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SUMMER SCHOOL IN HIGH ENERGY PHYSICS AND COSMOLOGY

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INTRODUCTION TO GUTs

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Introduction to GUTs

by

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1) Coupling "constants" do run!

2) A bottom-up approach to GUTs

3) Georgi-Glashow SU(5) GUT

4) Exercise: gauge coupling running in the supersymmetric standard model

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Suggested reviews:

Group theory : R. Slansky Phys. Rep. 79 (81)
GUTs : P. Langacker Phys. Rep. 72 (81)

Yang-Mills Theory + covariant interactions with fermions

Coupling constant RGE's (see for instance Ramond 8.6)

Bare Lagrangian = $\left[\mathcal{L}^{\infty}_{ren} = \mathcal{L}_{\mu} + \mathcal{L}_{C.T.}\right]$ (bare parameters infinite) Contains counterterms!

$$\mathcal{L}^{q}_{YM} = \bar\psi_0 i \not{D} \psi_0 - \frac{1}{4} F_{\mu\nu 0} F^{\mu\nu}_0 - m_0 \bar\psi_0 \psi_0$$

$$\hookrightarrow g_0 \bar\psi_0 \not{A}_0 \psi_0$$

$$D_\mu = \partial_\mu - i g_0 A_{\mu 0} \qquad A_\mu \equiv A^a_\mu T^a$$

$$F_{\mu\nu 0} = \partial_\mu A_{\nu 0} - \partial_\nu A_{\mu 0} - i g_0 [A_{\mu 0}, A_{\nu 0}]$$

The bare quantities are related to the renormalized ones by

$$\psi_0 = (1+K_2)^{1/2} \psi \equiv Z_2^{1/2} \psi$$

$$A_{\mu 0} = (1+K_3)^{1/2} A_\mu \equiv Z_3^{1/2} A_\mu$$

$\Rightarrow$ infinite, finite tree level !, counterterms

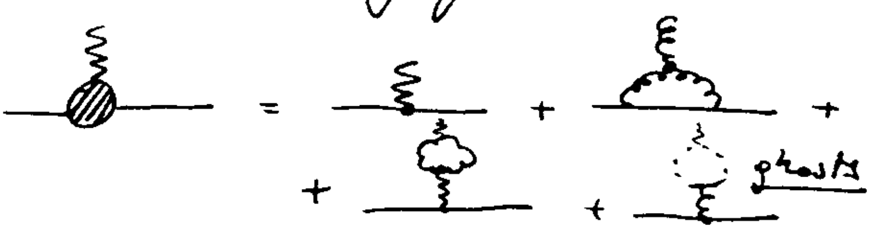
$$m_0 = m \frac{1+K_m}{1+K_2} \equiv m \frac{Z_m}{Z_2} \qquad m_0 \bar\psi_0 \psi_0 = \underbrace{m Z_m} \bar\psi \psi$$

$$g_0 = \mu^{\epsilon/2} g \frac{1+K_1}{Z_2 Z_3^{1/2}} \equiv \mu^{\epsilon/2} g \frac{Z_1}{Z_2 Z_3^{1/2}} \qquad \left[\mu^{\epsilon/2} \underbrace{g Z_1} \bar\psi \not{A} \psi\right]$$

or from quartic A_μ coupling $\sim g^2$ $\quad g^2$ see Ramond !

$$\left[g_0''\right]^2 = \left[g \mu^{\epsilon/2} \frac{Z_4^{1/2}}{Z_3}\right]^2$$ Ward Identities will ensure $g_0'' = g_0$!

Renormalization of g_0



from propagators

— ◆ —

Running coupling constants

Ward identities $Z_1 = Z_2$?

why [illegible] is left.

$$g_0 = g\mu^{\epsilon/2}\frac{Z_1}{Z_2 Z_3^{1/2}}$$

+ the counterterms $|_{\overline{MS}}$

$$Z_i\Big|_{\overline{MS}} = 1 + \underbrace{\frac{g^2}{(4\pi)^2}k_i \times \frac{\tilde{r}}{2}}_{K} + O(g^4)$$

← finite constant

$\rightarrow \left(k_1 - k_2 - \frac{k_3}{2}\right)\ln\frac{\mu^2}{M^2}$

$$\text{where } \tilde{r} \equiv \frac{2}{\epsilon} - \gamma_E + \ln 4\pi \quad \left(+\ln\frac{\mu^2}{M^2}\right)$$

this comes from $\mu^{\epsilon} g^2$

It is left in the finite $g(\mu)$ renormalized

$\rightarrow$

$$g_0 = g_{\overline{MS}}\,\mu^{\epsilon/2}\left[1 + \frac{g^2_{\overline{MS}}}{(4\pi)^2}\left[\frac{\tilde{r}}{2}\right]\left(k_1 - k_2 - \frac{k_3}{2}\right)\right] + O(g^5)$$

Ward Identities $k_1 = k_2$?

(Slavnov-Taylor)

$$0 = \mu\frac{\partial}{\partial\mu}g_0 \overset{\overline{MS}}{=} \frac{\epsilon}{2}\cancel{\mu^{\epsilon/2}}\left[g + \frac{g^3}{16\pi^2}\frac{\tilde{r}}{2}(\ldots)\right]$$

$$+\cancel{\mu^{\epsilon/2}}\mu\frac{\partial g}{\partial\mu} + \cancel{\mu^{\epsilon/2}}\mu\frac{\partial}{\partial\mu}\left(\frac{g^3}{16\pi^2}\right)\frac{\tilde{r}}{2}(\ldots)$$

$$= \frac{1}{2}\frac{g^3}{16\pi^2}(\ldots.) + \mu\frac{\partial g}{\partial\mu} + \frac{\epsilon}{2}g + \mu\frac{\partial}{\partial\mu}\frac{g^3}{16\pi^2}\frac{\tilde{r}}{2}(\ldots)$$

$$\cancel{\mu^{\epsilon/2} = e^{\frac{\epsilon}{2}\ln\mu} = 1 + \frac{\epsilon}{2}\ln\mu}$$

$$\mu\frac{\partial g}{\partial\mu} = -\frac{1}{2}\frac{g^3}{16\pi^2}(\ldots.) - \left(\mu\frac{\partial g}{\partial\mu}\right)\frac{\partial}{\partial g}\left[\frac{g^3}{16\pi^2}\frac{r}{2}(\ldots)\right] - \frac{\epsilon}{2}g$$

$$\overset{\epsilon\to 0}{=} \frac{1}{2}\left(g\frac{\partial}{\partial g} - 1\right)\left(\frac{g^3}{16\pi^2}\right)(\ldots.) + O(g^5)$$

$$\left[\mu\frac{\partial g}{\partial\mu} \simeq -\frac{\epsilon}{2}g + O(g^3)\right]$$

$\left(k_1 - k_2 - \frac{k_3}{2}\right)$

$\frac{2g}{2}$) $$\mu \frac{\partial g}{\partial \mu} \overset{(1)}{=} \frac{g^3}{16\pi^2} \underbrace{\left(k_1 + k_2 - \frac{k_3}{2}\right)}_{b_{0g}} \equiv \frac{\partial g}{\partial \ln \mu}$$

$$\equiv \beta(g)$$

this is the coefficient of $\log \frac{\mu}{M}$!

[see previous page]

Let us multiply by g both sides of eq.

$$\frac{\partial g^2}{\partial \ln \mu^2} = g \beta(g)$$

$\alpha \equiv \frac{g^2}{4\pi}$

$t = \frac{1}{4\pi} \ln \mu^2 = \frac{1}{2\pi} \ln \mu$

$$\boxed{\frac{\partial \alpha_i}{\partial t} = b_i \alpha_i^2}$$

b_i = beta coefficients

$\downarrow$

$$\boxed{\frac{1}{\alpha_i(m)} = \frac{1}{\alpha_i(M)} - \frac{b_i}{2\pi} \ln \frac{M}{m}}$$

$$t = \frac{1}{4\pi} \ln \mu^2 = \frac{1}{2\pi} \ln \mu$$

$$\frac{d}{dt} \alpha = b_1 \alpha^2$$

$$\int_{\alpha(m)}^{\alpha(M)} d\alpha \, \frac{1}{\alpha^2} = b_1 \int_{t_m}^{t_M} dt$$

$$-\frac{1}{\alpha}\Big|_{\alpha(m)}^{\alpha(M)} = \frac{b_1}{2\pi} \ln \frac{M}{m}$$

$$\boxed{\frac{1}{\alpha(m)} = \frac{1}{\alpha(M)} + \frac{b_1}{2\pi} \ln \frac{M}{m}}$$

1-loop RGE

In order to write [illegible] the coefficients
b_i (coefficients of $\boxed{\log \mu}$ in renormalized couplings:
in dimensional regularization $D = 4 - \epsilon$ we have
$$G_i \left[\frac{2}{\epsilon} - \gamma_E + \ln(4\pi) + \ln \frac{\mu^2}{M_W^2} \right] \to 2\left(\frac{1}{\epsilon} + \ln \frac{\mu}{M} \right) G_i$$
see 9(b)
coeff. of $\frac{1}{\epsilon}$ $\equiv$ coeff. of $\ln \mu$!)
we have to recall a bit of Group theory

Given a group G the index $T_2(G,R)$ of a repr. R of the algebra is given by

(1) $$\mathrm{Tr}\{R^\alpha(G) R^\beta(G)\} = T_2(G,R)\,\delta^{\alpha\beta}$$

$[T_a, T_b] = i f_{abc} T_c$

$\mathrm{Tr}(T_a T_b) = \mathrm{Tr}(T_b T_a)$

$[\text{if } \mathrm{Tr}\, T_i = 0]$

certainly for the Adj

$\boxed{(T_i)_{\alpha\beta} = f_{i\alpha\beta}}$

(2) $\left[\sum_\alpha (R^\alpha R^\alpha)_{ij} = C_2 \delta_{ij}\right]$

α, β run over the algebra of G. $\sum\limits_{\alpha\beta}^{(1)} \equiv \sum\limits_{i,j}^{(2)}$

$$\underbrace{\mathrm{Dim}(R)}_{\delta_{ii}}\, C_2(R) = \mathrm{Dim}(G)\, T_2(G,R)$$

eigenvalue of Casimir operator (fixed)!

$[\mathrm{Dim}(\mathrm{Adj}) \equiv \mathrm{Dim}(G)]$

If $R \equiv \mathrm{Adj}(G) \Rightarrow \underline{C_2(\mathrm{Adj}) = T_2(G, \mathrm{Adj})}$

For $SU(N)$ $\underline{T_2(\mathrm{Fund.}) = \frac{1}{2}}$

$$\Rightarrow \begin{cases} C_2(\mathrm{Fund}) = \dfrac{N^2-1}{N}\,\dfrac{1}{2} = \dfrac{N^2-1}{2N} \\ \text{One finds} \\ T_2(\mathrm{Adj}) = C_2(\mathrm{Adj}) = N \end{cases}$$

$\boxed{\text{exercise}}$ $\boxed{\text{prove this relation!}}$

$\frac{1}{2} D_\mu \phi_1 D^\mu \phi_1 + \frac{1}{2} D_\mu \phi_2 D^\mu \phi_2 \equiv (D_\mu \phi)^* (D^\mu \phi)$

S, F, G

for the two contractions

$\frac{1}{2}\,\phi_1\,\frac{1}{2}\,(\times 2) = \boxed{\frac{1}{2}}$ of couplings

For a Group (i) we have

$$\begin{cases} b_i^S = \frac{1}{6}\, T_{2i}^S & \text{real scalars (complex: } \times 2) \\ b_i^D = \frac{4}{3}\, T_{2i}^D & \text{Dirac fermions (Weyl: } \div 2) \equiv \frac{2}{3} T_{2i}^W \\ b_i^G = -\frac{11}{3}\, T_{2i}^G & \text{massless gauge bosons (e.g. gluons)} \\ \hline b_i^M = -\frac{7}{2}\, T_{2i}^M & \text{massive " " (massless + Goldstone bosons)} \end{cases}$$

$\boxed{\text{Tr over Dirac space}}$

$$-\frac{7}{2} = -\frac{11}{3} + \frac{1}{6}$$

In the symmetric $SU(3)_c \times SU(2)_L \times U(1)_Y$ (massless gauge bosons)

$$\Rightarrow \boxed{\frac{\partial \alpha_i}{\partial t} = \left[-\frac{11}{3} T_{2i}^G + \frac{2}{3} T_{2i}^W + \frac{1}{6} T_{2i}^S\right] \alpha_i^2}$$

$$t = \frac{1}{2\pi} \ln \frac{\mu}{\mu_0}$$

B. Beta function coefficients

We list in this appendix the beta function coefficients relative to the gauge sector in the SM, in its supersymmetric extension, and some related quantities; the two-loop coefficients are taken from ref. [33].

Coefficients of the SM 1-loop running, assuming N_g families and n_l light Higgs doublets (in square brackets the case $N_g = 3, n_l = 1$):

$$\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} 0 \\ -22/3 \\ -11 \end{pmatrix} + N_g \cdot \begin{pmatrix} 4/3 \\ 4/3 \\ 4/3 \end{pmatrix} + n_l \cdot \begin{pmatrix} 1/10 \\ 1/6 \\ 0 \end{pmatrix} \quad \left[= \begin{pmatrix} 41/10 \\ -19/6 \\ -7 \end{pmatrix} \right] \tag{B.1}$$

List of b-differences and the factor d (in square brackets the case $N_g = 3, n_l = 1$):

$$\begin{pmatrix} b_2 - b_3 \\ b_3 - b_1 \\ b_1 - b_2 \end{pmatrix} = \begin{pmatrix} 11/3 \\ -11 \\ 22/3 \end{pmatrix} + n_l \cdot \begin{pmatrix} 1/6 \\ -1/10 \\ -1/15 \end{pmatrix} \quad \left[= \begin{pmatrix} 23/6 \\ -111/10 \\ 109/15 \end{pmatrix} \right] \tag{B.2}$$

$$d = (b_1 - b_3) + \frac{3}{5}\,(b_2 - b_3) = \frac{3}{5}\,(22 + n_l/3) \quad [= 67/5] \tag{B.3}$$

Coefficients of the SM 2-loop running, assuming N_g families and n_l light Higgs doublets (in square brackets the case $N_g = 3, n_l = 1$):

$$\begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -136/3 & 0 \\ 0 & 0 & -102 \end{pmatrix} + N_g \cdot \begin{pmatrix} 19/15 & 3/5 & 44/15 \\ 1/5 & 49/3 & 4 \\ 11/30 & 3/2 & 76/3 \end{pmatrix}$$
$$+ \quad n_l \cdot \begin{pmatrix} 9/50 & 9/10 & 0 \\ 3/10 & 13/6 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \left[= \begin{pmatrix} 3.98 & 2.7 & 8.8 \\ 0.9 & 35/6 & 12 \\ 1.1 & 4.5 & -26 \end{pmatrix} \right] \tag{B.4}$$

Coefficients of the SUSY SM 1-loop running, assuming N_g families and n_d Higgs doublets; in n_d there are light and heavy doublets; $n_d = n_l + n_h$ (in square brackets the case $N_g = 3, n_d = 2$):

$$\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} 0 \\ -6 \\ -9 \end{pmatrix} + N_g \cdot \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} + 3 \cdot n_d \cdot \begin{pmatrix} 1/10 \\ 1/6 \\ 0 \end{pmatrix} \left[= \begin{pmatrix} \overset{(33/5)}{\cancel{66/10}} \\ 1 \\ -3 \end{pmatrix} \right] \tag{B.5}$$

List of b-differences and the factor d (in square brackets the case $N_g = 3, n_d = 2$):

$$\begin{pmatrix} b_2 - b_3 \\ b_3 - b_1 \\ b_1 - b_2 \end{pmatrix} = \begin{pmatrix} 3 \\ -9 \\ 6 \end{pmatrix} + 3 \cdot n_d \cdot \begin{pmatrix} 1/6 \\ -1/10 \\ -1/15 \end{pmatrix} \quad \left[= \begin{pmatrix} 4 \\ -48/5 \\ 28/5 \end{pmatrix} \right] \tag{B.6}$$

$$d = (b_1 - b_3) + \frac{3}{5}\,(b_2 - b_3) = \frac{3}{5}\,(18 + n_d) \quad [= 60/5] \tag{B.7}$$

Coefficients of the SUSY SM 2-loop running, assuming N_g families and n_d Higgs doublets (in square brackets the case $N_g = 3, n_d = 2$):

$$\begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -24 & 0 \\ 0 & 0 & -54 \end{pmatrix} + N_g \cdot \begin{pmatrix} 38/15 & 6/5 & 88/15 \\ 2/5 & 14 & 8 \\ 22/30 & 6/2 & 68/3 \end{pmatrix}$$
$$+ \; n_d \cdot \begin{pmatrix} 9/50 & 9/10 & 0 \\ 3/10 & 7/2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \left[= \begin{pmatrix} 7.96 & 5.4 & 17.6 \\ 1.8 & 25 & 24 \\ 2.2 & 9 & 14 \end{pmatrix} \right] \tag{B.8}$$

List of coefficients d_i of eq. (4.48), in the case $N_g = 3, n_d = 2, n_l = 1$:

$$\begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix} = \frac{2}{5} \begin{pmatrix} 25 \\ -100 \\ 56 \end{pmatrix} \tag{B.9}$$

List of coefficients σ_a of eq. (4.57), in the case $N_g = 3, n_d = 2$ (minimal SUSY $SU(5)$):

$$\begin{pmatrix} \sigma_T \\ \sigma_V \\ \sigma_\Phi \end{pmatrix} = \frac{12}{5} \begin{pmatrix} 3 \\ -2 \\ -1 \end{pmatrix} \tag{B.10}$$

Hypercharge Normalization

If a generation of fermions fill up completely a irr (or reducible) repres. of the GUT gauge group, then the normalization of the generators of the GUT group fixes the relative normalization of the generators of $SU(3)$, $SU(2)$, $U(1)$ originally.
$T^a_{(3)}$ T^a_2 Y

$Tr\left(T^a_{(3)} T^b_{(3)}\right) = \frac{1}{2}\delta^{ab}$ for the color triplets (fundamental)

$Tr\left(T^a_{(2)} T^b_{(2)}\right) = \frac{1}{2}\delta^{ab}$ for the weak doublets "

$Tr\left(Y_i^2\right) = Y_i^2$ each particle (i) is a repres. with different charge Y_i

same normalization holds for $SU(5)$ and $SO(10)$

Take one family

$(3,2,\frac{1}{6})$ $\begin{pmatrix} u \\ d \end{pmatrix}_L$ $(\bar{3},1,\frac{2}{3})$ u_R $(\bar{3},1,-\frac{1}{3})$ d_R $(1,2,-\frac{1}{2})$ $\begin{pmatrix} \nu \\ e^- \end{pmatrix}_L$ $(1,1,-1)$ e^-_R $\begin{bmatrix} (1,1,1) \\ \nu_R \end{bmatrix}$

If they fill a repr. of GUT group $G \to \tilde{T}^a$

$$Tr\left(\tilde{T}^a_{(3)} \tilde{T}^b_{(3)}\right) \propto Tr\left(T^a_{(3)} T^b_{(3)}\right) \times (\underset{\text{doublet}}{2} + \underset{u_R,\, d_R}{2}) = \boxed{2}\,\delta^{ab}$$
subalgebra

$$Tr\left(\tilde{T}^a_{(2)} \tilde{T}^b_{(2)}\right) \propto Tr\left(T^a_{(2)} T^b_{(2)}\right) \times (\underset{\text{color}}{3} + \underset{\text{lepton}}{1}) = \boxed{2}\,\delta^{ab}$$

$$Tr\left(\tilde{T}_{(1)}^2\right) \propto \left(\tfrac{1}{6}\right)^2(3\cdot 2) + \left(\tfrac{2}{3}\right)^2 + \left(-\tfrac{1}{3}\right)^2 + \left(-\tfrac{1}{2}\right)^2\cdot 2 + 1$$
$$= \frac{10}{3}$$

$$\Rightarrow \tilde{T}_{(1)}^2 \to \frac{3}{5}\tilde{T}_{(1)} \quad \Rightarrow \quad Tr\left(\tilde{T}_{(1)}^2\right) \propto \frac{10}{3}\,\frac{3}{5} = \boxed{2}$$

$$\Rightarrow \boxed{Y \to \sqrt{\tfrac{3}{5}}\,Y \quad \text{and} \quad g_1 \to \sqrt{\tfrac{5}{3}}\,g_1}$$

We require that have same normalization only; absolute normalization will be determined when G and reprs are given.

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The requirement of identical normalization of isospin T_3 generator and Y ($Tr\, T_3^2 = Tr\, Y^2$)

$$\Rightarrow \begin{cases} Y' = \sqrt{\frac{3}{5}}\, Y \\ g_1' = \sqrt{\frac{5}{3}}\, g_1 \end{cases} \qquad \mathcal{D}_\mu^{(1)} = \partial_\mu - i \underbrace{g_1}\, Y B_\mu$$

in covariant derivative: $\underline{g_1' Y' = g_1 Y}$ $\quad = \partial_\mu - i g_1' Y' B_\mu$

so that for GUT $g_1 \to g = g_2 = g_3$

$$\text{Remember}: \begin{cases} \alpha_1 = \boxed{\alpha/\cos\theta_W^2 = \frac{3}{5}\alpha_1'} \\ \boxed{\alpha_2 = \alpha/\sin\theta_W^2} \end{cases}$$

Our RGE are given for α_1' !

$$\boxed{Q = T_3 + \sqrt{\frac{5}{3}}\, Y'}$$

$$\boxed{Q = T_3 + Y}$$

If we had defined Y in the SM this way from the beginning it would fit naturally in GUT SU(5) or SO(10) scheme!

$$\frac{1}{\alpha_1'(M_Z)} = \frac{1}{\alpha_G} + \frac{b_1'}{2\pi}\ln\frac{M_{GUT}}{M_Z} \equiv \frac{1}{\alpha_G} + b_1' t_{GUT} = \frac{3}{5}\frac{1-s^2}{\alpha}$$

$$\frac{1}{\alpha_2(M_Z)} = \frac{1}{\alpha_G} + \frac{b_2}{2\pi}\ln\frac{M_{GUT}}{M_Z} = \frac{1}{\alpha_G} + b_2 t_{GUT} = \frac{s^2}{\alpha}$$

$$\frac{1}{\alpha_3(M_Z)} = \frac{1}{\alpha_G} + \frac{b_3}{2\pi}\ln\frac{M_{GUT}}{M_Z} = \frac{1}{\alpha_G} + b_3 t_{GUT}$$

1-loop predictions of M_G, α_G, $\sin^2\theta_W$ in standard $SU(3)_C \times SU(2)_L \times U(1)_Y$

$$\frac{1}{\alpha_i(M_Z)} = \frac{1}{\alpha(M_G)} + b_i\, t \qquad t = \frac{1}{2\pi} \ln \frac{M_G}{M_Z}$$

Initial conditions at M_Z for $\alpha_1, \alpha_2, \alpha_3$

What we know : are $\alpha_{\overline{MS}}(M_Z)$ ~~→~~ $\alpha_{3\,\overline{MS}}(M_Z)$ and $\sin^2\theta_{W\,\overline{MS}}(M_Z)$ (approximately)

$$\frac{1}{\alpha(M_Z)} = 127.9 \pm 0.1 \qquad \sim 1‰ \text{ accuracy}$$

$\boxed{\text{take central values}}$

$$\begin{cases} \alpha_3(M_Z) = 0.12 \pm 0.01 & \sim 10\% \text{ accuracy} \\ \sin^2\theta_W(M_Z) = 0.2316 \pm 0.0003 & \sim 1‰ \text{ accuracy} \end{cases}$$

$$\alpha = \alpha_2 \sin^2\theta_W \Rightarrow \frac{1}{\alpha_2} = \frac{\sin^2\theta_W}{\alpha} \Longleftarrow$$

$$\alpha = \frac{3}{5}\alpha_1 \Rightarrow \frac{1}{\alpha_1} = \frac{3}{5}\,\frac{1-\sin^2\theta_W}{\alpha} \Longleftarrow$$

We have three equations 2 unknowns (M_G, α_G)

Consistency check : let us use (α, $\sin^2\theta_W$) and predict

$$\alpha_G\,,\ M_G\,,\ \alpha_S$$

scheme(b)

Exercise : do it for scheme (a) $\boxed{\alpha, \alpha_3 \text{ input}}$ and find out in both schemes the results

From the solutions of the RGE (scheme a) $(\alpha, \alpha_3$ inputs)

$$\sin^2\theta_W(\mu) = \frac{1}{d}\left[(b_2-b_3)\frac{3}{5} + (b_1-b_2)\frac{\alpha(\mu)}{\alpha_3(\mu)}\right]$$

$$d = (b_1-b_3) + \frac{3}{5}(b_2-b_3) = 67/5 \quad (\text{for } n_H = 1)$$

Notice that the value of $\sin^2\theta_W$ (and $\frac{1}{\alpha_3}$ in scheme b) depends only on the differences $(b_i - b_j)$ and therefore does not depend on n_f fermions !

$$\begin{pmatrix} b_2-b_3 \\ b_3-b_1 \\ b_1-b_2 \end{pmatrix} = \begin{bmatrix} 11/3 \\ -11 \\ 22/3 \end{bmatrix} + n_H \begin{bmatrix} 1/6 \\ -1/10 \\ -1/15 \end{bmatrix} \overset{n_H=1}{=} \begin{bmatrix} 23/6 \\ -111/10 \\ 109/15 \end{bmatrix}$$

$\alpha^{-1} = 128$
$\alpha_3 = 0.12$

$$\underline{\sin^2\theta_W(m_Z)}_{\overline{MS}} = \frac{5}{67}\left[\frac{23}{10} + \frac{109}{15}\underbrace{\frac{\alpha(m_Z)_{\overline{MS}}}{\alpha_3(m_Z)_{\overline{MS}}}}_{6.51\times 10^{-2}}\right] = \underline{\underline{0.207}}$$

not good !

$\boxed{\text{EXP} = 0.232}$

(*) largest error is uncertainty in $\alpha_3 \sim 0.003$ (thresholds effects and 2-loop running are smaller !)

To find $\sin^2\theta_W(M_X)$ $\quad \alpha = \alpha_2 \sin^2\theta_W \quad [\alpha_2(M_X) = \alpha_3(M_X)]$

$$\sin^2\theta_W(M_X) = \frac{5}{67}\left[\frac{23}{10} + \frac{109}{15}\sin^2\theta_W(M_X)\right]$$

$$\sin^2\theta_W(M_X)\left[1 - \frac{109}{201}\right] = \frac{23}{134}$$

$$\sin^2\theta_W(M_X) = \frac{23}{134}\,\frac{201}{92} = \frac{4.623}{12.328} \overset{(1541)}{=} \frac{3}{8} \;!\; (= 0.375)$$

This is a general result which depends only on having assumed that fermions fill up a representation of the GUT group !!! (one gets the same replacing scalars ?)

In particular, when we have the SU(5) generators we can check it as a group identical result

$$\sin^2\theta_W = \frac{g_1'^2}{\frac{5}{3}g_2^2 + g_1'^2} \overset{g_1 = g_2}{\underset{M_X}{=}} \frac{3}{8} \;!\quad \text{semplice no?}$$

$$\sin^2\theta_W = \frac{g_1'^2}{g_2^2 + g_1'^2} = \frac{3/5\, g_1^2}{g_2^2 + 3/5\, g_1^2}$$

$g_2 = g_1$ at M_X

$\frac{1}{\alpha_3}$ in scheme (b) (α, $\sin^2\theta_W$ as inputs)

$$\frac{1}{\alpha_3(M_t)} = \frac{1}{b_1-b_2}\,\frac{1}{\alpha(M_t)}\left[\frac{3}{5}(b_3-b_2) + d\,\sin^2\theta_W(M_t)\right]$$

$$\underline{d} = (b_1-b_3) + \frac{3}{5}(b_2-b_3) = \frac{66+n_H}{5} = \begin{cases} \frac{67}{5} & n_H = 1 \\ \frac{68}{5} & n_H = 2 \end{cases}$$

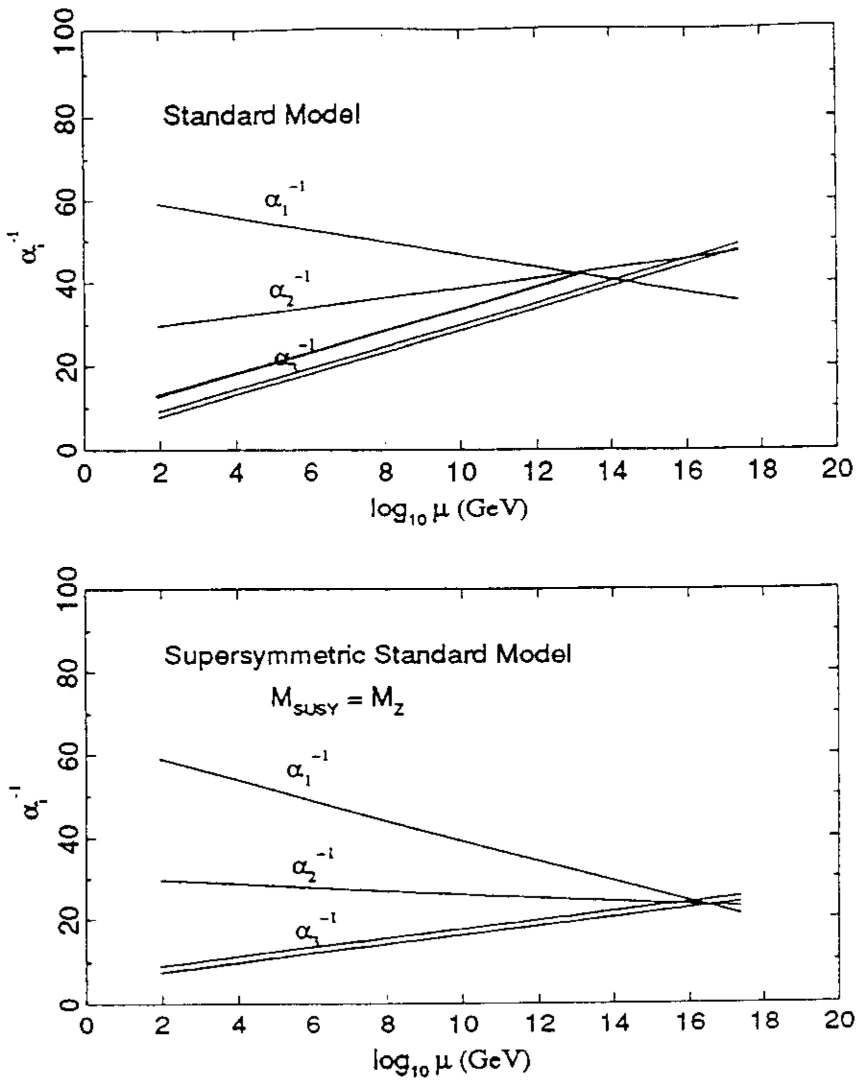
$$\underline{b_1-b_2} = \begin{cases} \frac{109}{15} \\ \frac{108}{15} \end{cases} \qquad \underline{b_3-b_2} = \begin{cases} -23/6 \\ -24/6 \end{cases}$$

$$\frac{1}{\alpha_3} = \frac{1}{\alpha}\,\frac{1}{5}\left(\frac{1}{b_1-b_2}\right)\left[3(b_3-b_2) + 5\cdot d\cdot\sin^2\theta_W\right]$$

$$= \frac{1}{\alpha}\begin{cases} 0.112 \\ 0.106 \end{cases} \qquad \begin{cases} 14.33 \\ 13.51 \end{cases} \text{ too large}$$

$$\alpha_3 = \begin{cases} 0.080 & n_H=1 \\ 0.074 & n_H=2 \end{cases} \quad ?!$$

[uncertainties from 2-loop running and thresholds $\lesssim 0.01$]



;ure 2: Running coupling constants in the standard model and in the minimal ›ersymmetric extension. From [23].

ich is a reasonable average of both the LEP and low energy determinations.

From these inputs one can plot the running couplings and see whether they meet. is seen in Figure 2 that they do not meet in the ordinary standard model, but 'y do meet with reasonable precision in the supersymmetric extension [19]-[21]. is provides evidence, independent of the nonobservation proton decay, that the ıplest versions of nonsupersymmetric grand unification are excluded, while the ›ersymmetric case is allowed.

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3.2 Implications of Coupling Unification

.e coupling constants unify in the supersymmetric extension of the standard model a scale

$$M_X \sim 3 \times 10^{16}\mathrm{GeV}, \tag{18}$$

t not in the ordinary standard model. Does this constitute proof of supersymmetry?

Table 3.6: Numerical predictions of t, $1/\alpha_G$, and $s_0^2(M_Z)$ in case (a). (Input parameters are indicated by brackets. No correction terms are included.)

	SM		MSSM	
	one-loop	two-loop	one-loop	two-loop
t	4.73	4.65	5.28	5.25
$\frac{1}{\alpha_G}$	41.46	41.32	24.18	23.49
$\frac{1}{\alpha(M_Z)}$	(127.9)			
$s_0^2(M_Z)$	0.2070	0.2100	0.2304	0.2335
$\alpha_s(M_Z)$	(0.120)			

Table 3.7: Numerical predictions of t, $1/\alpha_G$, and $\alpha_s(M_Z)$ in case (b). (Input parameters are indicated by brackets. No correction terms are included.) The near equality of cases (a) and (b) for the MSSM is a reflection of the success of the coupling constant unification.

	SM		MSSM	
	one-loop	two-loop	one-loop	two-loop
t	4.04	4.05	5.24	5.32
$\frac{1}{\alpha_G}$	42.41	42.22	24.38	23.12
$\frac{1}{\alpha(M_Z)}$	(127.9)			
$s_0^2(M_Z)$	(0.2316)			
$\alpha_s(M_Z)$	0.071	0.073	0.115	0.129

Table 3.8: The two-loop predictions of the SM and MSSM in case (a) for $\Delta F = 1$ additional fermion family and $\Delta n_H = 1$ or 2 additional light Higgs (super)multiplets. (Input parameters are indicated by brackets. No correction terms are included.)

	SM		MSSM	
	$\Delta F = 1$	$\Delta n_H = 1$	$\Delta F = 1$	$\Delta n_H = 2$
t	4.69	4.58	5.32	4.76
$\frac{1}{\alpha_G}$	34.83	40.83	11.52	22.09
$\frac{1}{\alpha(M_Z)}$	(127.9)			
$s_0^2(M_Z)$	0.2099	0.2141	0.2345	0.2562
$\alpha_s(M_Z)$	(0.120)			

Table 3.9: The two-loop predictions of the SM and MSSM in case (b) for $\Delta F = 1$ additional fermi family and $\Delta n_H = 1$ or 2 additional light Higgs (super)multiplets. (Input parameters are indicated brackets. No correction terms are included.)

	SM		MSSM	
	$\Delta F = 1$	$\Delta n_H = 1$	$\Delta F = 1$	$\Delta n_H = 2$
t	4.07	4.08	5.45	5.80
$\frac{1}{\alpha_G}$	36.63	41.63	10.34	15.74
$\frac{1}{\alpha(M_Z)}$	(127.9)			
$s_0^2(M_Z)$	(0.2316)			
$\alpha_s(M_Z)$	0.073	0.078	0.132	Non-perturbative

only low-energy input used is that of $\alpha(M_Z)$.] The prediction of the former two is compared wi the data. Observe that in superstring theory the relation between the string scale M_S and the stri coupling g_{string}, i.e., $M_S = 5.2 \times g_{string} \times 10^{17}$ GeV [90], is not consistent in the MSSM assuming dire unification, i.e., $M_S = M_G$ and $\alpha_{string} = \alpha_G$ (see also Antoniadis et al. [68]). In general, lower valu for the weak angle and a unification scale $M_G \gtrsim 10^{17}$ GeV imply non-perturbative values of $\alpha_s(M$ This, of course, can be rectified using large threshold corrections. Threshold corrections are discuss next (sections 3.3 and 3.4).

3.3 The Correction Terms: A Formal Discussion

This section will be devoted to the correction terms Δ_i,

$$\Delta_i = \Delta_i^{conversion} + \sum_{boundary}\sum_{\zeta}\frac{b_i^{\zeta}}{(2\pi)}(\ln(\frac{M_\zeta}{M_{boundary}}) - C^{J_\zeta}) + \Delta_i^{top} + \Delta_i^{Yukawa} + \Delta_i^{NRO}. \quad (3.$$

The first term is a constant (briefly discussed in section 2.2) which depends only on the gauge gro G_i [26],

$$\Delta_i^{conversion} \equiv -\frac{C_2(G_i)}{(12\pi)}, \quad (3.$$

where $C_2(G_i)$ is the quadratic casimir operator for the adjoint representation, $C_2(G_i) = N$ (0) for G_i $SU(N)$ [$U(1)$]. $\Delta_i^{conversion}$ results from the need to use the dimensional-reduction ($\overline{\mathrm{DR}}$) scheme in t MSSM, so that the algebra is kept in four dimensions [26]. Thus, we convert the $\overline{\mathrm{MS}}$ couplings abo M_Z,

$$\frac{1}{\alpha_{i_{\overline{MS}}}} = \frac{1}{\alpha_{i_{\overline{DR}}}} - \Delta_i^{conversion}. \quad (3.1$$

For consistency we will use $\overline{\mathrm{DR}}$ also in the SM case, though this is not required. α_G is then given in $\overline{\mathrm{DR}}$ definition.

The second term sums over the one-loop threshold corrections [84]. b_i^{ζ} is the (decoupled) contributi of a heavy field ζ to the β function coefficient b_i between M_ζ and $M_{boundary}$. C^{J_ζ} is a mass-independe number which depends on the spin J_ζ of ζ and on the regularization scheme used. In $\overline{\mathrm{MS}}$ (usi

1) $\frac{3}{5}\frac{1-s^2}{\alpha} = \frac{1}{\alpha_G} + b_1' t$ — 12 — scheme (b)

2) $\frac{s^2}{\alpha} = \frac{1}{\alpha_G} + b_2 t$ 3) $0 = \frac{1}{\alpha_G} + b_3 t - \frac{1}{\alpha_3}$

quantity. Today the situation is different, and the main experimental uncertainty is on the strong coupling constant. As a consequence it is worthwhile to use $\sin^2\theta_W$ as input, and quote the prediction for α_3. We will call *scheme a* the first way of proceeding, *scheme b* the second one.

From the mathematical point of view, we have to solve the following two linear systems

$$scheme\ a: \quad \begin{pmatrix} \frac{3}{5}\frac{1}{\alpha} \\ 0 \\ \frac{1}{\alpha_3} \end{pmatrix} = \begin{pmatrix} 1 & b_1 & \frac{3}{5} \\ 1 & b_2 & -1 \\ 1 & b_3 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\alpha_G} \\ t \\ \frac{\sin^2\theta_W}{\alpha} \end{pmatrix} \tag{3.13}$$

$$scheme\ b: \quad \begin{pmatrix} \frac{3}{5}\frac{1-\sin^2\theta_W}{\alpha} \\ \frac{\sin^2\theta_W}{\alpha} \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & b_1 & 0 \\ 1 & b_2 & 0 \\ 1 & b_3 & -1 \end{pmatrix} \begin{pmatrix} \frac{1}{\alpha_G} \\ t \\ \frac{1}{\alpha_3} \end{pmatrix} \tag{3.14}$$

After inverting the two matrices we obtain:

$$scheme\ a: \quad \begin{pmatrix} \frac{1}{\alpha_G} \\ t \\ \frac{\sin^2\theta_W}{\alpha} \end{pmatrix} = \frac{1}{d} \begin{pmatrix} -b_3 & -\frac{3}{5}b_3 & b_1 + \frac{3}{5}b_2 \\ 1 & \frac{3}{5} & -\frac{8}{5} \\ b_2 - b_3 & b_3 - b_1 & b_1 - b_2 \end{pmatrix} \begin{pmatrix} \frac{3}{5}\frac{1}{\alpha} \\ 0 \\ \frac{1}{\alpha_3} \end{pmatrix} \tag{3.15}$$

$$scheme\ b: \quad \begin{pmatrix} \frac{1}{\alpha_G} \\ t \\ \frac{1}{\alpha_3} \end{pmatrix} = \frac{1}{b_1 - b_2} \begin{pmatrix} -b_2 & b_1 & 0 \\ 1 & -1 & 0 \\ b_3 - b_2 & b_1 - b_3 & b_2 - b_1 \end{pmatrix} \begin{pmatrix} \frac{3}{5}\frac{1-\sin^2\theta_W}{\alpha} \\ \frac{\sin^2\theta_W}{\alpha} \\ 0 \end{pmatrix} \tag{3.16}$$

where $d = 3/5(b_2 - b_3) + (b_1 - b_3)$, and at the end we can write the 1-loop result as follows:

$$scheme\ a: \quad \begin{cases} \frac{1}{\alpha_G} &= \frac{1}{d}\left[(b_1 + \frac{3}{5}b_2)\frac{1}{\alpha_3} - \frac{3}{5}b_3\frac{1}{\alpha}\right] \\ t &= \frac{1}{d}\left[\frac{3}{5}\frac{1}{\alpha} - \frac{8}{5}\frac{1}{\alpha_3}\right] \\ \frac{\sin^2\theta_W}{\alpha} &= \frac{1}{d}\left[(b_2 - b_3)\frac{3}{5}\frac{1}{\alpha} + (b_1 - b_2)\frac{1}{\alpha_3}\right] \end{cases} \tag{3.17}$$

they do not depend on n_f: fermion contribution cancel in $\begin{cases} b_2 - b_3 \\ b_1 - b_3 \end{cases}$!

$$scheme\ b: \quad \begin{cases} \frac{1}{\alpha_G} &= \frac{1}{b_1 - b_2}\left[(b_1 + \frac{3}{5}b_2)\frac{\sin^2\theta_W}{\alpha} - \frac{3}{5}b_2\frac{1}{\alpha}\right] \\ t &= \frac{1}{b_1 - b_2}\left[\frac{3}{5}\frac{1}{\alpha} - \frac{8}{5}\frac{\sin^2\theta_W}{\alpha}\right] \\ \frac{1}{\alpha_3} &= \frac{1}{b_1 - b_2}\frac{1}{\alpha}\left[\frac{3}{5}(\cancel{b_2}_{3} - \cancel{b_3}_{2}) + d\right] \end{cases} \tag{3.18}$$

$\sin^2\theta_W$

We have been quite detailed in this analysis because it will be also useful when computing the two-loop contributions to the running and the threshold effects as corrections to the previous framework; this is in fact the case for SUSY $SU(5)$. In fact the couplings $\alpha_i(M_Z)$ are different from the one-loop values by little corrections Δ_i, which can be thought as modifying the input values in the right hand side of eqs. (3.15–3.16). The procedure will be detailed in the following.

$\boxed{\text{EX}}$ (*) Find out predictions [own formulas] in scheme (b) $\boxed{\text{for } n_H = 1, 2}$ ***

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SM couplings do not meet at a "point" (although they converge to a small region; and this is by itself noticeable).

Let us try to understand what kind of exotic particles should minimally appear at the GUT scale, in order to make the three SM couplings run with the same slope: this is what should happen if a GUT theory ~~exists~~ does exist.

Let us begin with the gauge sector:

$$b_1^G = 0 - \frac{11}{3} X_1$$

$$b_2^G = -22/3 - \frac{11}{3} X_2$$

$$b_3^G = -11 - \underbrace{\frac{11}{3} X_3}_{\text{additional gauge boson which would make}}$$

$$b_1^G = b_2^G = b_3^G$$ needed after unification!
(we know that $b_1^F = b_2^F = b_3^F$ already!)

Now: $b_3 = b_2 \Rightarrow \boxed{X_2 = 1 + X_3}$

Let us try to ~~[illegible]~~ find out what they can be by building them up from fundamental repr.

The fundamental repr. of $SU(N)$: $T_2 = \frac{1}{2}$

Let us consider the tensor product repr. $(3,2)$ of $SU(3) \times SU(2)$ (must "connect" the subgroups) $\frac{SU(5)}{SU(3)\times SU(2)\times U(1)_Y}$

$$X_2 \equiv \quad = \overset{1/2}{T_2(2)} \cdot 3 = \frac{3}{2} \qquad (3,2)$$

$$X_3 \equiv \quad = \overset{=1/2}{T_2(3)} \cdot 2 = 1 \qquad (3,2)$$

$\Rightarrow \boxed{X_2 - X_3 = \frac{1}{2}}_{(3,2)}$! "Half" of what we need !

Now, a repr. for vector gauge bosons must be real (Adj.) therefore we must consider

2 of $SU(2)$ is real !

$$\underbrace{(3,2) \oplus (\bar{3},2)}_{12 \text{ extra gauge bosons}}$$

$\Rightarrow X_2 - X_3 \Big|_{(3,2)\oplus(\bar{3},2)} = 1$! good candidates for completing the vector boson GUT multiplet ?

$\boxed{\begin{matrix} X_2 = 3 \\ X_3 = 2 \end{matrix}}$!

Find out hyperchange:

They have to carry hyperchange to obtain

$\underline{\underline{b_1}} \equiv \underline{\underline{b_3}} \Rightarrow X_1 = +\frac{3}{\cancel{11}}\cancel{11} + X_3 = +5$!

$+5 = 6(Y)^2 + 6(-Y)^2 \Rightarrow Y_{(3,2)} = \pm\frac{1}{2}\sqrt{\frac{5}{3}} \equiv -Y_{(\bar{3},2)}$ antiparity ?

choose a sign ! $\Rightarrow$ $\boxed{\left(3,2,-\frac{1}{2}\sqrt{\frac{5}{3}}\right) \oplus \left(\bar{3},2,+\frac{1}{2}\sqrt{\frac{5}{3}}\right)}$ 12 new Gauge bosons

Let us classify them according to Q_{em}

$$\boxed{Q = T_3 + \sqrt{\tfrac{5}{3}}\, Y_{GUT}}$$

$$\bar{Y}^{\mu 1}_{\alpha} \equiv \Big(3, +\tfrac{1}{2}, \underbrace{\tfrac{1}{2} - \tfrac{5}{6}}_{-1/3}\Big) \qquad Y^{\alpha=1,2,3}_{1} \equiv \Big(\bar{3}, -\tfrac{1}{2}, \underbrace{-\tfrac{1}{2} + \tfrac{5}{6}}_{1/3}\Big)$$

$$\bar{X}^{\mu 2}_{\alpha} \equiv \Big(3, -\tfrac{1}{2}, \underbrace{-\tfrac{1}{2} - \tfrac{5}{6}}_{-4/3}\Big) \qquad X^{\alpha}_{2} \equiv \Big(\bar{3}, +\tfrac{1}{2}, \underbrace{\tfrac{1}{2} + \tfrac{5}{6}}_{4/3}\Big)$$

fractional charge   mix color or weak

total number of gauge bosons

$8 + 3 + 1 + 12 = 24$? = Write box of p. 19 (decomposition of 24 !)

They fit in the 24 of SU(5) ?

Exercize

Write the 24

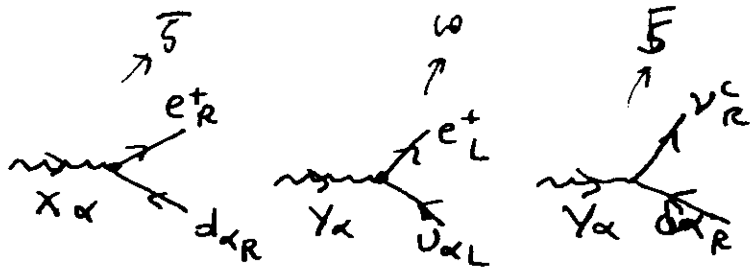
$$A_\mu = \sqrt{2} \sum_{i=1}^{24} A^i_\mu \frac{\lambda^i}{2}$$ — generators of SU(5)

— Write the 24 generators of SU(5) — (see 21a)

— —

Proton Decay

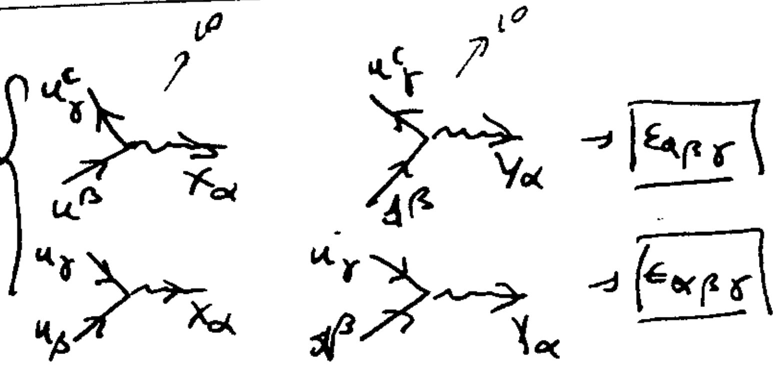
Leptoquarks



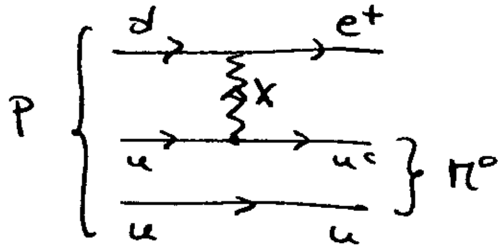
SU(3) $\quad T_3 \quad Q$

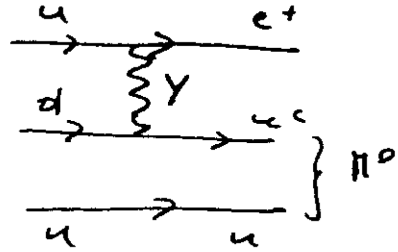
$$Y_\alpha^r = \left(\bar{3}, -\tfrac{1}{2}, \tfrac{1}{3}\right) \qquad X_\alpha^q = \left(\bar{3}, \tfrac{1}{2}, \tfrac{4}{3}\right)$$

diquarks

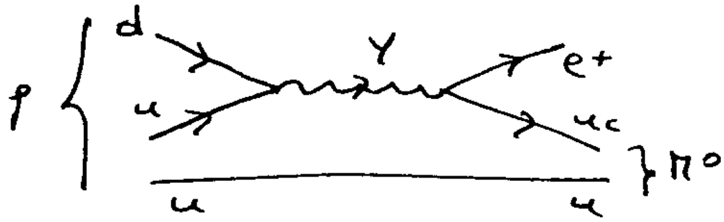


$$\boxed{\overline{\psi^c_R}\,\gamma_\mu\,\chi^c_R = -\bar{\chi}_L\,\gamma_\mu\,\psi_L}$$

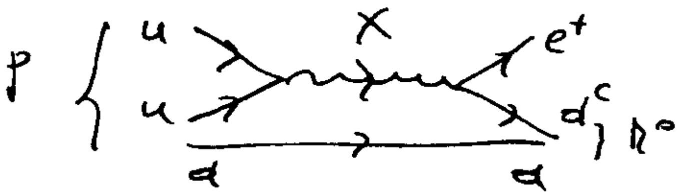


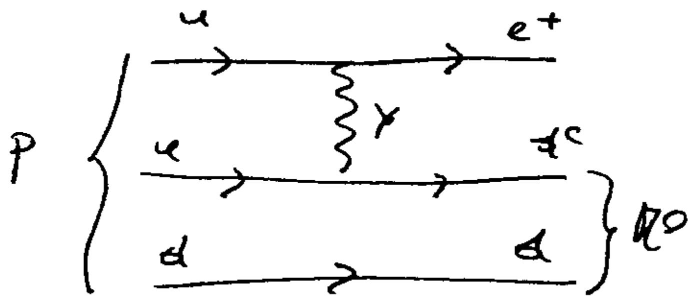


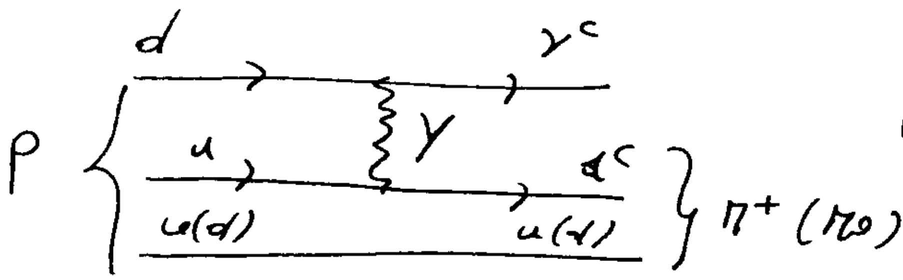
$M^{0\pm} \cong \pi^0$, [redacted] π^+, π^-



$P \to e^+_R + \pi^0$







$P \to \nu^c_R + \pi^0$

Estimate of proton lifetime

$$\Gamma_p \sim \frac{\alpha_{GUT}^2 \, m_p^5}{M_{GUT}^4}$$

$$\left.\begin{array}{l} \alpha_{GUT} \simeq 1/40 \\ M_{GUT} \simeq 10^{13}\ \mathrm{GeV} \end{array}\right. \Rightarrow \Gamma_p \simeq \frac{10^{-3}}{10^{52}}\ \mathrm{GeV}$$

$$\tau_p = \frac{1}{\Gamma_p} \simeq 10^{52}\ \mathrm{MeV}^{-1} = 10^{52}\, 10^{-21}\ \mathrm{sec}$$

$$= 10^{31}\ \mathrm{sec} \simeq 10^{24}\ \mathrm{year}$$

$(\hbar = 6.6 \times 10^{-22}\ \mathrm{MeV \cdot sec}, \quad 1\ \mathrm{Year} \simeq 10^{7}\ \mathrm{sec})$

$T_{p\,(p\to e^+\pi^0)}^{(exp)} > 10^{33}\ \mathrm{yr}$ 9 orders longer!

$$\Rightarrow M_{GUT} \gtrsim 10^{15}\ \mathrm{GeV}$$

The Higgs contribution to the beta coeffs. (**)
($n_H = 1$)

$$b_1^S = \frac{1}{10} + \frac{1}{6}\cancel{\tfrac{1}{2}}\, X_1$$
$$b_2^S = \frac{1}{6} + \frac{1}{6}\cancel{\tfrac{1}{2}}\, X_2$$
$$b_3^S = 0 + \frac{1}{6}\cancel{\tfrac{1}{2}}\, X_3$$

What is minimally missing to make them equal ?

$b_2 = b_3 \Rightarrow X_3 = X_2 + 1$ (a)

$b_3 = b_1 \Rightarrow X_3 = X_1 + \frac{3}{5}$ (b)

(a) a triplet of complex scalars $\Rightarrow T_2 = \frac{1}{2} \times 2 = 1 \equiv X_3$ (fundamental (3), complex)

$\boxed{X_2 = 0} \Rightarrow SU(2)_L$ singlets $\Rightarrow$ (3, 1)

$$\boxed{X_1 \equiv 2 \cdot 3 \cdot Y^2} = X_3 - \frac{3}{5} = 1 - \frac{3}{5} = \boxed{\frac{2}{5}}$$

$$\Rightarrow Y = \frac{1}{\sqrt{15}} = \frac{1}{3}\sqrt{\frac{3}{5}} \quad \Rightarrow \left\{ b_1^S = b_2^S = b_3^S = \frac{1}{6} \right. \ !$$

$Y_{SU} = Q$ (since $T_3 = 0$)

$Q = 1/3$

$$\Rightarrow \bar{5} = \begin{pmatrix} H_1 \\ H_2 \\ H_3 \\ \hline H^- \\ -H^0 \end{pmatrix}$$

analogously to $\bar{5}$ of fermions

$$\Rightarrow \frac{1}{6} T_2 = \frac{1}{6}\left(\frac{1}{2} \times 2\right) = \frac{1}{6}$$ w.r.to SU(5) !

*Then also the 24 ϕ is contributing equally (real)

$$\frac{1}{6} C_2 = \frac{1}{6} 5 = \frac{5}{6}$$ with respect to SU(5) !

(**) What is missing to make them equal?

Introduce the 24 ϕ after discussion of the breaking pattern and conditions ! p. 22a (4.2).

Grand Unified Group [dimension G : total number of linearly independent generators]
(Slansky PRep 79 (81) 1)

Requirements

$(2 + 1 + 1)$

1) G must embed $SU(3) \times SU(2) \times U(1)$
→ rank ≥ 4 [rank : maximum number of simultaneously diagonalizable generators of a single Lie Algebra G.]

ii) G must be single in order to really unify the couplings
(a product of identical single groups whose couplings constants can be set equal by some discrete symmetry can be viable as well)

iii) It must contain complex repres: real (self-conjugate) repre. allow for gauge invariant mass terms for fermions (the particles that we know are not equivalent to their antiparticles!) (except possibly for neutrinos) (lepton number)

---

i) $\begin{cases} U(1) : \text{rank } 1 \\ SU(2) : \text{rank } 1 \\ SU(3) : \text{rank } 2 \end{cases}$ only 1 parameter! T_3 is the only diagonalizable gen.

$\overline{4}$

The maximal abelian subalgebra of G, is the Cartan subalgebra of all the diagonalizable generators H_i
$i = 1, \dots, l$ $\quad l = \text{rank } G$

$$\boxed{[H_i, H_j] = 0 \quad i = 1, \dots, l}$$

$$\lambda_1 = \begin{pmatrix} 0&1&0\\1&0&0\\0&0&0 \end{pmatrix} \quad \lambda_4 = \begin{pmatrix} 0&0&1\\0&0&0\\1&0&0 \end{pmatrix} \quad \lambda_6 = \begin{pmatrix} 0&0&0\\0&0&1\\0&1&0 \end{pmatrix}$$

$$\lambda_2 = \begin{pmatrix} 0&-i&0\\i&0&0\\0&0&0 \end{pmatrix} \quad \lambda_5 = \begin{pmatrix} 0&0&-i\\0&0&0\\i&0&0 \end{pmatrix} \quad \lambda_7 = \begin{pmatrix} 0&0&0\\0&0&-i\\0&i&0 \end{pmatrix}$$

$$\lambda_3 = \begin{pmatrix} 1&0&0\\0&-1&0\\0&0&0 \end{pmatrix} \equiv \underline{H_1} \qquad \lambda_8 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1&0&0\\0&1&0\\0&0&-2 \end{pmatrix} = \underline{H_2}$$

Il termine Cinetico: $\sim \overline{\Psi}_1 \gamma_\mu \Psi_2 = \overline{\varphi}_1 \bar{\sigma}^\mu \varphi_2 - \overline{\varphi}_2 \sigma^\mu \varphi_1$

$\sim f_R \otimes f_L$

Group candidates for GUT [which satisfy criteria i) ii) and iii) of page 15] of rank 4:

$\underbrace{[SU(2)]^4, \; [O(5)]^2,}_{(a)} \; [SU(3)]^2, \; \underbrace{[G_2]^2, \; O(8), \; O(9)}$

$\underline{Sp(8), \; F_4,} \; SU(5)$ → See $\boxed{\text{Slanski, Phys Rep 79(81) (b) or Georgi (book) for group theory !}}$

(a) no SU(3) subgroups

(b) only real reprs

Left: (1) $[SU(3)]^2$

(2) $[SU(5)]$

(1) → No real unification of α_3 and α_{weak} ?

Quarks should be put in a $SU(3)_{weak} \subset U(1)_Q$

Tr(over fundamental of $SU(3)_{weak}$) = 0 ⇒ $\sum Q_f = 0$

⇒ (u, d, s) triplet ⇒ ~~violation of the generation structure!: e.g. (u, d) are~~

~~much more related than u, s~~

~~and~~ What about (c, t, b) ??? no way to insert ~~all~~ the known 6 quarks in a $SU(3)_{weak}$!

⇒ The only viable rank 4 candidate is

SU(5) !

$$\boxed{[Q,\psi_a] = -Q_{ab}\psi_b}$$

repres. of Q

$$\boxed{\psi^c \to -Q^t}$$

$$Y = \sqrt{\frac{3}{5}}\left(\begin{array}{ccc|cc} -1/3 & & & & \\ & -1/3 & & & 0 \\ & & -1/3 & & \\ \hline & & & 1/2 & \\ & 0 & & & 1/2 \end{array}\right)$$

$$T_3 = \left(\begin{array}{c|cc} 0 & & 0 \\ \hline 0 & 1/2 & \\ & & -1/2 \end{array}\right) \Rightarrow \boxed{Q = T_3 + \sqrt{\frac{5}{3}}\,Y}$$

$$Q = \left(\begin{array}{ccc|cc} -1/3 & & & & \\ & -1/3 & & & \\ & & -1/3 & & \\ \hline & & & 1 & \\ & & & & 0 \end{array}\right)$$

acting on a $5 = (3,1)_{-\frac{1}{3}} \oplus (1,2)_{\frac{1}{2}} \Rightarrow \begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ \hline e^+ \\ -\nu^c \end{pmatrix}_R$

for $\bar{5}$ we have to take $\boxed{-Y^T, \ -T_3^T}$ conjugate repr.

$$\bar{5} = (\bar{3},1,\tfrac{1}{3}) \oplus (1,\bar{2},-\tfrac{1}{2}) \Rightarrow \begin{array}{c} \begin{pmatrix} d_1^c \\ d_2^c \\ d_3^c \\ \hline e^- \\ -\nu \end{pmatrix}_L \end{array} \begin{array}{c} \bar{3} \\ \\ \bar{2} \end{array}$$

$(2,-\frac{1}{2})$

$\begin{pmatrix} \nu \\ e^- \end{pmatrix}_L \sim 2$

$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

$$i\sigma_2 \begin{pmatrix} \nu \\ e^- \end{pmatrix}_L = \begin{pmatrix} e^- \\ -\nu \end{pmatrix}_L \sim 2^* \qquad \psi_i^* \sim \epsilon_{ij}\psi_j \ \swarrow 2$$

transform as a 2^*

$$c \to \begin{pmatrix} e^+ \\ -\nu^c \end{pmatrix}_R$$

$\equiv$ spinor space

Insert discussion on $SU(2)$ invariants (appendix) & notes

n = dimension of repr.

$$\bar{\psi} \sim \epsilon_{1\ldots n-1}\,\psi^1 \ldots \psi^{n-1} \qquad \square \ \ \psi$$

$\left.\right\} n-1$ antisymmetric

$\bar{\psi}$

at and beyond M_X can be most simply fulfilled if G possesses twelve superheavy gauge bosons whose quantum numbers are given in Eq. (2.23). So, we concluded that G had $12+12=24$ generators. This leads us naturally to the choice of SU(5). Indeed, the adjoint $\underline{24}$-dimensional representation of SU(5) decomposes under $SU(3)_C \times SU(2)_L \times U(1)_Y$ as:

$$\begin{array}{ccccccccccc} \underline{24} = & (\underline{8},\underline{1},0) & + & (\underline{1},\underline{3},0) & + & (1,1,0) & + & (3,2,-5/6) & + & (\bar{3},2,5/6) \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ & G^{\alpha}_{\mu\beta} & & W^{\pm}_{\mu},\ W^{3}_{\mu} & & B_{\mu} & & A^{\alpha}_{\mu r} & & A^{r}_{\mu\alpha} \end{array} \tag{4.1}$$

where $\alpha,\beta = 1,2,3$ are used for colour and the remaining two indices $r = 4,5$ for weak isospin. Notice that only the twelve new bosons $A^{\alpha}_{\mu r}$ carry both colour (α) and flavour (r) indices. It is very useful for calculations to write down the adjoint representation in a matrix form:

$$A = \sqrt{2}\sum_{i=1}^{24} A^i \cdot \frac{\lambda^i}{2} \tag{4.2}$$

where A^i is the i-th element of the adjoint representation and the λ^i matrices are the generators of SU(5) (the analogous of the Pauli matrices for SU(2), or the Gell-Mann matrices for SU(3)). Since we want to evidentiate the SU(3)⊗SU(2)⊗U(1) substructure of SU(5), a convenient way of writing down the λ^i matrices is:

$$\lambda_{1\ldots8} = \left(\begin{array}{c|c} \begin{matrix}(SU(3))\\ \frac{\lambda_{1\ldots8}}{2}\end{matrix} & 0 \\ \hline 0 & 0 \end{array}\right) \longleftarrow SU(3) \tag{4.3a}$$

$$\lambda_{9\ldots11} = \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & \begin{matrix}(SU(2))\\ \frac{\tau_{1\ldots3}}{2}\end{matrix} \end{array}\right) \longleftarrow SU(2) \tag{4.3b}$$

$$\lambda_{24} = \frac{1}{\sqrt{15}} \left(\begin{array}{ccc|cc} -1 & & & & 0 \\ & -1 & & & \\ & & -1 & & \\ \hline & & & 3/2 & \\ 0 & & & & 3/2 \end{array}\right) \longleftarrow U(1)' \equiv Y' \tag{4.3c}$$

$-\frac{1}{3}\sqrt{\frac{3}{5}}$ $\Rightarrow \boxed{Y_{d^c_L} = \frac{1}{3} = Q_{d^c_L}}$

$\rightarrow \frac{1}{2}\sqrt{\frac{3}{5}}$ $\boxed{Y_{(e^-, \nu)} = -\frac{1}{2}}$

$$\lambda_{12} = \frac{1}{2}\left(\begin{array}{ccc|cc} & & & 1 & 0 \\ & 0 & & 0 & 0 \\ & & & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) ; \quad \lambda_{13} = \frac{1}{2}\left(\begin{array}{ccc|cc} & & & -i & 0 \\ & 0 & & 0 & 0 \\ & & & 0 & 0 \\ \hline i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

.... (6 positions) .... (6 positions)

$$\lambda_{22} = \frac{1}{2}\left(\begin{array}{ccc|cc} & & & 0 & 0 \\ & 0 & & 0 & 0 \\ & & & 0 & 1 \\ \hline 0 & 0 & 0 & & \\ 0 & 0 & 1 & & 0 \end{array}\right) ; \quad \lambda_{23} = \left(\begin{array}{ccc|cc} & & & 0 & 0 \\ & 0 & & 0 & 0 \\ & & & 0 & -i \\ \hline 0 & 0 & 0 & & \\ 0 & 0 & i & & 0 \end{array}\right)\frac{1}{2}$$

coset $\frac{SU(5)}{SU(3)\times SU(2)\times U(1)}$

(4.3d)

The first three indices have been used for colour and the remaining two for isospin. The above matrices are orthogonal and normalized so as: $\boxed{T_2(\lambda^a\lambda^b) = \frac{\delta^{ab}}{2}}$ In particular λ_{24} is determined by requiring that:

$$\hat{Y} \equiv \begin{pmatrix} a & & & & 0 \\ & a & & & \\ & & a & & \\ & & & b & \\ 0 & & & & b \end{pmatrix} \cdot \sqrt{\frac{1}{2(3a^2+2b^2)}} \tag{4.4}$$

$\boxed{Tr\,\hat{Y} = 0}$ $\Rightarrow (a,b) = (-1, 3/2)$

this choice is immaterial ?

with $3a = -2b$. Making the choice $a = -1$, we get the expression in Eq. (4.3c).
The Eqs. (4.2) and (4.3) allow us to collocate the gauge bosons in the following matrix.

$$A = \left(\begin{array}{ccc|cc} G_1^1 - \sqrt{\frac{2}{15}}\,B & G_2^1 & G_3^1 & \bar{X}^1 & \bar{Y}^1 \\ G_1^2 & G_2^2 - \sqrt{\frac{2}{15}}\,B & G_3^2 & \bar{X}^2 & \bar{Y}^2 \\ G_1^3 & G_2^3 & G_3^3 - \sqrt{\frac{2}{15}}\,B & \bar{X}^3 & \bar{Y}^3 \\ \hline X_1 & X_2 & X_3 & \frac{W^3}{\sqrt{2}} + \frac{3}{2}\sqrt{\frac{2}{15}}\,B & W^+ \\ Y_1 & Y_2 & Y_3 & W^- & -\frac{W^3}{\sqrt{2}} + \frac{3}{2}\sqrt{\frac{2}{15}}\,B \end{array}\right) \quad . \tag{4.5}$$

Before putting our money on the SU(5) model we must make sure that the ordinary fermions can be conveniently accommodated in a representation of SU(5). The irreducible 15-dimensional representation of SU(5) is readily discarded. Indeed its decomposition under SU(3) × SU(2):

$$\underset{\sim}{15} = (6,1) + (3,2) + (1,3) \tag{4.6}$$

shows that the $\underline{15}$ is certainly not suitable to accommodate the known fermions. We look for a representation of SU(5) which decomposes under SU(3) × SU(2) as:

$$\underset{u_R, d_R}{2 \times (\bar{3},1)} + \underset{u_L, d_L}{(3,2)} + \underset{\nu_L e_L}{(1,2)} + \underset{e_R}{(1,1)} \tag{4.7}$$

This property is enjoyed by the reducible $\underline{\bar{5} + 10}$ representation of SU(5), since:

$$\underline{\bar{5}} = \underset{\frac{1}{3}}{(\bar{3},1)} + \underset{-\frac{1}{2}}{(1,2)} \tag{4.8}$$

The final result is always the same (2 unknown (a,b) 2 constraints: ... Tr = 0)

and

$$\underline{10} = (3,2) + (\bar{3},1) + (1,1) \quad . \tag{4.9}$$

Incidentally, we invite the readers who are unfamiliar with the Young tableaux technique[10)] to derive the decomposition of the $\underline{24}$, $\underline{15}$ and $\underline{10}$ under SU(3) × SU(2) just starting from the obvious decomposition of $\bar{5}$ as given by Eq. (4.8), and considering the products $5 \times \bar{5}$ and 5×5. The only ambiguity in the fermion assignment concerns the $(\bar{3},1)$ components in $\underline{\bar{5}}$ and $\underline{10}$. Where should we put u_L^c and d_L^c? The doubt is readily dissipated by the fact that the electric charge is traceless over any irreducible representation, so, in particular, the sum of the electric charges of the fermions in the $\underline{\bar{5}}$ has to vanish. The (1,2) component is given by the $(e_L\ \nu_L)$ doublet and so it contributes a factor -1 to Q. Hence, the three entries $(\bar{3},1)$ are the $(d^c)_L$, in its three colours, to give $\sum_{f \in \underline{\bar{5}}} Q = 0$. In conclusion:

$$\underline{\bar{5}} \supset (d^c)_L + (e_L^-\ \nu_L) \tag{4.10}$$

and

$$\underline{10} \supset (u^c)_L + (u_L\ d_L) + (e^+)_L \quad . \tag{4.11}$$

It is useful and common to display the fermion assignment in a matrix form. The entries of the vector $\bar{5}$ can be readily established by writing down Q explicitly. From Eqs. (4.3), we have:

$$T_3 = \begin{pmatrix} 0 & & & & \\ & 0 & & 0 & \\ & & 0 & & \\ & & & +\frac{1}{2} & \\ & 0 & & & -\frac{1}{2} \end{pmatrix} \quad ; \tag{4.12}$$

$$Y' = \frac{1}{\sqrt{15}} \begin{pmatrix} -1 & & & & 0 \\ & -1 & & & \\ & & -1 & & \\ & & & +3/2 & \\ 0 & & & & +3/2 \end{pmatrix} \tag{4.13}$$

so

$$Q = T_3 + \sqrt{\frac{5}{3}}\, Y' = \begin{pmatrix} -1/3 & & & & 0 \\ & -1/3 & & & \\ & & -1/3 & & \\ & & & +1 & \\ 0 & & & & 0 \end{pmatrix} \tag{4.14}$$

when applied to the $\underline{5}$ representation. Thus, the values of Q over the entries of $\bar{5}$ are $+\frac{1}{3}$, $+\frac{1}{3}$, $+\frac{1}{3}$, -1, 0, and, therefore, we get: $\rightarrow$ repr. of T^i

$[T^i, \psi_a] = -t^i_{ab}\psi_b$

$[\bar{T}^i, \psi^\dagger_b] = -(-t^t)^i_{bL}\psi^\dagger_L$

$\bar{t} = -t^t$

$$\bar{5}:\ (\psi_L)_a = \begin{pmatrix} d_1^c \\ d_2^c \\ d_3^c \\ e^- \\ -\nu_e \end{pmatrix}_L , \qquad (a = 1,\dots,5) \tag{4.15}$$

$$\equiv\ 5: (\psi_R)_a = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ e^+ \\ -\bar{\nu}_e \end{pmatrix}_R$$

The minus sign in front of ν_e is due to the fact that $(\psi_L)_r$, $(r = a = 4,5)$, is a $\bar{2}$.[f]

The $\underline{10}$ is the antisymmetric part of the product 5×5: $10_{ab} = (5_a \times 5_b)_A$. For example: $10_{a5} = (5_a \times 5_5)_A = (5_a \times \bar{\nu})_A$, so the first

[f] Since we are presently working in the interaction basis, i.e. in the basis of the current eigenstates, the signs have no physical meaning. They shall get meaningful only after the diagonalization of the fermion mass matrix, i.e. when we go to the physical basis of the mass eigenstates.

three entries $(a = 1,2,3)$ have charge $+1/3$, the fourth $(a = 4)$ has charge $+1$, whilst 10_{55} is zero. Analogously: $10_{a4} = (5_a \times e^+)_A$, implying that the charge of 10_{a4} $(a = 1,2,3)$ is $-1/3 + 1 = +2/3$, the entry 10_{44} vanishes and the entry 10_{54} has a charge $0 + 1 = +1$. Going on with this simple exercise, the reader can obtain the following matrix form of the $\underline{10}$:

$$10:\ (\psi_L)_{ab} = \frac{1}{\sqrt{2}} \left(\begin{array}{ccc|cc} 0 & u_3^c & -u_2^c & -u^1 & -d^1 \\ -u_3^c & 0 & u_1^c & -u^2 & -d^2 \\ u_2^c & -u_1^c & 0 & -u^3 & -d^3 \\ \hline u^1 & u^2 & u^3 & 0 & -e^+ \\ d^1 & d^2 & d^3 & e^+ & 0 \end{array}\right)_L . \tag{4.16}$$

Therefore the structure of the $\underline{10}$ in terms of colour and isospin indices is:

$$10 = \begin{pmatrix} \underset{\alpha\beta_{\text{antis.}}}{\overset{\text{only colour}}{\downarrow}} & \overset{\text{colour}}{\uparrow}\ \alpha\tau \nearrow \text{isospin} \\ & \underset{\text{no index at all}}{\overset{45}{\uparrow}} \end{pmatrix} . \tag{4.17}$$

The signs in front of the fermionic entries are mostly conventional and such that the correct $SU(3) \times SU(2) \times U(1)$ interactions are reproduced.

The final aim of this section is the construction of the fermion-gauge boson interactions. The covariant derivatives for the fermions in the N, $\bar{N}$ and $N(N-1)/2$ antisymmetric representations of $SU(N)$ are given by:

$$(D_\mu \psi)^a = \left(\partial_\mu \delta^a_b - i \frac{g}{\sqrt{2}} (A_\mu)^a_b\right) \psi^b , \tag{4.18}$$

$$(D_\mu \psi)_a = \left(\partial_\mu \delta_a^b + \frac{ig}{\sqrt{2}} (A_\mu)_a^b\right)\psi_b \tag{4.19}$$

and

$$(D_\mu \psi)^{ab} = \partial_\mu \psi^{ab} - \frac{ig}{\sqrt{2}} (A_\mu)_c^a \psi^{cb} - \frac{ig}{\sqrt{2}} (A_\mu)_d^b \psi^{ad} \quad , \tag{4.20}$$

respectively. $(a,b,\ldots = 1,\ldots,N$ SU(N) indices).

From the Eqs. (4.5), (4.15) and (4.19), we get the gauge interactions of the fermions belonging to the $\bar{5}$ [11]:

$$\begin{aligned}
\mathscr{L}_{int} = g \Bigg\{ & \left[\bar{d}_L^c \gamma^\mu \vec{G}_\mu \frac{\vec{\lambda}}{2} d_L^c\right] \longleftarrow \text{SU(3) interactions} \\
& + \left[(\bar{\nu}_e\ \bar{e})_L \gamma^\mu \vec{W}_\mu \frac{\vec{\tau}}{2} \begin{pmatrix}\nu_e \\ e\end{pmatrix}_L \right] \longleftarrow \text{SU(2) interactions} \\
& + \sqrt{\frac{3}{5}}\left[-\frac{1}{2}\bar{\nu}_L \gamma^\mu B_\mu \nu_L - \frac{1}{2}\bar{e}_L \gamma^\mu B_\mu e_L \right. \\
& \qquad \left. + \frac{1}{3}\bar{d}_L^c \gamma^\mu B_\mu d_L^c\right] \longleftarrow \text{U(1) interactions} \\
& + \Bigg| \frac{1}{\sqrt{2}}\left[-\bar{e}_L \gamma^\mu \bar{X}_\mu^\alpha d_{\alpha_L}^c\right] \\
& \qquad + \frac{1}{\sqrt{2}}\left[+\bar{\nu}_L \gamma^\mu \bar{Y}_\mu^\alpha d_{\alpha_L}^c\right] + \text{h.c.}\Bigg|\Bigg\} \longleftarrow \text{new interactions}
\end{aligned} \tag{4.21}$$

Making use of the Eqs. (4.5), (4.16) and (4.20), we can get the analogous result for the fermions in the 10-dimensional representation[11]:

$$\begin{aligned}
\mathscr{L}_{int} = g \Bigg\{ & \left[\bar{u}_L \gamma^\mu \vec{G}_\mu \cdot \frac{\vec{\lambda}}{2} u_L + \bar{d}_L \gamma^\mu \vec{G}_\mu \cdot \frac{\vec{\lambda}}{2} d_L\right] \longleftarrow \text{SU(3) interactions} \\
& + \left[(\bar{u}\ \bar{d})_L \gamma^\mu \vec{W}_\mu \cdot \frac{\vec{\tau}}{2} \begin{pmatrix} u \\ d\end{pmatrix}_L \right] \longleftarrow \text{SU(2) interactions}
\end{aligned}$$

$$\begin{aligned}
& + \sqrt{\frac{3}{5}}\left[\frac{1}{6}\bar{u}_L \gamma^\mu B_\mu u_L + \frac{1}{6}\bar{d}_L \gamma^\mu B_\mu d_L \right. \\
& \qquad \left. - \frac{2}{3}\bar{u}_L^c \gamma^\mu B_\mu u_L^c + \bar{e}_L^c \gamma^\mu B_\mu e_L^c \right] \longleftarrow \text{U(1) interactions} \\
& + \Bigg[\frac{1}{\sqrt{2}}[\bar{d}_{L\alpha} \gamma^\mu \bar{X}_\mu^\alpha e_L^c + \varepsilon_{\alpha\beta\gamma} \bar{u}_L^{c\gamma} \gamma^\mu \bar{X}_\mu^\alpha u_L^\beta] \\
& \qquad + \frac{1}{\sqrt{2}}[-\bar{u}_{L\alpha} \gamma^\mu \bar{Y}_\mu^\alpha e_L^c + \varepsilon_{\alpha\beta\gamma} \bar{u}_L^{c\gamma} \gamma^\mu \bar{Y}_\mu^\alpha d_L^\beta] + \text{h.c.}\Bigg]\Bigg\} \longleftarrow \text{new interactions}
\end{aligned} \tag{4.22}$$

The X and Y bosons are often called lepto-quarks and, as it is apparent from the Eqs. (4.21) and (4.22), they mediate interactions which violate the lepton and baryon numbers[11]).

Finally, we stress that the fermion fields in Eqs. (4.21), (4.22) are the current eigenstates. The signs which appear in the matrix representation of fermions as well as the mixings between families are fixed only when the fields are reexpressed in the mass eigenstate basis (see Sec. 1.3)[12].

4.2 The Higgs Sector and the Hierarchy Nightmare

We require the Higgs sector to provide for:

(a) the breaking pattern $SU(5) \xrightarrow[M_X]{} SU(3) \times SU(2) \times U(1)_Y \xrightarrow[M_W]{} SU(3) \times U(1)_{em}$

(b) the fermion masses at the $SU(2)_L \times U(1)_Y \xrightarrow[M_W]{} U(1)_{em}$ breaking.

The Higgs scalar whose VEV breaks SU(5) at the M_X scale must be a singlet under $SU(3) \times SU(2) \times U(1)$. Since this latter group is a maximal (rank 4) subgroup of SU(5), this singlet must be in a real representation of SU(5). Indeed, there are no quantum numbers other than those of $SU(3) \times SU(2) \times U(1)$ which could differentiate this field from its conjugate. Apart from $\underline{1}$, the smallest real representation of SU(5) is the $\underline{24}$. It can be written as a traceless matrix:

$$\phi_a^b = \sum_{i=1}^{24} \varphi_i \frac{(\lambda^i)_a^b}{\sqrt{2}} \qquad (4.23)$$

24 components; 12 are eaten by Y, X gauge bosons

in a way analogous to the gauge boson matrix in Eq. (4.2). Since any power of 24 contains an SU(5) singlet, the most general renormalizable ϕ Higgs potential is:

$$V = -\frac{\mu_{24}^2}{2} \mathrm{Tr}(\phi^2) + \frac{1}{4} a (\mathrm{Tr}(\phi^2))^2 + \frac{1}{2} b\, \mathrm{Tr}(\phi^4) + \frac{1}{3} c\, \mathrm{Tr}(\phi^3) \ . \qquad (4.24)$$

The trace is taken since ϕ, being in the adjoint representation of SU(5), transforms as follows:

$$\phi \longrightarrow U^{+} \phi\, U \quad , \qquad U \in SU(5) \quad .$$

For simplicity the cubic term can be set to zero by imposing the discrete symmetry $\phi \to -\phi$.
By making use of SU(5) transformations we can put the VEV of ϕ in the convenient diagonal form. The breaking of the SU(N) groups was fully analyzed by L.F. Li[13]. Applying his general results to our specific case, we get the following two possibilities: Phys. Rev. D9 (74) 1723

$$\langle \phi \rangle = \begin{pmatrix} \nu & & & & 0 \\ & \nu & & & \\ & & \nu & & \\ & & & \nu & \\ 0 & & & & -4\nu \end{pmatrix} \qquad (4.25)$$

and

$$\langle \phi \rangle = \begin{pmatrix} \nu & & & & \\ & \nu & & & \\ & & \nu & & \\ & & & -\frac{3}{2}\nu & \\ & & & & -\frac{3}{2}\nu \end{pmatrix} \ . \qquad (4.26)$$

The two possibilities hold true for $b < 0$ and $b > 0$, respectively. Thus:

$$b < 0 \longrightarrow SU(5) \xrightarrow[M_X]{} SU(4) \times U(1) \qquad (4.27)$$

$$b > 0 \longrightarrow SU(5) \xrightarrow[M_X]{} SU(3) \times SU(2) \times U(1) \ . \qquad (4.28)$$

We remind that the adjoint representation VEV can never lower the rank of the initial group.
Physically we are interested in the breaking pattern (4.28), given by Eq. (4.26). The positivity condition[11]

$$\frac{1}{4} a\, (\mathrm{Tr} \langle \phi \rangle^2)^2 + \frac{1}{2} b\, \mathrm{Tr} \langle \phi \rangle^4 > 0 \qquad (4.29)$$

implies the bound:

$$\nu^4 \left[\frac{1}{4} a \frac{225}{4} + \frac{1}{2} b \frac{105}{8} \right] > 0 \qquad a > -\frac{7}{15} b \qquad (4.30)$$

The value of V at the minimum is:

$$\langle V \rangle = -\frac{\mu^2}{2} \frac{15}{2} \nu^2 + \frac{1}{4} a \cdot \frac{225}{4} \nu^4 + \frac{1}{2} b \cdot \frac{105}{8} \nu^4 \ . \qquad (4.31)$$

The minimum condition:

$$\frac{\partial V}{\partial \nu} = 0 \longrightarrow \nu \left[-\frac{15}{2} \mu^2 + \nu^2 \left(\frac{225}{4} a + \frac{105}{4} b \right) \right] = 0 \qquad (4.32)$$

yields

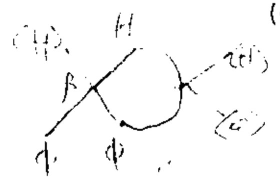
$$\nu^2 = \frac{2\mu^2}{15a + 7b} \ . \qquad (4.33)$$

To realize the second breaking step, $SU(3)_C \times SU(2)_L \times U(1)_Y \longrightarrow SU(3)_C \times U(1)_{em}$ we make use of a 5 of Higgs scalars (H). H contains a colour triplet H^α ($\alpha = 1,2,3$) which is singlet under $SU(2)_L$ and a colourless isospin doublet $\varphi \equiv H^r$ ($r = 1,2$) which plays the same role as the Higgs doublet of the Standard Model (Sec. 1.3). The most general potential containing only H reads:

$$V(H) = -\frac{\mu_5^2}{2} H^+ H + \frac{\lambda}{4} (H^+ H)^2 \quad . \tag{4.34}$$

Assuming colour unbroken, the VEV of H is:

$$\langle H \rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ v_0/\sqrt{2} \end{pmatrix} , \tag{4.35}$$



where $v_0^2 = \frac{2\mu_5^2}{\lambda}$ and $M_W^2 = g^2 \frac{v_0^2}{2}$.

However, the potential $V = V(\phi) + V(H)$ does not give rise to a phenomenologically viable model[11]. Indeed, as it is evident from Eq. (4.34), the colour triplet H^α would remain at the same mass as φ. More precisely, there is a combination of H^α with a colour triplet in ϕ which is eaten by the superheavy bosons. This combination is mainly constituted of the triplet in ϕ. The orthogonal combination which is mainly made up of H^α remains at the M_W scale. This is simply catastrophic: from Sec. 3.2, we know that H^α possesses the correct quantum numbers to mediate a proton decay that arises from six-dimensional QQQL operators. Therefore H^α must be substantially heavier then M_W.

This problem can find a solution if we consider also the following ϕ - H cross terms which are allowed by the SU(5) gauge invariance[11]:

$$V(\phi, H) = \alpha H^+ H \,\mathrm{Tr}\, \phi^2 + \beta H^+ \phi^2 H + \cancel{\delta\, H^+ \phi H} \quad . \tag{4.36}$$

The last term will be omitted because of the $\phi \to -\phi$ symmetry. We notice that in any case such cross terms would have arisen as a result of the radiative corrections to $V(\phi) + V(H)$. We look for the minimization of the full potential:

$$V = V(\phi) + V(H) + V(\phi, H) \quad .$$

Now that ϕ and H are coupled, $\langle \phi \rangle$ may also break $SU(2)_L$, whilst $SU(3)_C$ must be rigorously unbroken. So we look for solutions with:

$$\langle \phi \rangle = \mathrm{diag}\left(v \;\; v \;\; v \;\; \left(-\frac{3}{2} - \frac{\varepsilon}{2}\right) v \;\; \left(-\frac{3}{2} + \frac{\varepsilon}{2}\right) v \right) \quad . \tag{4.37}$$

In the absence of ϕ - H mixings, i.e. $\cancel{\alpha} = \beta = 0$, ε must vanish. This means that in the limit $\cancel{\alpha}, \beta \to 0$ we should find an extremum of the combined potential which connects up with the previous partial solutions. The solution with these properties has[11]:

$$\varepsilon = \frac{3}{\cancel{20}\; 10} \frac{\beta v_0^2}{b v^2} + O\left(\frac{v_0^4}{v^4}\right) \quad . \tag{4.38}$$

Since we require $v \sim O(M_X)$ and $v_0 \sim O(M_W)$, we get $\varepsilon v \ll v_0$. This fact is very welcome. Indeed, this implies that the SU(2) breaking due to $\langle \phi \rangle$ (Eq. (4.37)) is much smaller than that due to $\langle H \rangle$. Since the component of ϕ which breaks the $SU(2)_L$ (without breaking colour) is an isotriplet (see the decomposition (4.1)), it must be much smaller than v_0 if we want to respect the $\rho = 1$ relation (Sec. 1.3). There are now more complicated expressions for the relations of μ^2 and μ_5^2 to v_0 and v[11]:

$$\mu_{24}^2 = \frac{15}{2} a v^2 + \frac{7}{2} b v^2 + \alpha v_0^2 + \frac{9}{30} \beta v_0^2 \tag{4.39}$$

and

$$\mu_5^2 = \frac{1}{2} \lambda v_0^2 + 15 \alpha v^2 + \frac{9}{2} \beta v^2 - 3 \varepsilon \beta v^2 \quad . \tag{4.40}$$

Equation (4.39) shows only a minor modification with respect to Eq. (4.33), being $v_0 \ll v$. What is worrying is the result exhibited by Eq. (4.40). Since μ_5^2 is $O(M_W^2)$, i.e. $\mu_5^2 \ll v$ the natural thing to happen would be for v_0 to take a value comparable with v in order to make the r.h.s. of Eq. (4.40) of $O(M_W^2)$. In other words, without putting any particular constraint on α and β, we expect $v_0 \sim O(v)$. Obviously that would completely spoil the hierarchy between M_W and M_X with the Fermi scale coinciding with the grand unification scale. If we want to avoid such a disaster, we must neglect our aesthetical feeling and fine tune α and β to one part in $(v^2/v_0^2) \sim 10^{24}$! Even more disgustingly, such an adjustment must be repeated at any order of perturbative theory, since radiative corrections are to displace α and β from their initial values for more than one part in 10^{24}.

The colour triplet in H is finally removed from the dangerous M_W area. Indeed, the combination of H^α and ϕ_5^α which is "eaten up" to give mass to the Y superheavy bosons $(Q = 1/3)$ is given by[11]:

$$\phi_5^\alpha - \frac{\sqrt{2}\, v_0}{5v} H^\alpha + 0\left(\frac{v_0^2}{v^2}\right) \quad , \tag{4.41}$$

whereas the orthogonal combination

$$H_3 \equiv H^\alpha + \frac{\sqrt{2}\, v_0}{5v} \phi_5^\alpha + 0\left(\frac{v_0^2}{v^2}\right) \tag{4.42}$$

is the physical colour triplet Higgs field with a mass[11]:

$$m_{H_3}^2 = -\frac{5}{3}\beta v^2 + 0(v_0^2) \quad . \tag{4.43}$$

from 4.36 & 4.42

A detailed discussion on the mass spectrum of the scalar and gauge bosons of SU(5) in the presence of the full potential $V = V(\phi) + V(H) + V(\phi, H)$ will be given in the second part of the book[14]. Our analysis here, apart from pointing out the salient feature of the minimization of V, had the primary purpose of providing an explicit example of the hierarchy problem. In fact, as we have seen, what is known under this name presents two different aspects:

i) the difficulty of explaining the immense gap between M_W and M_P. This can be realized at the tree level by a miraculous fine tuning of the parameters in the Higgs potential, however in traditional GUT's we completely lack any "raison d' être" of the M_W/M_P ratio. In the second and third parts of the book we shall see that present $N = 1$ local supersymmetric theories may shed some light on this first aspect of the hierarchy problem;

ii) the second aspect is sometimes known as the "technical" aspect of the hierarchy problem. The tree level fine tuning is not stable, for radiative corrections spoils it. Again, the reader must be patient until the second and third parts to see how supersymmetry is particularly helpful in dealing with this technical problem.

5. LOW ENERGY PHENOMENOLOGICAL CONSEQUENCES OF SU(5)

How can we make our mind up about the validity of the SU(5) model? The low energy tests reduce essentially to the determination of $\sin^2\theta_W$ and the instability of matter. Some other circumstantial evidence can be drawn from the fermionic mass spectrum, but, unfortunately, in a very loose way, as we shall see. A precise prediction for $\sin^2\theta_W$ and proton lifetime requires a more sophisticated treatment then the single step approximation for the RGE which we adopted in Sec. 2. Keeping faithful to our spirit of furnishing an elementary introduction to GUT's in this first part of the book, we shall discuss the matching (or, alas mismatching) between the theoretical and experimental results concerning $\sin^2\theta_W$ (Sec. 5.1) and the nucleon decay (Sec. 5.4) in a more elementary way, borrowing some numerical results from the detailed analysis of the second part. The section on the proton decay comes after examining the fermion mass spectrum (Sec. 5.2) and the mixing angles and phases (Sec. 5.3), since some elements of these two sections bear some consequence on the calculation of τ_p. A peculiar aspect of the fermionic spectrum as predicted by SU(5) concerns the neutrino mass. The correlation between the neutrino masslessness and the conservation of B-L will be examined in Sec. 5.5. Obviously, in agreement with the expectation of the sportive readers, we conclude our excursion to the SU(5) land by making a sort of "final judgement" in the SU(5) model (Sec. 5.6) to be compared with the score of the Standard Model given in Sec. 1.3.

5.1 The Successful $\sin^2\theta_W$ Prediction of SU(5)

In Sec. 2.3 we derived the value of $\sin^2\theta_W$ at M_X, $\sin^2\theta_W = 3/8$, making the only assumptions of the usual $SU(3) \times SU(2) \times U(1)$ assignment where n_H and N_F denote the number of light Higgs isodoublets and families, respectively. This result is in excellent, and we would add astonishing agreement with the experimental value of $\sin^2\theta_W$ at $\sqrt{20}$ GeV:

$$\sin^2\theta_W(q_0 = \sqrt{20}\ \mathrm{GeV})\Big|_{\mathrm{exp}} = 0.216 \pm 0.010 \quad . \tag{5.9b}$$

From our analysis of the symmetry breaking pattern in Sec. 4.2 we see that $SU(2)_L \times U(1)_Y$ is broken down to $U(1)_{em}$ predominantly through a weak isodoublet. So we do not depart from the Standard Model prediction for ρ. The $\rho = 1$ value is radiative corrected. With one Higgs doublet and taking $\sin^2\theta_W = 0.23$, $m_{Higgs} = m_Z$, $m_t = 20$ GeV and $\langle q^2\rangle = -20\,\mathrm{GeV}^2$ the ρ parameter in $\nu_\mu N$ scattering is[15]:

$$\rho^2 = 0.983$$

to be compared with the experimental value $\rho^2 = 0.998 \pm 0.050$. Clearly the SU(5) predictions concerning $\sin^2\theta_W$ and ρ are in extraordinary agreement with the present experimental data. Now that we start producing directly the $W^\pm$ and Z^0 it will be exciting to see if the SU(5) model can survive further improvements in the determination of these neutral current parameters. Anyway, as we shall see in the next sections, the experimental clouds on the SU(5) model are not to come from this direction, but rather from the other crucial prediction concerning the matter instability.

5.2 Fermion Masses in SU(5)

We have already seen in the previous section that the fermions of each generation are assigned to the $\underline{\bar{5}} + \underline{10}$ representation of SU(5):

$$\Psi_{La} \equiv \underline{\bar{5}} \quad , \qquad \Psi_L^{[ab]} \equiv \underline{10} \quad , \tag{5.10}$$

where $a, b = 1,\ldots,5$ are SU(5) indices and the $\underline{10}$ representation is the antisymmetric part of the $5_a \times 5_b$ representation product. The Higgs fields which can provide fermion masses through their VEV's must give rise to SU(5) invariant Yukawa terms with the following fermion-fermion operators:

$\overline{\psi^c_L}\,\psi_L = \psi^T_{L\,\bar{5}}\, C\, \psi_{L\,\bar{5}}$

$$\psi^T_{La}\, C\, \psi_{Lb} \longrightarrow \bar{5} \times \bar{5} = \overline{10} + \overline{15} \tag{5.11a}$$

$$\psi^T_{La}\, C\, \psi_L^{bc} \longrightarrow \bar{5} \times 10 = 5 + \overline{45} \tag{5.11b}$$

$$\psi_L^{Tab}\, C\, \psi_L^{cd} \longrightarrow 10 \times 10 = \bar{5} + 45 + 50 \quad , \tag{5.11c}$$

where C denotes the charge conjugation matrix. Indeed, given two left-handed fields, ψ_L and χ_L, a mass term is of the form:

$$\psi_L^T\, C\, \chi_L + \mathrm{h.c.} = \chi_L^T\, C\, \psi_L + \mathrm{h.c.} = \psi_R^c\, \chi_L + \mathrm{h.c.} \quad ,$$

where the charge conjugation matrix C satisfies the condition $C\gamma_\mu C^{-1} = -\gamma_\mu^T$.

The SU(5) content of the above fermion-fermion operators shows that the Higgs scalar ϕ in the adjoint representation does not couple to fermions. So, even letting alone the need for some other Higgs field to account for the correct $SU(5) \to SU(3) \times SU(2) \times U(1) \to SU(3) \times U(1)$ breaking (Sec. 4.2), we are forced to introduce some new scalar field if we want to provide fermion masses. The most economical choice to achieve both the aims, correct symmetry breaking pattern and fermionic mass spectrum, consists in the introduction of a Higgs in the 5-dimensional representation, H^a. Then, we can construct the SU(5) singlets:

$$\mathcal{L}_Y = g_{1mn}\, \Psi^T_{Lma}\, C\, \psi^{ab}_{Ln}\, \bar{H}_b + g_{2mn}\, \Psi^{Tab}_{Lm}\, C\, \Psi^{cd}_{Ln}\, H^e\, \varepsilon_{abcde} + \mathrm{h.c.} \quad , \tag{5.12}$$

where ε_{abcde} is the totally antisymmetric tensor, m and n are generation indices and $\bar{H}_b$ in the $\bar{5}$ representation is the Hermitian conjugate of the field H^b. Equation (4.36) gives:

$$\langle H^e \rangle = \frac{v_0}{\sqrt{2}}\, \delta^{e5} \quad , \tag{5.13}$$

so that, inserting the above VEV in Eq. (5.12), we get the fermion masses:

$$M_{mn} = g_{1mn}\, \Psi^{T}_{Lma}\, C\, \Psi^{a5}_{Ln} \langle \bar{H}_5 \rangle +$$

$$+ g_{2mn}\, \Psi^{Tab}_{Lm}\, C\, \Psi^{cd}_{Ln} \langle H^5 \rangle\, \varepsilon_{abcd5} + \text{h.c.} \qquad (5.14)$$

Since the fifth column and row of ψ^{ab} (see Eq. (4.16)) contains d and e^+, the second term of (5.14) provides a mass only for the $Q = +2/3$ quarks. On the other hand, the first term in (5.14) yields the $Q = -1/3$ quark masses for $a = 1,2,3$ and the $Q = -1$ lepton masses for $a = 4$. Notive that because of the vanishing entry ψ^{55}, no neutrino mass is possible. In conclusion:

$10_{ferm} \times 10_{ferm} \times \bar{H} \;\rightarrow\;$ u-quark masses

$5_{ferm} \times 10_{ferm} \times H \;\rightarrow\;$ d-quark and e-lepton masses.

An interesting implication of the second Yukawa coupling in Eq. (5.14) is the equality:

$$M^d_{mn} = M^e_{mn} = g_{1mn} \langle \bar{H}_5 \rangle \quad , \qquad (5.15)$$

where M^d and M^e denote the $Q = -1/3$ and $Q = -1$ fermion mass matrices, respectively.

Clearly this result continues to hold true in the presence of several five-dimensional Higgs fields. Therefore, if fermion masses arise from VEV's of only Higgs 5-plets, we get

$$m_d = m_e \;, \qquad m_s = m_\mu \;, \qquad m_b = m_\tau \quad . \qquad (5.16)$$

This equality holds true only at $q^2 > M_X^2$, i.e. in the SU(5) symmetric limit. Analogously to the gauge coupling constants, also for the masses there is an energy dependence. The detailed calculation of the renormalization of these masses from M_X down to M_W [11] will be presented in the second part. The major result is that:

$$m_b = m_\tau \Big|_{M_X} \longrightarrow m_b \simeq 3 m_\tau \Big|_{M_W} \quad , \qquad (5.17)$$

in striking agreement with the experimental value. This has been accounted for as one of the major successes of the minimal SU(5) model, i.e. the model where only a ψ and a H are present in the Higgs sector.

Unfortunately, this enthusiasm is readily dissipated when we focus our attention on the first two generations. There the relation (5.16) yields:

$$\frac{m_d}{m_s} = \frac{m_e}{m_\mu} \quad . \qquad (5.18)$$

Since d and s (or e and μ) present the same interactions they are renormalized in the same way and so the equality (5.18) holds at any momentum.[9] The result is clearly catastrophic: current algebra gives $m_d/m_s \approx 1/24$ [16] in sharp contrast with $m_e/m_\mu = 1/207$.

Two possible ways out have been pointed out to keep the good $m_b = m_\tau \big|_{M_X}$ relation, while modifying the bad prediction (5.18): i) enlargement of the Higgs spectrum by the addition of a $\underline{45}$ Higgs representation[17,18]; ii) introduction of non-renormalizable effective interactions giving rise to mass contributions in the MeV range[19]. Let us analyze briefly both these suggestions.

i) From the decompositions (5.11b) and (5.11c) we see that beside the representation $\underline{5}$ also the 45-dimensional representation is suitable to provide a mass for all the fermions. The Higgs 45-plet H^{ab}_c ($H^{ab}_c = -H^{ba}_c$; $\sum_a H^{ab}_a = 0$) presents the following Yukawa couplings[17,18]:

$$\mathcal{L}^{(45)}_Y = f_{1mn}\, \Psi^T_{Lam}\, C\, \Psi^{bc}_{Ln}\, H^{+a}_{bc}$$

$$+ f_{2mn}\, \Psi^{Tab}_{Lm}\, C\, \Psi^{cd}_{Ln}\, H^{ef}_d\, \varepsilon_{abcef} + \text{h.c.} \qquad (5.19)$$

To calculate the mass contribution from (5.19) we need to know the

[9] A possible objection to this reasoning shall be presented in the second part.

Let us list the superfields which extend minimally the SM. (SUSY SU(5))

$\boxed{\text{Chiral superfields}}$

$Q \equiv (3, 2, \frac{1}{6})$: $(\tilde{u}_L, u_L ; \tilde{d}_L, d_L)^T$

$U^c \equiv (\bar{3}, 1, -\frac{2}{3})$: $(\tilde{u}_R, u^c_L)$

$D^c \equiv (\bar{3}, 1, \frac{1}{3})$: $(\tilde{d}_R, d^c_L)$

$L \equiv (1, 2, -\frac{1}{2})$: $(\tilde{\nu}_L, \nu_L ; \tilde{e}_L, e_L)^T$

$E^c \equiv (1, 1, 1)$: $(\tilde{e}_R, e^c_L)$

$H_1 \equiv (1, 2, -\frac{1}{2})$: $(H_1, \tilde{h}_1)$

$H_2 \equiv (1, 2, \frac{1}{2})$: $(H_2, \tilde{h}_2)$

$\boxed{\text{Vector superfields}}$

$G_\mu \equiv (8, 1, 0)$: $(\tilde{g}, g)_\mu$

$W^{\pm}_\mu \equiv (1, 3, 0)$: $(\tilde{w}, W)_\mu$

$B^0 \equiv (1, 1, 0)$: $(\tilde{b}, B)_\mu$

---

2 Higgs superfields needed with opposite hypercharge (2 and 2*) : Superpotential is a function of z — to give masses to up and down type fermions!

not of z and z*

$z z^*$ is not a chiral superfield !

$\theta\theta\bar{\theta}\bar{\theta}$ component → Kinetic terms

z^n → $\theta\theta$ component $\boxed{\text{supersymmetric first}}$

— ● —

~~gauge coupling~~ unifications

Beta coefficients

$$\boxed{\frac{1}{\alpha_i(\mu)} = \frac{1}{\alpha_i(M)} + b_i t \qquad t = \frac{1}{2\pi} \ln \frac{M}{\mu}}$$

$$\begin{cases} b_i^G = -\frac{11}{3} T_{2i}^G & \text{massless gauge boson} \\ b_i^W = \frac{2}{3} T_{2i}^W & \text{Weyl fermion} \\ b_i^S = \frac{1}{6} T_{2i}^S & \text{real scalar} \end{cases}$$

Compute b_i for superfields

chiral superfields

$$b_i^{\chi} = \left(\frac{2}{3} + \frac{2}{6}\right) T_{2i}^{\chi} = 1 \cdot T_{2i}^{\chi}$$

vector superfields

$$b_i^{V} = \left(-\frac{11}{3} + \frac{2}{3}\right) T_{2i}^{V} = -3 \cdot T_{2i}$$

Suppose $\boxed{n_g}$ generations of fermions (sfermions) and n_H doublets of Higgs scalars (higgsinos)

$$\Rightarrow \boxed{b_i = -3 T_{2i}^{V} + n_g T_{2i}^{\chi_F} + n_H T_{2i}^{\chi_H}}$$

Problem $\boxed{\text{final}}$ ~~examination~~ (take-home)

$b_i^V \quad b_i^{\chi_F} \quad b_i^{\chi_H}$

1) Compute $(T_{2i}^{V}), (T_{2i}^{\chi_F}), (T_{2i}^{\chi_H})$ for $i = 1,2,3$

with the SUSY-SM content

find $\Rightarrow (b_1, b_2, b_3)$ SUSY as a function of (n_g, n_H)

Hints

for $T_{2i} \sim \underbrace{Tr\, T_i^a T_i^b}$ Σ over all particles

with correct normalisation for $\boxed{Y' = \sqrt{\frac{3}{5}} Y}$

(in particular

$T_{(2)1}^{\chi_F} = T_{(2)2}^{\chi_F} = T_{(2)3}^{\chi_F} = 2$?)

---

2) Then impose unification

assuming

a) $M_{SUSY} = M_Z$ $\boxed{\text{SUSY SM}}$ from M_Z to M_{GUT}

i) use scheme (a) or (b)

inputs: (α, s^2) | (α, α_3)

$\Downarrow$ | $\Downarrow$

output: (α_3, α_G, t) | (s^2, α_G, t)

ii) do it for $\boxed{n_H = 2}$ ~~4~~

same procedure as for SM SU(5) ?

---

b) Assume $M_Z^{(n_H=1)} < M_{SUSY}^{(n_H=2)} < M_{GUT}$?

Run α_i SM from M_Z to M_S ($n_H = 1$)

complete $\alpha_i(M_S)$ formally

then repeat (a) for $\boxed{\alpha_i(M_S) \to \alpha_G(M_G)}$

use as input α, s^2, α_3

to determine $\boxed{M_S, \alpha_G, t}$ — ? —

Table 13
Simple irreps of simple Lie algebras

Algebra	Dynkin designation	Dimensionality
A_n	(10 ... 0)	$n+1$
	or (0 ... 01)*	$\overline{n+1}$
B_n	(10 ... 0)*	$2n+1$
	(000 ... 01)	2^n
C_n	(10 ... 0)	$2n$
D_n	(10 ... 0)*	$2n$
	(00 ... 01)	2^{n-1}
	or (00 ... 010)*	2^{n-1}
G_2	(01)	7
F_4	(0001)	26
E_6	(100000)	27
	or (000010)*	$\overline{27}$
E_7	(0000010)	56
E_8	(00000010)	248

* This irrep can be constructed from products of the unstarred irrep.

Table 14
Maximal subalgebras of classical simple Lie algebras with rank 8 or less

Rank 1	
$SU_2 \supset U_1$	(R)
(SU_2, SO_3, Sp_2, all isomorphic)	
Rank 2	
$SU_3 \supset SU_2 \times U_1$	(R)
$\quad \supset SU_2$	(S)
$Sp_4 \supset SU_2 \times SU_2$; $SU_2 \times U_1$	(R)
$\quad \supset SU_2$	(S)
(SO_5 isomorphic to Sp_4, $SO_4 \sim SU_2 \times SU_2$)	
Rank 3	
$SU_4 \supset SU_3 \times U_1$; $SU_2 \times SU_2 \times U_1$	(R)
$\quad \supset Sp_4$, $SU_2 \times SU_2$	(S)
$SO_7 \supset SU_4$, $SU_2 \times SU_2 \times SU_2$; $Sp_4 \times U_1$	(R)
$\quad \supset G_2$	(S)
$Sp_6 \supset SU_3 \times U_1$; $SU_2 \times Sp_4$	(R)
$\quad \supset SU_2$; $SU_2 \times SU_2$	(S)
(SO_6 is isomorphic to SU_4)	
Rank 4	
$SU_5 \supset SU_4 \times U_1$, $SU_2 \times SU_3 \times U_1$	(R)
$\quad \supset Sp_4$	(S)
$SO_8 \supset SO_6$; $SU_2 \times SU_2 \times Sp_4$, $SU_2 \times SU_4$; $SO_7 \times U_1$	(R)
$\quad \supset SU_3$, $SU_2 \times SU_2$	(S)
$Sp_8 \supset SU_4 \times U_1$; $SU_2 \times Sp_6$; $Sp_4 \times Sp_4$	(R)
$\quad \supset SU_2$; $SU_2 \times SU_2 \times SU_2$	(S)
$SO_9 \supset SU_2 \times SU_2 \times SU_2 \times SU_2$; $SU_4 \times U_1$	(R)
$\quad \supset SU_2$, SO_7; $SU_2 \times Sp_4$	(S)

Table 14 (continued)

Rank 5		
SU_6	$\supset SU_5 \times U_1;\ SU_2 \times SU_4 \times U_1;\ SU_3 \times SU_3 \times U_1$	(R)
	$\supset SU_3;\ SU_4;\ Sp_6;\ SU_2 \times SU_3$	(S)
SO_{11}	$\supset SO_{10};\ SU_2 \times SO_8;\ Sp_4 \times SU_4;\ SU_2 \times SU_2 \times SO_7;\ SO_9 \times U_1$	(R)
	$\supset SU_2$	(S)
Sp_{10}	$\supset SU_5 \times U_1;\ SU_2 \times Sp_8;\ Sp_4 \times Sp_6$	(R)
	$\supset SU_2;\ SU_2 \times Sp_4$	(S)
SO_{10}	$\supset SU_5 \times U_1;\ SU_2 \times SU_2 \times SU_4;\ SO_8 \times U_1$	(R)
	$\supset Sp_4;\ SO_9;\ SU_2 \times SO_7;\ Sp_4 \times Sp_4$	(S)
Rank 6		
SU_7	$\supset SU_6 \times U_1;\ SU_2 \times SU_5 \times U_1;\ SU_3 \times SU_4 \times U_1$	(R)
	$\supset SO_7$	(S)
SO_{13}	$\supset SO_{12};\ SU_2 \times SO_{10};\ Sp_4 \times SO_8;\ SU_4 \times SO_7;\ SU_2 \times SU_2 \times SO_9;\ SO_{11} \times U_1$	(R)
	$\supset SU_2$	(S)
Sp_{12}	$\supset SU_6 \times U_1;\ SU_2 \times Sp_{10};\ Sp_4 \times Sp_8;\ Sp_6 \times Sp_6$	(R)
	$\supset SU_2;\ SU_2 \times SU_4;\ SU_2 \times Sp_4$	(S)
SO_{12}	$\supset SU_6 \times U_1;\ SU_2 \times SU_2 \times SO_8;\ SU_4 \times SU_4;\ SO_{10} \times U_1$	(R)
	$\supset SU_2 \times Sp_6;\ SU_2 \times SU_2 \times SU_2;\ SO_{11};\ SU_2 \times SO_9;\ Sp_4 \times SO_7$	(S)
Rank 7		
SU_8	$\supset SU_7 \times U_1;\ SU_2 \times SU_6 \times U_1;\ SU_3 \times SU_5 \times U_1;\ SU_4 \times SU_4 \times U_1$	(R)
	$\supset SO_8;\ Sp_8;\ SU_2 \times SU_4$	(S)
SO_{15}	$\supset SO_{14};\ SU_2 \times SO_{12};\ Sp_4 \times SO_{10};\ SO_7 \times SO_8;\ SU_4 \times SO_9;\ SU_2 \times SU_2 \times SO_{11};\ SO_{13} \times U_1$	(R)
	$\supset SU_2;\ SU_4;\ SU_2 \times Sp_4$	(S)
Sp_{14}	$\supset SU_7 \times U_1;\ SU_2 \times Sp_{12};\ Sp_4 \times Sp_{10};\ Sp_6 \times Sp_8$	(R)
	$\supset SU_2;\ SU_2 \times SO_7$	(S)
SO_{14}	$\supset SU_7 \times U_1;\ SU_2 \times SU_2 \times SO_{10};\ SU_4 \times SO_8;\ SO_{12} \times U_1$	(R)
	$\supset Sp_4;\ Sp_6;\ G_2;\ SO_{13};\ SU_2 \times SO_{11};\ Sp_4 \times SO_9;\ SO_7 \times SO_7$	(S)
Rank 8		
SU_9	$\supset SU_8 \times U_1;\ SU_2 \times SU_7 \times U_1;\ SU_3 \times SU_6 \times U_1;\ SU_4 \times SU_5 \times U_1$	(R)
	$\supset SO_9;\ SU_3 \times SU_3$	(S)
SO_{17}	$\supset SO_{16};\ SU_2 \times SO_{14};\ Sp_4 \times SO_{12};\ SO_7 \times SO_{10};\ SO_8 \times SO_9;\ SU_4 \times SO_{11};$ $SU_2 \times SU_2 \times SO_{13};\ SO_{15} \times U_1$	(R)
	$\supset SU_2$	(S)
Sp_{16}	$\supset SU_8 \times U_1;\ SU_2 \times Sp_{14};\ Sp_4 \times Sp_{12};\ Sp_6 \times Sp_{10};\ Sp_8 \times Sp_8$	(R)
	$\supset SU_2;\ Sp_4;\ SU_2 \times SO_8$	(S)
SO_{16}	$\supset SU_8 \times U_1;\ SU_2 \times SU_2 \times SO_{12};\ SU_4 \times SO_{10};\ SO_8 \times SO_8;\ SO_{14} \times U_1$	(R)
	$\supset SO_9;\ SU_2 \times Sp_8;\ Sp_4 \times Sp_4;\ SO_{15};\ SU_2 \times SO_{13};\ Sp_4 \times SO_{11};\ SO_7 \times SO_9$	(S)