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Lecture II

SUMMER SCHOOL IN HIGH ENERGY PHYSICS AND COSMOLOGY

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MONOPOLE SOLUTIONS IN SUPERSYMMETRIC THEORIES

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Please note: These are preliminary notes intended for internal distribution only.

Lecture 2

Last time found $N \neq 0$ topology of Higgs + finite E

$$\Rightarrow \; g = \int \vec{B} \cdot d\vec{S} = \frac{4\pi N}{e}$$

made stupid factor of 2 mistake, all formulae except last were correct.

Gives $ge = 4\pi N$. compare w/ Dirac $ge = 2\pi n$

Factor of 2 is because although we have been doing SO(3), we could take gauge group to be SU(2) with particles in 2 carrying charge $e/2$. Dirac then requires

$$ge = 4\pi N \quad \text{if minimal charge is } e/2.$$

- also comment on g vs. $g/4\pi$ conventions - Coleman.

#

We expect finite-energy sol'ns carrying mag. charge. We can either construct them by

1) Symmetry plus numerics
2) going to special limit ~~←~~ related to SUSY.

In Higgs vac. $F^{\mu\nu} = \frac{1}{v} \Phi^a F^{a\mu\nu}$ and

$$g = \int_{S^2_\infty} \vec{B} \cdot d\vec{S} = \frac{1}{v} \int_{S^2_\infty} \Phi^a \vec{B}^a \cdot d\vec{S} = \frac{1}{v} \int \vec{B}^a \cdot (\vec{D}\Phi)^a \, d^3r$$

using Bianchi $\vec{D} \cdot \vec{B}^a = 0$ and IBP.

For static config. w/ vanishing E field mass is

$$M = \int d^3r \left\{ \tfrac{1}{2} (\vec{B}^a \cdot \vec{B}^a + \vec{D}\Phi^a \cdot \vec{D}\Phi^a) + V(\Phi) \right\}$$

$$\geq \int d^3r \left\{ \tfrac{1}{2} \vec{B}^a \cdot \vec{B}^a + \vec{D}\Phi^a \cdot \vec{D}\Phi^a \right\}$$

$$= \tfrac{1}{2} \int d^3r \; (\vec{B}^a - \vec{D}\Phi^a) \cdot (\vec{B}^a - \vec{D}\Phi^a) \quad + \; vg$$

so $M \geq vg$ w/ equality iff. $\vec{B}^a = \vec{D}\Phi^a$ Bog. eqn.

this is for $g>0$; if $g<0$ $M \geq -vg$ $\vec{B}^a = -\vec{D}\Phi^a$

equality iff $\boxed{\begin{array}{l} V(\Phi) \equiv 0 \\ \vec{B}^a = \vec{D}\Phi^a \end{array}}$

Ex. 4. For a dyon with elec, mag. charge (q, g) prove the bound

$$\boxed{M \geq v(q^2 + g^2)^{1/2}}$$

Can we saturate this bound? 1^{st} need $V(\Phi) \equiv 0$.
We can still/ Then $\langle \Phi \rangle$ is arbitrary, we can impose sym. breaking by imposing b.c.

$$|\Phi| = \sqrt{\Phi^a \Phi^a} \to v \quad \text{as } r \to \infty.$$

then Bog. eqn $\vec{B}^a = \vec{D}^a \Phi$ is only <u>1^{st} order</u> not 2^{nd} order like eqns of motion.

It is still non-trivial to find general sol'n, but sol'n for $g = 4\pi/e$ can be found explicitly:

Ansatz (sph sym + discrete sym)

$$\Phi^a = \frac{\hat{r}^a}{er} H(ver)$$

$$A_i^a = -\epsilon^a{}_{ij} \frac{\hat{r}^j}{er} \left(1 - K(ver)\right)$$

gives after subst into $B = D\Phi$,

$$y \frac{dK}{dy} = -KH \quad ; \quad y\frac{dH}{dy} = H - (K^2 - 1) \qquad y = ver$$

can check sol'n $\quad H(y) = y \coth y - 1$

$\quad K(y) = y/\sinh y$

Comments: C1. as $y \to \infty$, $H \to y = ver$

$\Phi^a \to v\hat{r}^a$ ✓

$K \to 2ye^{-y}$

$A_i^a \to \frac{-\epsilon^a{}_{ij}\hat{r}^j}{er}$

C2. Since $H \to y(\coth y - \frac{1}{y}) = y - 1 + O(e^{-2y})$

$$\Phi^a = \frac{\hat{r}^a}{er}\left(ver - 1 + O(e^{-2ver})\right)$$

$$= v\hat{r}^a - \frac{\hat{r}^a}{er} + O(e^{-2ver})$$

Physical interp.

In this limit the physical spectrum consists of

	Mass	$C(Q_e, Q_m)$
Higgs	0	(0,0)
photon	0	(0,0)
$W^\pm$	ve	(1,0)
Monopole	vg	(0,1)

- Higgs having $\frac{1}{r}$ long-range tail is only possible in BPS limit - it is massless. When $V(\Phi) \neq 0$, Higgs massive tail would be $e^{-m_H r}$

- corrections e^{-ver} are $e^{-M_W r}$ due to massive $W^\pm$ in the core of the monopole.

C3. Since for BPS monopoles $M = vg$, a n-mon. sol'n w/ $g = ng_0$, $g_0 = 4\pi/e$ has mass $M_n = v ~~g~~ n g_1 = n M_1$. Thus there must be no force between such monopoles - with same ~~charge~~ sign magnetic charge.

We will see how this can happen soon.

<u>Collective Coordinates</u>

Given a classical soliton sol'n in field theory one often finds it is part of multi-param family of solutions. Param z_i are called collective coordinates. Space they parametrize is called moduli space of solutions.

Before discussing gen'l situation, discuss 4 collective coordinates of charge 1 monopole. These are

- $\vec{X}$ from translations.

that is in sol'n $\Phi^a(\vec{r})$, $A_i^a(\vec{r})$ took monopole at origin

but / C/V the "breaks" translation inv. but by acting on sol'n w/ translations we get a new sol'n w/ same energy $\Phi^a(\vec{r} + \vec{X})$, $A_i^a(\vec{r} + \vec{X})$ to make monopole move we consider $\Phi^a(\vec{r} + \vec{X}(t))$, $A_i^a(\vec{r} + \vec{X}(t))$

The remaining coll. coor. is more subtle.

recall in gauge theory config. space is

$$\mathcal{C} = \mathcal{A} / \mathcal{G}$$

$\mathcal{A}$: space of gauge fields; $\mathcal{G}$: ~~gauge transf. vanishing at spatial infinity.~~

but gauge-transf not vanishing at infinity give different points in $\mathcal{C}$ when acting on $\mathcal{A}$

Here we have ~~monopoles~~ config space $\mathcal{C}_M = \mathcal{A}_M / \mathcal{G}$ with $\mathcal{A}_M = (A, \Phi)$ ~~xxxxxxxxxxxx~~

Now let's work in $A_0^a = 0$ gauge and try to find a ~~configuration~~ deformation δA_i^a, $\delta\Phi^a$ of monopole which is both physical and does not change the energy.

Lets also simplify by writing $A_\mu = e A_\mu^a T^a$

$$\Phi = \Phi^a T^a$$

w/ T^a anti-hermitian gen. of $SU(2)$ (e.g. $T^a = -\frac{i}{2}\sigma^a$)

Then in $A_0 = 0$ gauge must impose Gauss Law

$$D_i \dot{A}_i + [\Phi, \dot{\Phi}] = 0$$

If we try to construct a deformation as a time-dep. g.t.

$$\delta A_i = D_i \Lambda(t) \quad , \quad \delta\Phi = [\Phi, \Lambda(t)]$$

Gauss requires $D_i \delta\dot{A}_i + [\Phi, [\Phi, \dot{\Lambda}]] = 0$

or

$$\underbrace{D_i D_i \dot{\Lambda} + [\Phi, [\Phi, \dot{\Lambda}]]}_{} = 0$$

pos-def. op so no soln if Λ vanishes at ∞ - as expected

But can solve by $\Lambda = \chi(t)\,\Phi$ with χ arb. fcn. of time.

since $[\Phi, \Phi] = 0$ and

$$D_i D_i \Phi = D_i B_i = 0 \text{ by Bog. eqn. + Bianchi.}$$

So we have found 1 more physical collective coord.

Equivalently, start w/ PS solution and gauge transform by $g = e^{-\chi\Phi}$ - χ is a physical collective coordinate.

before

So for 1-monopole we have moduli space

$$\mathcal{M}_1 = \underset{\underset{\chi}{\uparrow}}{S^1} \times \underset{\underset{\vec{X}}{\uparrow}}{\mathbb{R}^3}$$

Insert (I) here

Now to explain the situation more generally note the following connection:

If we write $A_4 = \Phi$ ~~and for ... let ya~~ and let $a, b, .. = 1 \ldots 4$

then the 4-d Euc. Self dual eqn.

$$F_{ab} = \tfrac{1}{2} \epsilon_{abcd} F_{cd} = {}^*F_{ab} \qquad F_{ab} = \partial_a A_b - \partial_b A_a + [A_a, A_b]$$

$$\cancel{A_a = (A_a, \Phi)}$$

is equivalent to, assuming $A_a(\vec{x})$ ind. of x_4

$$F_{i4} = D_i A_4 = D_i \Phi$$

and $F_{ab} = {}^*F_{ab} \iff B_i = D_i \Phi$

Gauss Law is $D_a \dot{A}_a = 0$

and gauge transf. on (A_i, Φ) is $\delta A_a = D_a \Lambda$
for Λ ind. of x_4

Because Bog. eqn's can be written in terms of trans. inv. self-dual eqn's there are many connections between the techniques used to solve them.

(I) What is the interp. of χ? hard deformation

$\delta A_i = D_i \Lambda(t) = \chi(t) D_i \Phi$

gives rise to $\delta E_i = \delta F_{0i} = \partial_0 \delta A_i = \dot{\chi} D_i \Phi = \dot{\chi} B_i$

that is monopole acquires an electric field proportional to $\dot{\chi}$ and existing B_i

Also since gauge group is compact, is compact.
In particular $\chi, \chi + 2\pi$ both rel. to g.t.
which is 1 at ∞ - since subset ∞ part of gauge group $\mathcal{G}$ and config identified

$\Rightarrow$ quantization of dyon charge when coll. coor. are treated quan mechanically

$$\boxed{\begin{matrix} & \text{SU}^{\text{?}} & \\ \mathfrak{a}^? & \overset{\ }{\to} & \mathcal{C} \\ & \downarrow & \\ & \mathcal{X}_{g.t.} & \end{matrix}}$$

Now given config. space $\mathcal{C}$, the k-monopole moduli:

$$\mathcal{M}_k = \{ A_a : F_{ab} = {}^*F_{ab},\ \text{top. charge } k \}$$

Tangent vectors to $\mathcal{M}_k$ are deformations of soln $A_a(\vec{x})$ in k-monopole sector which obey linearized Bog. eq'n. If $A_a \to A_a + \delta_\alpha A_a$

$\delta F_{ab} = D_a \delta A_b - D_b \delta A_a$ so require

$$D_a \delta_\alpha A_b - D_b \delta_\alpha A_a = \tfrac{1}{2} \epsilon_{abcd} (D_c \delta_\alpha A_d - D_d \delta_\alpha A_c) \qquad (1)$$

and for deformation to be $\perp$ to gauge transf.

$$D_a \delta_\alpha A_a = 0 \qquad (2)$$

Now suppose we have a config. $A_a(\vec{x}, z^\alpha(t))$

coll. coor. on $\mathcal{M}_k$

We can get an action for coll. coor as follows.

We get a deformation by diff wrt. z^α $\frac{\partial}{\partial z^\alpha} A_a \equiv \partial_\alpha A_a$

Now in general to satisfy (2) we will have to accompany this by a g.t. to construct

$$\delta_\alpha A_a = \partial_\alpha A_a - D_a \epsilon_\alpha \quad \text{w/} \quad D_a \delta_\alpha A_a = 0$$

if we take $A_a(\vec{x}, z^\alpha(t))$, $A_0 = \dot{z}^\alpha \epsilon_\alpha$

then $F_{0a} = \partial_0 A_a - \partial_a A_0 + [A_0, A_a] = \dot{z}^\alpha \partial_\alpha A_a - \dot{z}^\alpha [\epsilon_\alpha, A_a] - \dot{z}^\alpha \partial_a \epsilon_\alpha = \dot{z}^\alpha \delta_\alpha A_a$

action is

$$S = -\frac{1}{4}\int d^3x\,dt\,\mathrm{Tr}\,F_{0a}F_{0a} = \frac{1}{4}\int dt\,\mathcal{G}_{\alpha\beta}\,\dot{z}^\alpha\,\dot{z}^\beta$$

where $\mathcal{G}_{\alpha\beta} = -\int d^3x\ \mathrm{Tr}\,\delta_\alpha A_a\,\delta_\beta A_a$ is a natural metric on $\mathcal{M}_k$.

In principle this determines a metric on $\mathcal{M}_k$ given explicit k-monopole soln. Unfortunately sufficiently explicit soln's are not known for $k>1$

Ex. 5 a) Follow this to determine metric on $\mathcal{M}^1$ - answer - it is flat
$S \propto \int dt\ c_1\dot{x}^2 + c_2\dot{\bar{\mathbb{X}}}\,\dot{\mathbb{X}}$

b) ~~[illegible]~~
View ~~[illegible]~~ ϵ_α as a connection on $\mathcal{M}_k$ w/ cov. deriv. $\delta_\alpha = \partial_\alpha + [\epsilon_\alpha,\ ]$
Show $\delta_\alpha A_a$ is mixed comp. of curvature on $\mathbb{R}^4 \times \mathcal{M}_k$ w/ connection (A_a, ϵ_α)