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I & II

SUMMER SCHOOL IN HIGH ENERGY PHYSICS AND COSMOLOGY

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CHIRAL FERMIONS ON THE LATTICE

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Please note: These are preliminary notes intended for internal distribution only.

Lecture I :

1. Chiral fermions :

Basic objects in model building

	ϕ	ψ	A_μ
Spin	0	1/2	1

We shall be exclusively concerned with ψ and A.

The fundamental building blocks are Weyl spinors

$$\psi_L = \frac{1-\gamma_5}{2}\psi \qquad \gamma_5^{\dagger} = \gamma_5$$

$$\psi_R = \frac{1+\gamma_5}{2}\psi \qquad \gamma_5^{2} = 1$$

ψ = 4-component Dirac spinor

Under $SO(1,3)$ – i.e. excluding P: $(t, \underline{x}) \to (t, -\underline{x})$ –

ψ_L and ψ_R do not mix. $\Rightarrow$ We can treat them

as independent objects.

For example in $SU(2)_L \times U(1)_Y$ - model for each

family of Leptons, e, ν_e family, say, we have

$$\begin{pmatrix} \nu_e \\ e_L \end{pmatrix} \sim (2,-1) \qquad e_R \sim (1,-2)$$

That: $Y(e_L) = -1 \neq Y(e_R) = -2$, does not conflict with Lorentz invariance is due to the fact that e_L and e_R do not mix under $SO(1,3)$

We can also write down Lorentz invariant coupling to vector fields

$$\nabla_\mu \Psi_L = \left(\partial_\mu - i W_\mu^A \frac{\tau_a}{2} - i B_\mu\right) \Psi_L \qquad \Psi_L = \begin{pmatrix} \nu_L \\ e_L \end{pmatrix}$$

$$\nabla_\mu e_R = (\partial_\mu - i\, 2\, B_\mu)\, e_R$$

$$\mathcal{L} = \bar{\Psi}_L \not{\nabla} \Psi_L + \bar{e}_R \not{\nabla} e_R + \cdots$$

A mass term
$$m_e (\bar{e}_L e_R + \bar{e}_R e_L)$$
is not compatible with $SU(2)_L \times Y(1)_Y$ invariance

Chiral theories:

The left handed and the right handed spinors belong to unitarily inequivalent representations of the gauge group G.

Vector-like theories:

For any left-handed spinor in some representation R of G there is a right handed spinor in the same repn R of G.

<u>Examples</u> 1) $SU(2)_L \times U(1)_Y$ is chiral, because $Y(e_R) = -2$ and there is no left handed spinor with the same Y-charge

2) QCD with $G = SU(3)$ is vector like. For example with one flavour of quark, say u,

$$\mathcal{L}_{QCD} = \bar{u} \not{D} u$$
$$= \bar{u}_L \not{D} u_L + \bar{u}_R \not{D} u_R$$

Since u_L and u_R belong both to $\underline{3}$ of $SU(3)$, a mass term can be introduced

$$m(\bar{u}_L u_R + \bar{u}_R u_L)$$

If $m = 0$, $\mathcal{L}_{QCD}$ has a global invariance of $U(1)_L \times U(1)_R$

$$u_L(x) \rightarrow u'_L(x) = e^{i\alpha_L} u_L$$
$$u_R(x) \rightarrow u'_R(x) = e^{i\alpha_R} u_R \qquad \alpha_L \,\&\, \alpha_R \text{ const.}$$

If $m \neq 0$, $\mathcal{L}_{QCD}$ is inv. only if $\alpha_L = \alpha_R$ i.e.

$$U(1)_L \times U(1)_R \longrightarrow U(1)_V$$

2. Lattice formulation :

$$H_{Dirac} = \int d^3x \; \bar{\psi} \left(i\gamma^i \partial_i + m \right) \psi$$

$\gamma^{0\,2} = 1 = -\gamma^{i\,2}$

$$= \int d^3\underline{x} \; \psi^+ \left(i\gamma^0 \gamma^i \partial_i + m\gamma^0 \right) \psi$$

$\gamma^{0\,+} = \gamma^0$, $\gamma^{i\,+} = -\gamma^i$

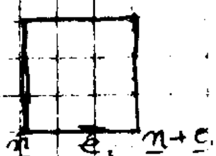
$$= \int d^3\underline{x} \; \psi^+ \left[\frac{i}{2} \gamma^0 \gamma^i \left(\overrightarrow{\partial_i} - \overleftarrow{\partial_i} \right) + m\gamma^0 \right] \psi$$

+ tot. deriv.

Discretize :

lattice spacing = 1

$$\partial_i \psi(\underline{x}) \longrightarrow \psi(\underline{n} + \underline{e}_i) - \psi(\underline{n})$$



$$\int d^3\underline{x} \longrightarrow \sum_{\underline{n}}$$

$$H_{Dirac} = \sum_{\underline{n}} \Big[\psi^+(\underline{n}) \frac{i}{2} \gamma^0 \gamma^i \left(\psi(\underline{n} + \underline{e}_i) - \psi(\underline{n}) \right)$$

$$- \left(\psi^+(\underline{n} + \underline{e}_i) - \psi^+(\underline{n}) \right) \frac{i}{2} \gamma^0 \gamma^i \psi(\underline{n}) +$$

$$+ m \, \psi^+(\underline{n}) \gamma^0 \psi(\underline{n}) \Big]$$

Fourier expand

$$\tilde{\psi}(\underline{p}) = \sum_{n} \psi(\underline{n}) \, e^{-i\underline{p}\cdot\underline{n}}$$

$$= \tilde{\psi}(\underline{p} + 2\pi \underline{e}_i) \qquad i = 1, 2, 3$$

$$\Rightarrow \quad p_i \in [0, 2\pi)$$

$$\psi(\underline{n}) = \int \frac{d^3p}{(2\pi)^3} \; \tilde{\psi}(\underline{p}) \, e^{i\underline{p}\cdot\underline{n}}$$

$$H_{Dirac} = \int \frac{d^3p}{(2\pi)^3} \frac{d^3p'}{(2\pi)^3} \left[\tilde{\psi}^\dagger(p) \frac{i}{2} \gamma^0 \gamma^i \left((e^{ip_i'} - 1) - (e^{-ip_i} - 1) \right) \tilde{\psi}(p') + m \tilde{\bar\psi}(p) \gamma^0 \tilde{\psi}(p') \right] \underbrace{\sum_n e^{i\underline{n}\cdot(p'-p)}}_{= (2\pi)^3 \delta_3(p-p')}$$

$$= \int \frac{d^3p}{(2\pi)^3} \psi^\dagger(p) \gamma^0 \left(\gamma^i \sin p_i + m \right) \psi(p)$$

This has the same formal structure as the continuum H_{Dirac}, except that p_i has been replaced by $\sin p_i$. Therefore the single particle energies are

$$E(p) = \pm \left(\sum_{i=1}^{3} \sin^2 p_i + m^2 \right)^{1/2}$$

<u>Low E states</u>: For p around one of

$$p = (0,0,0)$$

$$(\pi,0,0), \quad (0,\pi,0), \quad (0,0,\pi)$$

$$(\pi,\pi,0), \quad (\pi,0,\pi), \quad (0,\pi,\pi)$$

$$(\pi,\pi,\pi)$$

$E(p)$ behaves like in continuum

$$E(p) = \pm (p^2 + m^2)^{1/2}$$

$\Rightarrow$ Our lattice formalism describes 8 – continuum states.

For vector like theories, like QCD, it is possible to remove the degeneracy by modifying the lattice H_{Dirac} to H_{Wilson}

$$H_{Wilson} = \int \frac{d^3p}{(2\pi)^3}\, \psi^\dagger(p)\, \gamma^0 \left(\gamma^i \sin p_i + m + r \sum_{i=1}^{3} (1 - \cos p_i) \right) \psi(p_i)$$

$$E(p) = \pm \left[\sum_i \sin^2 p_i + \left(m + r \sum_{i=1}^{3} (1 - \cos p_i) \right)^2 \right]^{1/2}$$

$$E(0,0,0) = \pm m$$

$$E(\pi,0,0) = \pm \left(m + 2r \right)^{1/2}, \text{ etc.}$$

If we restore the lattice spacing a

$$E(p) = \pm \left[\frac{1}{a^2} \sum_i \sin^2 a p_i + \left(m + \frac{r}{a} \sum_{i=1}^{3} (1 - \cos a p_i) \right)^2 \right]^{1/2}$$

As $a \to 0$, only $p = (0,0,0)$, describes a light particle. The doublers, acquire masses of the order of lattice cut-off.

3. Nielsen - Ninomiya theorem :

" Under certain assumptions, it is impossible to obtain continuum chiral fermions from Lattice fermions."

Set $m=0$ in H_{Dirac} and write it as $H_L + H_R$

$$H_{Dirac} = \int \frac{d^3p}{(2\pi)^3} \tilde{\psi}^\dagger(p)\, \gamma^0 \gamma^j \sin p_j \, \tilde{\psi}(p)$$

$$= \int \frac{d^3p}{(2\pi)^3} \tilde{\psi}^\dagger(p)\, \gamma^0 \gamma^j \sin p_j \left(\frac{1+\gamma_5}{2} + \frac{1-\gamma_5}{2}\right) \tilde{\psi}(p)$$

$$= H_R + H_L$$

$$H_R = \int \frac{d^3p}{(2\pi)^3} \tilde{\psi}_R^\dagger(p)\, \gamma^0 \gamma^j \sin p_j \, \tilde{\psi}_R$$

$$E(p) = \pm \left(\sum_{j=1}^{3} \sin^2 p_j \right)^{1/2}$$

Examine the single particle H near $E(p) = 0$

$$h_R = \gamma^0 \gamma^j \sin p_j \, \frac{1+\gamma_5}{2}$$

For $p \approx 0$ we have

$$h_R = \gamma^0 \gamma^j p_j \frac{1+\gamma_5}{2} + O(p^2)$$

Ignoring $O(p^2)$, we have the relativistic Hamiltonian of a right handed Weyl fermion.

Now consider $p = (\pi, 0, 0)$

 (3.2)

$$p_1 = q_1 + \pi \qquad p_2 = q_2 \qquad p_3 = q_3 \qquad q_i \simeq 0$$

$$\sin(q_1 + \pi) = -\sin q_1 \cong -q_1 + O(q^2)$$

$$h_R = \gamma^0 \left(-\gamma^1 q_1 + \gamma^2 q_2 + \gamma^3 q_3 \right) \frac{1+\gamma_5}{2} + O(q^2)$$

Apart from the sign in front of $\gamma^1 q_1$, we still have the relativistic h of a Weyl spinor. However we need to bring the sign into canonical form. This can be done by a similarity transformation on γ^μ.

$$\gamma^\mu \to S \gamma^\mu S^+$$

Take $\quad S = \gamma^5 \gamma^1 = S^+ = S^{-1}$

then

$$S \gamma^0 \gamma^1 \frac{1+\gamma_5}{2} S^+ = -\gamma^0 \gamma^1 \frac{1-\gamma_5}{2}$$

while

$$S \gamma^0 \gamma^i \frac{1+\gamma_5}{2} S^+ =$$

$$= \gamma^0 \gamma^i \frac{1-\gamma_5}{2} \qquad i = 2, 3$$

$$S\, h_R\, S^\dagger = \gamma^0 \gamma^i q_i \frac{1-\gamma_5}{2}$$

$\Rightarrow$ The excitations arround $p \approx (\pi, 0, 0)$ describe a left handed Weyl fermion.

the same is true for $p = (0, \pi, 0)$ and $p = (0, 0, \pi)$

Now look at $p = (\pi, \pi, 0)$

$$p_1 = \pi + q_1 \qquad p_2 = \pi + q_2 \,, \qquad p_3 = q_3 \qquad q_i \approx 0$$

$$\gamma^i \sin p_i = -\gamma^1 \sin q_1 - \gamma^2 \sin q_2 + \gamma^3 \sin q_3$$

Choose $\quad S = \gamma^0 \gamma^3 \,,\; S^\dagger = S^{-1}$

$$S\, \gamma^0 \gamma^i \frac{1+\gamma_5}{2} S^\dagger = $$

$$= -\gamma^0 \gamma^i \frac{1+\gamma_5}{2} \qquad i = 1, 2$$

while
$$S\, \gamma^0 \gamma^3 \frac{1+\gamma_5}{2} S^\dagger = \gamma^0 \gamma^3 \frac{1+\gamma_5}{2}$$

Therefore

$$S\, h_R\, S^\dagger = \gamma^0 \gamma^j \sin q_j \frac{1+\gamma_5}{2}$$

Similarly $p = (\pi, 0, \pi)$, $(0, \pi, \pi)$ $\Rightarrow$ right handed spinors

Finally let us look at the excitations arround $p = (\pi, \pi, \pi)$

$$p_j = q_j + \pi \qquad j = 1, 2, 3 \qquad q_j \approx 0$$

$$\gamma^0 \gamma^j \sin p_j = - \gamma^0 \gamma^j q_j + \cdots$$

choose $S = \gamma^0 = S^+$

$$S h_R S^+ = \gamma^0 \gamma^0 \gamma^j \sin p_j \frac{1+\gamma_5}{2} \gamma^0$$

$$= \gamma^0 \gamma^j q_j \frac{1-\gamma_5}{2}$$

<u>Summary</u>

$(0, 0, 0)_R$

$(\pi, 0, 0)_L$, $(0, \pi, 0)_L$, $(0, 0, \pi)_L$

$(\pi, \pi, 0)_R$, $(\pi, 0, \pi)_R$, $(0, \pi, \pi)_R$

$(\pi, \pi, \pi)_L$

There are equal number of left and right handed Weyl spinors. If the original Ψ was supporting some linear representation R of an internal symmetry group all these eight states will belong to the same representation R. $\Rightarrow$ The spectrum is vector like.

4. Generalization

To generalize the above negative result, it is more convenient to go to 4-dimensional Euclidean formulation. In the continuum

$\gamma_\mu^2 = 1$

$$S = \int d^4x \; \bar{\psi} \, \gamma^\mu \partial_\mu \frac{1+\gamma_5}{2} \psi \qquad \gamma^{\mu\dagger} = \gamma^\mu$$

$$= \int d^4x \; \bar{\psi} \, \tfrac{1}{2} \gamma^\mu \left(\overrightarrow{\partial_\mu} - \overleftarrow{\partial_\mu} \right) \frac{1+\gamma_5}{2} \psi$$

$$\rightarrow \int \frac{d^4p}{(2\pi)^4} \; \bar{\psi}(p) \, \gamma^\mu \sin p_\mu \, \frac{1+\gamma_5}{2} \, \psi(p)$$

The Euclidean inverse propagator is

$$G^{-1}(p) = \gamma^\mu \sin p_\mu \frac{1+\gamma_5}{2} \qquad p_\mu \in T^4$$

Arround $p_\mu = 0$, this looks like the continuum

$$G^{-1}(p) = \gamma^\mu p_\mu \frac{1+\gamma_5}{2}$$

Arround $p_\mu = (0,0,0,\pi)$, write $p_i = q_i$ & $p_4 = \pi + q_4$

and define a similarity transf. $S = \gamma_4 \gamma_5$

$$S \, \gamma_4 \frac{1+\gamma_5}{2} S^\dagger =$$

$$= - \gamma_4 \frac{1-\gamma_5}{2}$$

$$S \gamma_i \frac{1+\gamma_5}{2} S^+ =$$

$$= \gamma_i \frac{1-\gamma_5}{2} \qquad i = 1, 2, 3$$

Therefore

$$S G^{-1}(p) S^+ = \gamma^\mu q_\mu \frac{1-\gamma_5}{2}$$

$\Rightarrow$ The excitations around $(0,0,0,0)$ are right-handed, while those around $(0,0,0,\pi)$ are left-handed.

$G^{-1}(p)$ vanish at $\underline{16}$ points, which are grouped into 8 right-handed and 8-left handed Weyl fermions.

$\Rightarrow$ In the cont. limit we have a vector-like theory of 8-Dirac fermions

To generalize this result somewhat, let us first note that $p_\mu \in S^1 \times S^1 \times S^1 \times S^1 \cong T^4$, and $\sin p_\mu$ is a vector field on T^4. This vector field has a property that is shared by all continuous vector fields on T^4, namely,

$$\sum_{\sin p_\mu = 0} \text{Sign Det} \frac{\partial \sin p_\mu}{\partial p_\nu} = 0$$

$$\frac{\partial \sin p_\mu}{\partial p_\nu} = \delta_{\mu\nu} \cos p_\mu$$

$$\text{Det} \frac{\partial \sin p_\mu}{\partial p_\nu} = \cos p_1 \cos p_2 \cos p_3 \cos p_4$$

$p \rightarrow$	$(0,\dots,0)$	$(\pi,0,0,0)$ + 4 perm	$(\pi,\pi,0,0)$ + 6 perm	$(\pi,\pi,\pi,0)$ + 4 perm	(π,π,π,π)
$\text{Det} \frac{\partial \sin p_\mu}{\partial p_\nu}$	1	-1	$+1$	-1	$+1$

$$1 + 4(-1) + 6(+1) + 4(-1) + 1 = 8 - 8 = 0$$

Now take any continuous vector field $C_\mu(p)$ on T^4

$$\sum_{\text{isolated } C_\mu(p) = 0} \text{Sign} \frac{\partial C_\mu(p)}{\partial p_\nu} = 0 \qquad \text{(Hopf - Poincaré)}$$

Note that the zeroes must be isolated, otherwise the theorem is not true, e.g.

$$C_i(p) = \sin p_i \qquad i = 1, 2, 3$$

$$C_4(p) = \sin^2 \frac{p_1}{2} + \sin^2 \frac{p_2}{2} + \sin^2 \frac{p_3}{2}$$

This $C_\mu(p)$ has a continuum of zeroes

$$p_i = 0 \quad \& \quad p_4 \in [0, 2\pi)$$

and the theorem is violated!

Assume we replace $\sin p_\mu$ by an arbitrary continuous vector field $C_\mu(p)$ on T^4

$$G^{-1}(p) = \gamma^\mu C_\mu(p) \frac{1+\gamma_5}{2}$$

Isolated zeroes of $C_\mu(p)$ will describe relativistic particles in the continuum limit. But Hopf-Poincaré will force these particles to group together as Dirac particles rather than Weyl fermions. So if there is an internal symmetry in the problem, the theory will be vector like rather than chiral.

5. Comments:

1) Can we reduce the number of doublers ?

The answer is yes. For example we can modify the above degenerate example to (Wilczek '87)

$$C_i(p) = \sin p_i$$

$$C_4(p) = \sin^2\frac{p_1}{2} + \sin^2\frac{p_2}{2} + \sin^2\frac{p_3}{2} + \sin p_4$$

$C_\mu = 0$ has only two solutions

$$p_\mu = (0,0,0,0)_R \quad \& \quad p_\mu = (0,0,0,\pi)_L$$

2) Can we make the doublers heavy by a Wilson mass term ?

In the continuum a mass term is not compatible with γ_5 invariance. On the lattice there is no Lorentz symmetry. One may think we can modify $G^{-1}(p)$ to

$$G^{-1}(p) = \gamma^\mu C_\mu(p)\,\frac{1+\gamma_5}{2} + B(p)$$

and choose $B(p)$ s.t.

$$B(0) = 0 \quad \text{but} \quad B(\bar{p}) \neq 0$$

$$\text{where} \quad C_\mu(\bar{p}) = 0 \quad \bar{p} \neq 0$$

If there are no symmetries on the lattice, such a B-term could be allowed. But if the lattice has no symmetries at all, it may not have a decent Lorentz covariant continuum limit!

Usually the lattice has some crystallographic symmetry group. In this case one has to see if $B(p)$ is compatible with the restrictions imposed by this symmetry. It is very likely that a B-term will be excluded on this ground.

3) What happens if we allow for singularities in $C_\mu(p)$ (L.H. Karsten 1981)

$$C_j(p) = \frac{\sin p_j}{\sqrt{\cos^2\left(\frac{p_4}{2}+\frac{\pi}{4}\right)+\sum_{j=1}^{3}\sin^2\frac{p_j}{2}}}$$

$$C_4(p) = \frac{\sum_{\mu=1}^{4}\cos p_\mu - 3}{\sqrt{\cos^2\left(\frac{p_4}{2}+\frac{\pi}{4}\right)+\sum_{j=1}^{3}\sin^2\frac{p_j}{2}}}$$

Without the $\sqrt{}$ in the denominators, this would describe a right handed fermion about $(0,0,0,-\frac{\pi}{2})$, while near $P_F = (0, 0, 0, \frac{\pi}{2})$ the low E modes would be a left handed fermion. However with the $\sqrt{}$ in the denominators, the left handed fermion at $(0, 0, 0, \frac{\pi}{2})$ disappears and C_F becomes ill defined at this point. Such choices of C_F lead to non-local theories.

Non local theories have also been considered by C. Rebbi Phys. Lett. B 186 (1987) 200, in the context of Lattice QCD. He has shown that if we allow for non-locality there is no need for the introduction of a Wilson mass term. The advantage of this scheme is that the lattice preserves the chiral invariance explicitly, nevertheless as has been shown by Rebbi the chiral anomaly is correctly reproduced in his scheme.

1. Now we would like to try something really wild! Since we know how to handle the doublers in a vector like theory, e.g. by the addition of a Wilson term, let us try to obtain our chiral theory as a limiting case of a vector like theory. This is the scheme I will explain in more detail in the next lecture and show that it correctly passes the most important perturbation test, namely, it reproduces the non-abelian chiral anomalies correctly.

To motivate this approach, which is called the overlap approach, let us consider the Euclidean path integral in the continuum

$$e^{-\Gamma(A)} = \int (D\bar\psi\, D\psi)\, e^{\int d^4x\, \bar\psi \not{D} \frac{1+\gamma_5}{2} \psi}$$

$$= \text{“} \mathrm{Det}\, \not{D} \frac{1+\gamma_5}{2} \text{”} \qquad \not{D}\psi = (\not\partial - i\not{A})\psi$$

The result of integrating over ψ and $\bar\psi$ is a complex functional of A, denoted by $\Gamma(A)$, which satisfies certain properties. For example $\mathrm{Re}\,\Gamma(A)$ can be regularized in a gauge invariant way, while $\mathrm{Im}\,\Gamma(A)$ changes under local gauge transformations, unless the fermions are in the anomaly free representations of G. Hence

$$\delta A_\mu^a = (\nabla_\mu \alpha)^a$$

$$\delta\Gamma(A) = \int d^4x \, \frac{\delta\Gamma}{\delta A_\mu^a(x)} (\nabla_\mu \alpha)^a$$

$$= -\int d^4x \, \nabla_\mu \frac{\delta\Gamma}{\delta A_\mu^a(x)} \, \alpha^a(x)$$

$$= -\int d^4x \, \nabla_\mu \langle J_\mu^a \rangle \, \alpha^a(x)$$

$$\nabla_\mu \langle J_\mu^a(x) \rangle = \text{Consistent chiral anomaly}$$

The overlap approach gives a definition for $\Gamma(A)$ on the lattice (or in the continuum). The way this is done in the continuum is to start from two operators $H_\pm(A,\Lambda)$, which depend on the background gauge field A and on a mass parameter Λ, s.t.

$$H_+(A,\Lambda) = H_-(A,-\Lambda)$$

(5.6)

Let $|A\pm\rangle$ denote the eigenstates of $H_\pm(A,\Lambda)$, obtained by filling the negative Energy states of $H_\pm$.

Define $\Gamma(A)$ by:

$$e^{-\Gamma(A)} = \lim_{\Lambda\to\infty} \frac{\langle A+|A-\rangle}{\langle +|-\rangle}$$

where
$$|\pm\rangle = |A=0,\pm\rangle$$

The idea is to show that

$$\Gamma(A) = \mathrm{Det}\, \not{D}\, \frac{1+\gamma_5}{2}$$

Ref.

1) H.B. Nielsen and M. Ninomiya NPB 185 (1981) 20 & NPB 193 (1981) 173

Lecture II

1. H_{Dirac} in $1+4$ - dimensions

In the last lecture we considered the Dirac Hamiltonian in a 3-dimensional lattice. Its general form is

$$H = \int \frac{d^3p}{(2\pi)^3} \tilde{\psi}^+(p)\, \gamma^0 \left(\gamma^i C_i(p) + B(p) + m \right) \psi(p)$$

where $C_i(p)$ is a vector field on T^3 and $B(p)$ is a scalar function on T^3. The usual choice for these functions is

$$C_i(p) = \sin p_i$$

$$B(p) = r \sum_{i=1}^{3} (1 - \cos p_i)$$

Out of the eight points at which $C_i(p)$ vanishes, only one of them, namely $p_1 = p_2 = p_3 = 0$ is also a zero of $B(p)$. This is how Wilson makes the doublers heavy.

We would like now to examine a Hamiltonian of the above type on a 4-dimensional lattice. This will correspond to a continuum Dirac theory in $1+4$ dimensions.

In $1+4$ dimensions, there are 5 γ-matrices, which can be choosen to be $i\gamma_\mu$, $\mu = 1, \dots 4$ and $\gamma_5 = \gamma_1 \gamma_2 \gamma_3 \gamma_4$

where
$$\gamma_\mu^{\dagger} = \gamma_\mu \qquad \gamma_5^{\dagger} = \gamma_5$$

$$\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}$$

$$\{\gamma_\mu, \gamma_5\} = 0$$

With this choice the 5-dimensional Γ-matrices $\Gamma_\mu = i\gamma_\mu$, $\mu = 1 \ldots 4$ and $\Gamma_0 = \gamma_5$ satisfy

$$\{\Gamma_A, \Gamma_B\} = 2\eta_{AB} = 2\,\mathrm{diag}(1, -1, -1, -1, -1)$$

$$\Gamma_\mu^{\dagger} = -\Gamma_\mu \quad , \quad \Gamma_0^{\dagger} = \Gamma_0$$

Correspondingly the Hamiltonian becomes

$$H = \int \frac{d^4p}{(2\pi)^4}\, \tilde{\Psi}^{\dagger}(p)\, \Gamma^0 \left(\Gamma^\mu C_\mu(p) + B(p) + m\right) \Psi(p)$$

$$= \int \frac{d^4p}{(2\pi)^4}\, \tilde{\Psi}^{\dagger}(p)\, \gamma^5 \left(i\gamma^\mu C_\mu(p) + B(p) + m\right) \Psi(p)$$

where $C_\mu(p)$ is a vector field on T^4 and $B(p)$ is a scalar function on T^4. We shall demand that as $p_\mu \to 0$

$$C_\mu(p) \cong p_\mu + \ldots$$

$$B(p) \cong r p^2 + \ldots$$

The Hopf-Poincaré theorem then demands that $C_\mu(p)$ has at least one other zeros somewhere else. Like the Wilson choice $C_\mu(p) = \sin p_\mu$ and $B(p) = r\sum_{\mu=1}^{4}(1-\cos p_\mu)$, we shall demand in general that $B(p)$ does not vanish at other zeroes of $C_\mu(p)$.

Note that in 5-dimensions — in fact in any odd number of dimensions — there is no notion of chirality. This is due to the fact that $\Gamma_5 = \Gamma_0\Gamma_1\Gamma_2\Gamma_3\Gamma_4 = \gamma_5\gamma_1 \ldots \gamma_4 = \gamma_5^2 = 1$. The group $SO(1,4)$ has only one spinor representation which is 4-dimensional. Our ψ's now belong to this 4-dimensional representation. This observation has a subtle consequence on the sign of the mass term, namely, in 1+3-dimensions we can always go from $\gamma^0(\gamma^i p_i + m)$ to $\gamma^0(\gamma^i p_i - m)$ by a similarity transformation

$$\gamma^5\gamma^0(\gamma^i p_i + m)\gamma^5 = \gamma^0(\gamma^i p_i - m)$$

This is not possible in 1+4-dimensions. Therefore if $|m|$ is a given positive number we can construct two distinct operators $H_\pm$, viz.

$$H_\pm = \int \frac{d^4 p}{(2\pi)^4}\ \psi^\dagger(p)\, \gamma^5 \left(i\gamma^\mu C_\mu(p) + B(p) \pm |m| \right) \psi(p)$$

The mode decomposition of $\psi(p)$ can be performed on the basis of eigenvectors of $\gamma^5 \left(i\not{C} + B(p) + |m| \right)$ or $\gamma^5 \left(i\not{C} + B(p) - |m| \right)$. Let's then consider the eigenvalue equations

$$\gamma^5 \left(i\not{C} + B(p) \pm |m| \right) u_\pm(p,\sigma) = \omega_\pm(p)\, u_\pm(p,\sigma)$$

$$\gamma^5 \left(i\not{C} + B(p) \pm |m| \right) v_\pm(p,\sigma) = -\omega_\pm(p)\, v_\pm(p,\sigma)$$

where

$$\omega_\pm(p) = \sqrt{C^\mu(p) C_\mu(p) + (B(p) \pm |m|)^2}$$

We shall always assume $\omega_\pm(p) > 0$. Thus $u_\pm$ are the positive eigenvalue solutions and $v_\pm$ are the negative eigenvalue solutions. We can easily construct normalized eigenvectors

$$u_\pm(p) = \frac{\omega_\pm + B \pm |m| - i\not{C}}{\sqrt{2\omega_\pm (\omega_\pm + B \pm |m|)}}\ \chi(\sigma)$$

$$v_\pm(p) = \frac{\omega_\pm - B \mp |m| + i\not{C}}{\sqrt{2\omega_\pm (\omega_\pm - B \mp |m|)}}\ \chi(\sigma)$$

where

$$\gamma_5\, \chi(\sigma) = \chi(\sigma), \quad \sigma = 1, 2 \quad \text{and} \quad \chi^\dagger(\sigma)\, \chi(\sigma') = \delta_{\sigma\sigma'}$$

In QFT we expand $\psi(p)$ in a complete basis

$$\psi(p) = \sum_{\sigma=1}^{2} \left(b_\pm(p,\sigma)\, u_\pm(p,\sigma) + d_\pm^\dagger(p,\sigma)\, v_\pm(p,\sigma) \right)$$

From

$$\{ \psi^\dagger(p),\ \psi(p') \} = (2\pi)^4\, \delta_4(p-p')$$

$$\Rightarrow \quad \{ b_\pm(p,\sigma),\ b_\pm^\dagger(p',\sigma') \} = \delta_{\sigma\sigma'}\, (2\pi)^4 \delta_4(p-p'),$$

$$\{ d_\pm(p,\sigma),\ d_\pm^\dagger(p',\sigma') \} = \delta_{\sigma\sigma'}\, (2\pi)^4\, \delta_4(p-p')$$

Note that the + and − operators do not commute

For instance

$$\{ d_+^\dagger(p,\sigma),\ d_-(p',\sigma') \} = (2\pi)^4\, \delta_4(p-p')\, v_+^\dagger(p,\sigma)\, v_-(p,\sigma')$$

The ground states of each one of the Hamiltonians is obtained by filling their respective negative E states

Let us denote these ground states by $|\pm\rangle$. They may be represented as

$$|\pm\rangle = \prod_p \prod_\sigma d_\pm(p,\sigma)\,|0\rangle$$

where

$$b_\pm(p,\sigma)\,|0\rangle = 0 = d_\pm^\dagger(p,\sigma)\,|0\rangle$$

$$b_\pm |\pm\rangle = 0 \quad \text{but} \quad d_\pm^\dagger |\pm\rangle \neq 0$$

We can calculate the overlap $\langle +|-\rangle$

$$\langle +|-\rangle = \prod_p \cos^2\beta(p)$$

where

$$\cos\beta(p) = \sqrt{\frac{\omega_+ - B - |m|}{2\omega_+}\,\frac{\omega_- - B + |m|}{2\omega_-}} + \sqrt{\frac{\omega_+ + B + |m|}{2\omega_+}\,\frac{\omega_- + B - |m|}{2\omega_-}}$$

2. Chiral fermions :

Define a 2-point function

$$G(p) = \frac{1}{|m|} \frac{\langle + | \tilde{\psi}(p) \tilde{\psi}(p)^\dagger | - \rangle}{\langle + | - \rangle}$$

Using the mode decomposition given above we can simplify

$$G(p) = \frac{1}{|m|} \frac{1}{\cos\beta(p)} \sum_\sigma u_+(p,\sigma)\, u_-^\dagger(p,\sigma)$$

The singularities of $G(p)$ will correspond to propagating particles. Since $u_\pm$ are normalized vectors, the only source of singularities for $G(p)$ is the vanishing points of $\cos\beta(p)$. Since $\cos\beta(p)$ is a sum of two positive quantities it can vanish only if each one of them vanishes separately. This happens if

$$\omega_+ = |B + |m|| \quad \& \quad \omega_- = |B - |m||$$

From $\omega_\pm = \sqrt{C^2 + (B \pm |m|)^2}$ we see that $\cos\beta(p) = 0$ implies that

$$C_\mu(p) = 0$$

However we also need to require that $B + |m|$ and $B - |m|$ have opposite signs, i.e. $B^2 < m^2$

We have demanded that near $p_\mu = 0$, $C_p(p) \cong p_\mu$ and $B(p) \cong p^2$.

The Hopf - Poincaré theorem demands that if $C_p(p) = 0$ at the origin, it should necessarily vanish at least at one other point $\bar{p}_\mu$. If $B^2(\bar{p}) < m^2$, then $\cos p(\bar{p}) = 0$, and this point will correspond to a doubler. In order to avoid this unwanted pole it is sufficient to require that $B^2(\bar{p}) > m^2$. We should impose this condition at all other zeroes of $C_p(p)$. The only propagating pole will then be at $p_\mu = 0$.

What kind of a particle does this pole describe? To answer this let us expand $\tilde{G}(p)$ about $p_\mu = 0$. From the expressions given above we can easily deduce

$$\cos p(p) \cong \sqrt{\frac{p^2}{m^2}}\left(1 + \cdots\right)$$

$$u_+(p,\sigma) \cong \left(1 - \frac{i\not{p}}{2|m|} + \cdots\right) \chi(\sigma)$$

$$u_-(p,\sigma) \cong \left(-\frac{i\not{p}}{\sqrt{p^2}} + \frac{\sqrt{p^2}}{2|m|} + \cdots\right) \chi(\sigma)$$

Substitute in $\tilde{G}(p)$

$$\tilde{G}(p) = \frac{1}{|m|} \sqrt{\frac{m^2}{p^2}}\left(1+\cdots\right)\sum_{\sigma}\left(1-\frac{i\not{p}}{2|m|}+\cdots\right)\chi(\sigma)\chi^{+}(\sigma)\left(\frac{+i\not{p}}{\sqrt{p^2}}+\cdots\right)$$

$$= \frac{1}{p^2}\sum_{\sigma=1}^{2}\chi(\sigma)\chi(\sigma)^{+}\left(+i\not{p}\right) + O\left(\frac{p^2}{m^2}\right)$$

$$= \frac{1+\gamma_5}{2}\,\frac{i\not{p}}{p^2} + O\left(\frac{p^2}{m^2}\right)$$

Thus as $p \to 0$, $\tilde{G}(p)$ describes a massless right-handed particle

3. Coupling of gauge fields

So far we have discussed only free fermions. Now we would like to couple the gauge fields and see if the system will produce the correct features of a chiral gauge theory in the continuum limit.

To couple the gauge fields we return to position space and write the Hamiltonian as

$$H_{\pm} = \sum_{n,m} \Psi^{+}(n)\, H_{\pm}(n-m)\, \Psi(m)$$

where $H_{\pm}(n-m)$ is defined in terms of its Fourier mode

$$\sum_{n} H_{\pm}(n)\, e^{-ipn} = \gamma^{5}\left(i\,\not{C}(p) + B(p) \pm |m|\right)$$

We couple the gauge fields through the link variables

$$U(n,m) = P \exp i \int_{m}^{n} A$$

$$= P \exp i \int_{0}^{1} dt\, \dot{\xi}^{\mu}(t)\, A_{\mu}(\xi(t))$$

$$\xi(t) = t n^{\mu} + (1-t) m^{\mu}$$

A_μ is a slowly varying, Lie algebra valued vector potential

$$H_\pm(A) = \sum_{n,m} \psi^\dagger(n)\, H_\pm(n-m)\, U(n,m)\, \psi(m)$$

Under a gauge transformation

$$A_\mu(x) \longrightarrow A_\mu^\theta(x) = e^{i\theta(x)} \left(A_\mu(x) + i\partial_\mu \right) e^{-i\theta(x)}$$

$$U(n,m) \longrightarrow U^\theta(n,m) = e^{i\theta(n)}\, U(n,m)\, e^{-i\theta(m)}$$

$$H_\pm(A) \longrightarrow H_\pm(A^\theta) = \sum_{n,m} \psi^\dagger(n)\, H_\pm(n-m)\, e^{i\theta(n)}\, U(n,m)\, e^{-i\theta(m)}\, \psi(m)$$

$$= U_\theta H_\pm U_\theta^{-1}$$

where U_θ is a unitary operator that acts on ψ's

$$U_\theta \psi(n) U_\theta^{-1} = e^{-i\theta(n)} \psi(n)$$

$$U_\theta \psi^\dagger(n) U_\theta^{-1} = \psi^\dagger(n)\, e^{i\theta(n)}$$

$$U_\theta = \exp i \sum_n \psi^\dagger(n)\, \theta(n)\, \psi(n)$$

Let $|A\pm\rangle$ denote the Dirac ground states. They satisfy

$$H_\pm(A)|A\pm\rangle = E_\pm(A)|A\pm\rangle$$

We shall normalize $|A\pm\rangle$

$$\langle A\pm|A\pm\rangle = 1$$

The phases of $|A\pm\rangle$ are arbitrary. They can be constrained by imposing extra conditions on $|A\pm\rangle$. We shall choose

$$\langle + | A+\rangle > 0 \quad \text{and} \quad \langle - | A-\rangle > 0$$

where $$|\pm\rangle = |A=0, \pm\rangle .$$

Now consider the functional

$$e^{-\Gamma(A)} = \lim_{\partial A \ll A \ll 1} \frac{\langle A+|A-\rangle}{\langle +|-\rangle}$$

What is meant by the limit is that, in any process with an external A line carrying momentum k we should demand that $k \ll A \ll 1$, where 1 denotes the lattice cut off in our units (Recall that the lattice spacing has been set to unity)

The claim is that $\Gamma(A)$ is a good representation for the chiral determinant, i.e. in the continuum

$$e^{-\Gamma(A)} = \int (D\bar\psi\, D\psi)\, e^{\int d^4x\, \bar\psi\, \not{D}\, \frac{1+\gamma_5}{2}\, \psi}$$

4. Checks on overlap:

If $\Gamma(A)$ is a good representation of chiral determinant, it should pass certain tests. The most important one is that it should respond to gauge transformations in a correct way.

We know that in continuum gauge theories, if the fermions are not in an anomaly free representation of gauge group then under an infinitesimal gauge transformation

$$\delta A_\mu^a = (\nabla_\mu \alpha)^a$$

Γ is not invariant. In fact

$$\delta \Gamma(A) = \int d^4x \, \frac{\delta \Gamma(A)}{\delta A_\mu^a(x)} \, \delta A_\mu^a(x)$$

$$= \int d^4x \, \langle J_\mu^a(x) \rangle \, \nabla_\mu \alpha^a$$

$$= \int d^4x \, \nabla_\mu \langle J_\mu^a(x) \rangle \, \alpha^a(x)$$

$$\nabla_\mu \langle J_\mu^a(x) \rangle = \text{chiral anomaly}$$

Infact $\mathrm{Re}\,\Gamma(A)$ can be regularized gauge invariantly. Anomalies usually reside in $\mathrm{Im}\,\Gamma(A)$.

Does our definition of $\Gamma(A)$ reproduce the anomaly correctly?

To awnser this question we need to examine the response of $\Gamma(A)$ to gauge transformations.

We saw above that under a gauge transformation $A \rightarrow A^\theta$, the Hamiltonians transform as

$$H_\pm(A) \longrightarrow H_\pm(A^\theta) = U_\theta\, H_\pm(A)\, U_\theta^{-1}$$

Therefore

$$H_\pm(A)\,|A\pm\rangle = E_\pm(A)\,|A\pm\rangle$$

$$U_\theta H_\pm(A) U_\theta^{-1} U_\theta |A\pm\rangle = E_\pm(A)\, U_\theta |A\pm\rangle$$

$$H_\pm(A^\theta)\, U_\theta |A\pm\rangle = E_\pm(A)\, U_\theta |A\pm\rangle$$

This indicates that $U_\theta|A\pm\rangle$ is an eigenstate of $H_\pm(A^\theta)$

$$U_\theta\,|A\pm\rangle = c\,|A^\theta\pm\rangle$$

where $|c| = 1$, i.e. $c = \exp i\Phi_\pm(\theta, A)$, with $\Phi_\pm$ a real functional of θ and A. We can determine this functional from the requirement that $\langle \pm | A^\theta \pm \rangle > 0$

$$\langle + | A^\theta + \rangle = \langle + | U_\theta | A + \rangle \, e^{-i \Phi_+}$$

$$\langle + | A^\theta + \rangle > 0 \iff \ln \frac{\langle + | A^\theta + \rangle}{\langle + | A^\theta + \rangle^*} = 0$$

$$0 = -i \Phi_+ + \ln \langle + | U_\theta | A + \rangle - i \Phi_+ - \ln \langle A + | U_\theta | + \rangle$$

$$\Phi_+ = \frac{1}{2i} \left(\ln \langle + | U_\theta | A + \rangle - c.c. \right)$$

$$= \operatorname{Im} \ln \langle + | U_\theta | A + \rangle$$

For an infinitesimal θ we can write

$$U_\theta = 1 + Q_\theta \qquad Q_\theta = i \sum_n \psi^\dagger(n) \, \theta(n) \, \psi(n)$$

$$\Phi_+ = \operatorname{Im} \ln \left(\langle + | A + \rangle + \langle + | Q_\theta | A + \rangle \right)$$

$$= \underbrace{\operatorname{Im} \ln \langle + | A + \rangle}_{=0} + \operatorname{Im} \ln \left(1 + \frac{\langle + | Q_\theta | A + \rangle}{\langle + | A + \rangle} \right)$$

$$\Phi_+(\theta, A) = \operatorname{Im} \frac{\langle + | Q_\theta | A + \rangle}{\langle + | A + \rangle}$$

Similarly

$$\Phi_-(\theta, A) = \operatorname{Im} \frac{\langle - | Q_\theta | A - \rangle}{\langle - | A - \rangle}$$

Chiral Anomalies:

Now let us examine the gauge dependence of the overlap

$$\langle A^\theta + | A^\theta - \rangle = e^{i(\Phi_+ - \Phi_-)} \langle A + | A - \rangle$$

$$\ln \langle A^\theta + | A^\theta - \rangle = \ln \langle A + | A - \rangle + i(\Phi_+ - \Phi_-)$$

$$\Gamma(A^\theta) = \Gamma(A) + i(\Phi_+ - \Phi_-)$$

$\Rightarrow$

$$\mathrm{Re}\,\Gamma(A^\theta) = \mathrm{Re}\,\Gamma(A)$$

$$\mathrm{Im}\,\Gamma(A^\theta) = \mathrm{Im}\,\Gamma(A) + \left(\Phi_+(\theta, A) - \Phi_-(\theta, A)\right)$$

If our scheme is correct, we must obtain the standard answer for chiral anomaly in the continuum limit. To calculate it we must assume that θ is ∞-mal, and look at $\delta\Gamma(A)$ for slowly varying A's. The answer should read

$$\delta\Gamma(\theta, A) = \int d^4x \, \frac{1}{24\pi^2} \, \varepsilon^{\mu\nu\lambda\tau} \, \mathrm{Tr}\left(\partial_\lambda A_\mu \partial_\tau A_\nu \theta(x)\right) + \cdots$$

In order to reproduce this answer in the overlap picture we need to develop a perturbation theory.

Let us write $H_\pm(A) = H_\pm(0) + V(A)$, and consider the eigenvalue equation:

$$H_\pm(A)\,|A\pm\rangle = E_\pm(A)\,|A\pm\rangle$$

$$\left(H_+(0) + V(A)\right)|A+\rangle = \left(E_+(0) + \Delta E_+(A)\right)|A+\rangle$$

$$\left(V - \Delta E_+\right)|A+\rangle = \left(E_+(0) - H_+(0)\right)|A+\rangle \qquad (*)$$

The general solution can be written as

$$|A+\rangle = |+\rangle\,\alpha_+(A) + G_+\left(V - \Delta E_+\right)|A+\rangle$$

where $\left(E_+(0) - H_+(0)\right)|+\rangle = 0 \qquad (**)$

$$G_+ = \frac{1 - |+\rangle\langle +|}{E_+(0) - H_+(0)}$$

$\alpha_+(A)$ is a normalization. We can rewrite the above as

$$\left[1 - G_+(V - \Delta E_+)\right]|A+\rangle = |+\rangle\,\alpha_+(A)$$

$$|A+\rangle = \alpha_+(A)\left[1 - G_+(V - \Delta E_+)\right]^{-1} |+\rangle$$

Note that from our phase convention $\langle +|A+\rangle > 0$. It follows that $\alpha_+(A)$ is real and positive. We can determine this function from the normalization condition

$$\langle A+|A+\rangle = 1.$$

For example if we expand $|A+\rangle$ up to second order terms in A we obtain

$$|A+\rangle = \alpha_+(A)\left[1 + G_+(V - \Delta E_+) + G_+(V - \Delta E_+)\, G_+(V - \Delta E_+) + \cdots\right] |+\rangle$$

$$1 = \langle A+|A+\rangle = \alpha_+^2(A)\, \langle +|\left[1 + (V - \Delta E_+) G_+ + (V - \Delta E_+) G_+ (V - \Delta E_+) G_+ + \cdots\right] \left[1 + G_+(V - \Delta E_+) + G_+(V - \Delta E_+)\, G_+(V - \Delta E_+) + \cdots\right] |+\rangle$$

$$= \alpha_+^2(A)\, \langle +|\left(1 + (V - \Delta E_+)\, G_+^2\, (V - \Delta E_+) + \cdots\right) |+\rangle$$

$$\alpha_+(A) = 1 - \frac{1}{2} \langle +|V G_+^2 V|+\rangle + \cdots$$

Likewise to determine ΔE_+ multiply ⊛ from left by $\langle +|$ and make use of ⊛⊛

$$0 = \langle + | (V - \Delta E_+) | A + \rangle$$

$$\Delta E_+ = \frac{\langle + | V | A + \rangle}{\langle + | A + \rangle}$$

Now we can use all this information to calculate the anomaly $\Phi_+ - \Phi_-$

$$\Phi_+(\theta, A) - \Phi_-(\theta, A) = \operatorname{Im}\left(\frac{\langle + | Q_\theta | A + \rangle}{\langle + | A + \rangle} - \frac{\langle - | Q_\theta | A - \rangle}{\langle - | A - \rangle} \right)$$

$$= \sum_n \operatorname{Im}\, i \left(\frac{\langle + | \psi^+(n) \theta(n) \psi(n) | A + \rangle}{\langle + | A + \rangle} - \frac{\langle - | \psi^+(n) \theta(n) \psi(n) | A - \rangle}{\langle - | A - \rangle} \right)$$

We have used this formula to calculated:

1) Non-abelian consistent chiral anomalies in 2 and 4-dimensional lattice

2) gravitational anomalies in 2-dimensions in Einstein-Maxwell theory in the continuum formulation

3) 4-dimensional anomalies in the continuum formulation

Apart from the anomaly calculation there has also been a number of numerical tests in support of the overlap formalism by Narayanan and Neuberger.

Finally I should mention an exact evaluation of $\Gamma(A)$ in a simple background in $D=2$.

We take a 2-dimensional torus as a model of our Euclidean space time and consider the fermions in the background of a constant $U(1)$ gauge field then the continuum overlap can be exactly calculated and it gives the correct answer, which has been known in the mathematical literature as the Quillen determinant. The same exercise on the lattice also produces the correct answer. These confirmation have been obtained numerically by Narayanan and Neuberger and analytically by C. Fosco and myself.

Ref. 1) R. Naryanan and H. Neuberger NPB 443 (1995)
2) S. Randjbar-Daemi and J. Strathdee NPB 443 (1995)