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I.C.T.P., P.O. BOX 586, 34100 TRIESTE, ITALY, CABLE: CENTRATOM TRIESTE

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Lecture II

SUMMER SCHOOL IN HIGH ENERGY PHYSICS AND COSMOLOGY

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N = 2 SUPERSYMMETRY

A. VAN PROEYEN
Instituut voor Theoretische Fysica
Universiteit Leuven
B-3001 Leuven
BELGIUM

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MAIN BUILDING STRADA COSTIERA, 11 TEL. 2240111 TELEFAX 224163 TELEX 460392 ADRIATICO GUEST HOUSE VIA GRIGNANO, 9 TEL. 224241 TELEFAX 224531 TELEX 460449
MICROPROCESSOR LAB. VIA BEIRUT, 31 TEL. 224471 TELEFAX 224600 TELEX 460392 GALILEO GUEST HOUSE VIA BEIRUT, 7 TEL. 22401 TELEFAX 2240310 TELEX 460392

In our case we have that the z^i can be chosen to be X^A (special coordinates), and the index Λ is also A. For these coordinates we will call the metric $G_{A\bar{B}}$

$$\begin{aligned} G_{A\bar{B}}(X,\bar{X}) &= -iF_{AB} + c.c. = 2\mathrm{Im}\ F_{AB} = \partial_A\partial_{\bar{B}}K(X,\bar{X}) \\ K(X,\bar{X}) &= i(\bar{F}_A(\bar{X})X^A - F_A(X)\bar{X}^A) \end{aligned} \tag{3.29}$$

Furthermore we have $\mathcal{N}_{AB} = \bar{F}_{AB}$.

Note action zero for quadratic polynomials.

More general gauging for $\delta F = gy^A C_{A,BC}X^B X^C$. For gauge invariance a Chern–Simons term has to be added.

$$\mathcal{L}_{CS} = -\tfrac{2i}{3}g\epsilon^{\mu\nu\rho\sigma}C_{A,BC}W_\mu^A W_\nu^B\left(\partial_\rho W_\sigma^C - \tfrac{3}{16}g[W_\rho, W_\sigma]^C\right) . \tag{3.30}$$

Curvature

$$R_{A\bar{B}C\bar{D}} = -F_{ACE}g^{E\bar{F}}\bar{F}_{\bar{F}\bar{B}\bar{D}}$$

3.4 Coordinate independent description

We can use more general variables: $X^A(z^i)$, and $F_A(z^i)$. We have then

$$g_{i\bar{j}} = e_i^A\bar{e}_{\bar{j}}^B\partial_A\bar{\partial}_B K \ , \tag{3.31}$$

where $e_i^A = \frac{\partial}{\partial z^i}X^A$. We define now also $G^{A\bar{B}}$ as the inverse of $G_{A\bar{B}}$, and e_A^i as the inverse of e_i^A,... . We obtain then

$$G^{A\bar{B}}e^i_A g_{i\bar{j}} = \bar{e}_{\bar{j}}^B \tag{3.32}$$

$$g_{i\bar{j},k} = \partial_k e_i^A \cdot e_A^k g_{k\bar{j}} + e_i^A\bar{e}_{\bar{j}}^B + \partial_C G_{A\bar{B}}\cdot e_k^C \ , \tag{3.33}$$

as $\bar{e}$ is antiholomorphic. This gives as connection [1]

$$\begin{aligned} \Gamma_{ij}^k &= \hat{\Gamma}_{ij}^k + T_{ij}^k \\ \hat{\Gamma}_{ij}^k &= e_A^k\partial_j e_i^A \\ T_{ij}^k &= e_C^k G^{C\bar{D}}\partial_B G_{A\bar{D}}e_j^B e_i^A \ . \end{aligned} \tag{3.34}$$

The connection $\hat{\Gamma}$ is a flat connection (the curvature tensor vanishes for this one). We define the symmetric tensor (using (3.29))

$$C_{ijk} \equiv e_i^A e_j^B e_k^C F_{ABC} = ie_i^A e_j^B e_k^C\partial_C G_{A\bar{B}} \ . \tag{3.35}$$

Then

$$T_{ij}^k = -ie_A^k C_{ij\ell}g^{\ell\bar{m}}\bar{e}_{\bar{m}}^A \ . \tag{3.36}$$

The covariant derivatives are for a vector V_i

$$\mathcal{D}_i V_j \equiv \partial_i V_j - \Gamma_{ij}^k V_k \ ; \qquad \hat{\mathcal{D}}_i V_j \equiv \partial_i V_j - \hat{\Gamma}_{ij}^k V_k \ . \tag{3.37}$$

[1] In general for Kähler manifolds, the connection is only non–zero when all indices are holomorphic or antiholomorphic, and in the curvature tensor with all indices down, 2 should be holomorphic and 2 antiholomorphic. The nonzero components are $\Gamma_{ij}^k = g^{k\bar{\ell}}\partial_j g_{i\bar{\ell}}$, and $R_{\bar{\ell}i}{}^k{}_j = \partial_{\bar{\ell}}\Gamma_{ij}^k$ with all its other forms related by the symmetries of the curvature tensor.

Observe that by definition the full covariant derivative gives zero on the metric $g_{i\bar{j}}$, but

$$\hat{\mathcal{D}}_i e_j^A = 0 \ ; \qquad \mathcal{D}_i e_j^A = iC_{ij\ell} g^{\ell\bar{m}} \bar{e}_{\bar{m}}^A \ . \tag{3.38}$$

Let us compute the same equations for the derivatives of F_A. First of all, it is also holomorphic and thus $\bar{\partial} F_A = 0$, and defining $h_{iA} = \partial_i F_A = e_i^B F_{AB}$, which is also holomorphic of course, satisfies then

$$\begin{aligned}
\mathcal{D}_i h_{jA} &= iC_{ij\ell} g^{\ell\bar{m}} \bar{e}_{\bar{m}}^B F_{AB} + e_i^B e_j^C F_{ABC} \\
&= iC_{ij\ell} g^{\ell\bar{m}} \bar{e}_{\bar{m}}^B \left(F_{AB} - iG_{A\bar{B}}\right) \\
&= iC_{ij\ell} g^{\ell\bar{m}} \bar{e}_{\bar{m}}^B \bar{F}_{AB} = iC_{ij\ell} g^{\ell\bar{m}} \bar{h}_{\bar{m}A} \ ,
\end{aligned} \tag{3.39}$$

where first (3.32) was used, and then (3.29). This equation (3.39) looks similar as (3.38).

Introducing $V = (X^A(z), F_A(z))$, which is holomorphic $\bar{\partial}_{\bar{i}} V = 0$, we got differential equations on

$$U_i \equiv \partial_i V = (\partial_i X^A, \partial_i F_A) \ . \tag{3.40}$$

Namely:

$$\begin{aligned}
\mathcal{D}_i U_j &= iC_{ijk} g^{k\bar{\ell}} \bar{U}_{\bar{\ell}} \\
\partial_i \bar{U}_{\bar{j}} &= 0 \ .
\end{aligned} \tag{3.41}$$

These differential equations can also be used to define the geometry. This is also the appropriate setting to make contact with the geometry of the moduli of Riemann surfaces.

Observe that

$$\begin{aligned}
K &= i\bar{V}^T \Omega V \\
C_{ijk} &= U_k^T \, \Omega \, \mathcal{D}_i U_j
\end{aligned}$$

where

$$\Omega = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix} \ . \tag{3.42}$$

curvature: $R_{i\bar{j}k\bar{l}} = -C_{ikp} \bar{C}_{\bar{j}\bar{l}\bar{p}} g^{p\bar{p}}$.

4 Symplectic transformations

4.1 Duality symmetries for the vectors

Kinetic action for the spin–1 fields (coupled to scalars)

$$\mathcal{L}_1 = +\tfrac{1}{2}\mathrm{Im}\ \left(\mathcal{N}_{\Lambda\Sigma}\mathcal{F}^{+\Lambda}_{\mu\nu}F^{+\mu\nu\ ,\Sigma}\right) = -\tfrac{i}{4}\mathcal{N}_{\Lambda\Sigma}\mathcal{F}^{+\Lambda}_{\mu\nu}F^{+\mu\nu\ ,\Sigma} + \tfrac{i}{4}\bar{\mathcal{N}}_{\Lambda\Sigma}\mathcal{F}^{-\Lambda}_{\mu\nu}F^{-\mu\nu\ ,\Sigma}\ . \tag{4.1}$$

The field equation for the vector is (from now on we omit the non–abelian parts)

$$\frac{\partial\mathcal{L}}{\partial W^\Lambda_\mu} = 2\partial_\nu\frac{\partial\mathcal{L}}{\partial\mathcal{F}^\Lambda_{\mu\nu}} = 2\partial_\nu\left(\frac{\partial\mathcal{L}}{\partial F^{+\Lambda}_{\mu\nu}} + \frac{\partial\mathcal{L}}{\partial F^{-\Lambda}_{\mu\nu}}\right) \tag{4.2}$$

We then define

$$G^{\mu\nu}_{+\Lambda} \equiv 2i\frac{\partial\mathcal{L}}{\partial\mathcal{F}^{+\Lambda}_{\mu\nu}} = \mathcal{N}_{\Lambda\Sigma}\mathcal{F}^{+\Sigma\,\mu\nu}\ ; \qquad G^{\mu\nu}_{-\Lambda} \equiv -2i\frac{\partial\mathcal{L}}{\partial\mathcal{F}^{-\Lambda}_{\mu\nu}} = \bar{\mathcal{N}}_{\Lambda\Sigma}\mathcal{F}^{-\Sigma\,\mu\nu} \tag{4.3}$$

Observe that these relations are only consistent for symmetric $\mathcal{N}$. So far, this is an obvious remark, as in (4.1) we can choose $\mathcal{N}$ to be symmetric.

Then the Bianchi identities and equations of motions can be written as

$$\begin{array}{rcll} \partial^\mu\mathrm{Im}\ \mathcal{F}^{+\Lambda}_{\mu\nu} & = & 0 & \text{Bianchi identities} \\ \partial_\mu\mathrm{Im}\ G^{\mu\nu}_{+\Lambda} & = & 0 & \text{Equations of motion} \end{array} \tag{4.4}$$

These equations are invariant under $GL(2m,\mathrm{I\!R})$:

$$\begin{pmatrix}\tilde{\mathcal{F}}^+\\ \tilde{G}_+\end{pmatrix} = \mathcal{S}\begin{pmatrix}\mathcal{F}^+\\ G_+\end{pmatrix} = \begin{pmatrix}A & B\\ C & D\end{pmatrix}\begin{pmatrix}\mathcal{F}^+\\ G_+\end{pmatrix} \tag{4.5}$$

To be consistent :

$$\tilde{G}^+ = (C + D\mathcal{N})F^+ = (C + D\mathcal{N})(A + B\mathcal{N})^{-1}\tilde{F}^+ \tag{4.6}$$

$$\rightarrow \boxed{\widetilde{\mathcal{N}} = (C + D\mathcal{N})(A + B\mathcal{N})^{-1}} \tag{4.7}$$

Condition that this is symmetric:

$$\begin{array}{rcl} (C + D\mathcal{N})(A + B\mathcal{N})^{-1} & = & (A + B\mathcal{N})^{-1T}(C + D\mathcal{N})^T \\ (A + B\mathcal{N})^T(C + D\mathcal{N}) & = & (C + D\mathcal{N})^T(A + B\mathcal{N}) \end{array} \tag{4.8}$$

or

$$0 = C^TA - A^TC + \mathcal{N}\left(D^TA - B^TC) + (C^TB - A^TD\right)\mathcal{N} + \mathcal{N}(D^TB - B^TD) \tag{4.9}$$

For general $\mathcal{N}$ the above condition implies that the first and the last term are separately zero; furthermore we assume that the identity is the only constant matrix that commutes with $\mathcal{N}$. This implies thus

$$A^TC - C^TA = 0\ ,\quad B^TD - D^TB = 0\ ,\quad A^TD - C^TB = \mathbb{1}\ . \tag{4.10}$$

These equations imply that $\mathcal{S} \in Sp(2m, \mathbb{R})$:

$$\mathcal{S}^T \Omega \mathcal{S} = \Omega \qquad \text{where} \qquad \Omega = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix} . \tag{4.11}$$

Note that this symmetry was only for abelian sector: $g = 0$.

These transformations are symmetries of solutions of field equations and Bianchi identities. However, they do not leave the action invariant in general. **Vector kinetic lagrangian transforms** as follows

$$\text{Im } \mathcal{F}^{+\Lambda} \mathcal{N}_{\Lambda\Sigma} \mathcal{F}^{+\Sigma} = \text{Im } \mathcal{F}^{+\Lambda} G_{+\Lambda} \to \text{Im } \tilde{\mathcal{F}}^{+\Lambda} \tilde{G}_{+\Lambda} \tag{4.12}$$

$$= \text{Im } \left(\mathcal{F}^{+} G_{+}\right) \tag{4.13}$$

$$+\text{Im } \left(2\mathcal{F}^{+}(C^T B) G_{+} + \mathcal{F}^{+}(C^T A)\mathcal{F}^{+} + G_{+}(D^T B) G_{+}\right) \tag{4.14}$$

If $C = B = 0$ the lagrangian is invariant.
If $C \neq 0, B = 0$ it is invariant up to a four–divergence.

These transformations generically rotate electric into magnetic fields and vice versa. Such rotations, which are called duality transformations because electric and magnetic fields are dual to each other (in the sense of Poincaré duality), cannot be implemented on the vector potentials, at least not in a local way. Therefore, the use of these symplectic transformations is only legitimate for zero gauge coupling constant. From now on we deal exclusively with Abelian gauge groups.

In quantum theory transformations must be in $Sp(2m, \mathbb{Z})$. (See also lectures of J. Harvey).

4.2 Action on the scalar fields

In general the above transformations change $\mathcal{N}$, which are coupling constants. Another such symmetry is the diffeomorphism group: it acts as

$$z \longrightarrow \hat{z}(z) \; ; \qquad \hat{g}_{ij}(\hat{z}(z)) \frac{\partial \hat{z}^i}{\partial z^k} \frac{\partial \hat{z}^j}{\partial z^l} = g_{kl}(z)$$

$$\hat{\mathcal{N}}(\hat{z}(z)) = \mathcal{N}(z) \tag{4.15}$$

Such transformations which change the coupling constants, will be called **'Pseudo–symmetries'**:

$$D_{pseudo} = Diff(\mathcal{M}) \times Sp(2m, \mathbb{R}) \tag{4.16}$$

Combination:

$$\hat{\tilde{\mathcal{N}}}(\hat{z}(z)) = \tilde{\mathcal{N}}(z) = (C + D\mathcal{N})(A + B\mathcal{N})^{-1}$$

Isometries

$\hat{g}_{ij}(z) = g_{ij}(z)$ are proper symmetries of scalar action.

If isometries used together with symplectic transformations such that

$$\hat{\tilde{\mathcal{N}}}(z) = \mathcal{N}(z) \tag{4.17}$$

then this is a proper symmetry

We must have $Iso(\mathcal{M}) \subset Sp(2m; \mathbb{R})$

$$D_{prop} = Iso(\mathcal{M}) \subset Iso(\mathcal{M}) \times Iso(\mathcal{M}) \subset D_{pseudo} \tag{4.18}$$

Examples: S and T dualities
$N = 4$ in parametrization of Sen:

scalars: $\lambda = \lambda_1 + i\lambda_2$ and M

where $M = M^T$ and $M\eta M = \eta^{-1}$
$\eta = \eta^T$ metric of $O(6, 22)$

couplings to vectors:

$$\mathcal{N} = \lambda_1 \eta + i\lambda_2 \eta M \eta$$

should transform as $\widetilde{\mathcal{N}} = (C + D\mathcal{N})(A + B\mathcal{N})^{-1}$; with

$$A^T C - C^T A = 0 \quad , \quad B^T D - D^T B = 0 \quad , \quad A^T D - C^T B = \mathbb{1}$$

- T dualities:

$$\begin{aligned} \tilde{\mathcal{F}}^+ &= A\mathcal{F}^+ \\ \widetilde{M} &= AMA^T \qquad \text{with} \qquad \eta = A^T \eta A \\ \Rightarrow \widetilde{\mathcal{N}} &= \left(A^T\right)^{-1} \mathcal{N} A^{-1} \end{aligned}$$

as required with $\mathcal{S} \equiv \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & \left(A^T\right)^{-1} \end{pmatrix}$.

- S dualities:

$$\begin{aligned} \tilde{\mathcal{F}}^+ &= s\mathcal{F}^+ + r\eta^{-1}\mathcal{N}\mathcal{F}^+ \\ \tilde{\lambda} &= \frac{p\lambda + q}{r\lambda + s} \qquad \text{with} \qquad sp - qr = 1 \\ \Rightarrow \widetilde{\mathcal{N}} &= (p\mathcal{N} + q\eta)(r\eta^{-1}\mathcal{N} + s) \end{aligned}$$

or $\mathcal{S} = \begin{pmatrix} s\mathbb{1} & r\eta^{-1} \\ q\eta & p\mathbb{1} \end{pmatrix}$
non–perturbative