

$$\begin{aligned} 0 &= \left(R_{\kappa\lambda\mu\nu} + R_{\nu\lambda\kappa\mu} + R_{\mu\lambda\nu\kappa}\right)\gamma^\mu\gamma^\nu\gamma^\kappa\gamma^\lambda \\ &= R_{\kappa\lambda\mu\nu}[\gamma^\mu\gamma^\nu\gamma^\kappa + \gamma^\kappa\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\kappa\gamma^\mu]\gamma^\lambda , \end{aligned} \tag{.52}$$

and standard γ matrix-ology, such as

$$\gamma^\nu\gamma^\kappa\gamma^\mu = 2g^{\kappa\mu}\gamma^\nu - 2g^{\nu\mu}\gamma^\kappa + \gamma^\mu\gamma^\nu\gamma^\kappa \qquad (.53)$$

to write all the products of three gamma matrices in the order $\gamma^\mu\gamma^\nu\gamma^\kappa$. One may now deduce that

$$R_{\kappa\lambda\mu\nu}\gamma^\mu\gamma^\nu\gamma^\kappa\gamma^\lambda = -2g^{\kappa\mu}g^{\lambda\nu}R_{\kappa\lambda\mu\nu} = -2R\,. \qquad (.54)$$

Appendix 2: Differential Form Conventions

The basis for differential forms on a manifold X will be denoted by exterior products of dx^μ. Recall that

$$dx^\mu dx^\nu = -dx^\nu dx^\mu. \qquad (.55)$$

A one form $\alpha \in \Omega^1(X, \mathbb{R})$ is given in local coordinates by

$$\alpha = \alpha_\mu dx^\mu. \qquad (.56)$$

Likewise an -n-form $\beta \in \Omega^n(X, \mathbb{R})$ is given by

$$\beta = \beta_{\mu_1\dots\mu_n} dx^{\mu_1}\dots dx^{\mu_n} \qquad (.57)$$

and the 'coefficients' $\beta_{\mu_1\dots\mu_n}$ are totally antisymmetric tensor fields.

There is always a little confusion with the field strength. A gauge field A is denoted by

$$A = A_\mu dx^\mu \qquad (.58)$$

and it is natural to define the field strength as $F = dA$. However, in components we find

$$\begin{aligned} F = dA &= \partial_\mu A_\nu dx^\mu dx^\nu \\ &= \frac{1}{2}(\partial_\mu A_\nu - \partial_\nu A_\mu)\, dx^\mu dx^\nu \\ &= \frac{1}{2}F_{\mu\nu}dx^\mu dx^\nu \end{aligned} \qquad (.59)$$

so that there is a factor of $1/2$ in the definition of the components.

Hodge Dual

If the dimension of X is n, we can define the Hodge star $*$ which maps p forms, α, to $n-p$ forms, $*\alpha$, by

$$*\,\alpha = \frac{1}{p!}\sqrt{g}\,\epsilon_{\mu_1\dots\mu_{n-p}}{}^{\nu_1\dots\nu_p}\alpha_{\nu_1\dots\nu_p}dx^{\mu_1}\dots dx^{\mu_{n-p}}. \qquad (.60)$$

With this definition

$$*^2 = (-1)^{(n-p)p}. \qquad (.61)$$

With these conventions under control we deduce that

$$\int_X F * F = \frac{1}{2}\int_X d^4x\sqrt{g}F_{\mu\nu}F^{\mu\nu}\,, \quad \int_X FF = \frac{1}{4}\int_X d^4x\sqrt{g}\epsilon_{\alpha\beta\mu\nu}F^{\alpha\beta}F^{\mu\nu}. \qquad (.62)$$

Homology and Cohomology

To be continued

Poincare Dual

Theorem: Given any k-cycle S, there exits an $n-k$ form η, called the Poincare dual of S, such that

$$\int_S \omega = \int_X \omega\eta\,, \qquad (.63)$$

for all closed $\omega \in \Omega^k(X, \mathbb{R})$.

From De Rham's theorems we know that if we are given a basis $\{\gamma_k^i\}$ for $H_k(X, \mathbb{R})$ there exists a dual basis $\{\gamma_{n-k}^j\}$ of $H_{n-k}(X, \mathbb{R})$ satisfying

$$\int_X \gamma_p^i\gamma_{n-p}^j = \delta^{ij}. \qquad (.64)$$

To be continued

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