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SMR.858 - 6

Lecture II

SUMMER SCHOOL IN HIGH ENERGY PHYSICS AND COSMOLOGY

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BASICS OF CFT, STRING COMPACTIFICATIONS, MIRROR SYMMETRY AND SPACETIME TOPOLOGY CHANGE

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LECTURE 1

REVIEW OF LECTURE 0

- STRING THEORY AS MOTIVATION FOR STUDY OF CFT
  - COORD INVARIANCE
  - WEYL INVARIANCE

  $\Big\}$ CONFORMAL REP. INV

$$z \longmapsto f(z) = z + \varepsilon(z)$$
$$\bar{z} \longmapsto \bar{f}(\bar{z}) = \bar{z} + \bar{\varepsilon}(\bar{z})$$

- CFT:
  - ABOVE SYMMETRY YIELDS CONFORMAL WARD ID'S
    $\rightarrow$ CONSTRAINS CORRELATION FUNCTIONS
  - ABOVE SYMMETRY YIELDS CONVENIENT & POWERFUL WAY TO ORGANIZE HILBERT SPACE

Key Concepts:

2.3

- 1) E-M tensor : $T^{\alpha\beta}(x) = 2\, \delta S / \delta h_{\alpha\beta}(x)$

- 2) Weyl Invariance $\Rightarrow (\delta S/\delta h_{\alpha\beta})(h_{\alpha\beta}) = 0 \Rightarrow T^{\alpha}{}_{\alpha}(x) = 0$

- 3) In complex coords :
  - $T_{zz}(z,\bar{z})$ is hol $\Rightarrow$ write $T(z)$
  - Similarly $\bar{T}(\bar{z})$
  - $T_{z\bar{z}} = 0$

- 4) Conformal Ward Identities : (General)

$$\sum_{j=1}^{n} \langle \theta_1(z_1\bar{z}_1) \cdots \widehat{\delta\theta}_j(z_j\bar{z}_j) \cdots \theta_n(z_n\bar{z}_n) \rangle = \langle T(\omega)\,\theta_1 \cdots \theta_n \rangle$$

$\widehat{\delta\theta}_j(z_j,\bar{z}_j)$ = variation of θ_j under $z_j \to z_j + 1/(z_j - \omega)$

- 5) Primary Fields : $\Phi(\omega,\bar{\omega})$ is primary of wt $(\underline{h},\bar{h})$ if

$$T(z)\,\Phi(w,\bar{w}) = \underline{h}/(z-w)^2\; \Phi(w,\bar{w}) + 1/(z-w)\, \partial_w \Phi + \cdots$$

- 6) Highest Wt representation : $\Phi(0)|0\rangle \equiv |\Phi\rangle$
  - $L_0 |\Phi\rangle = h|\Phi\rangle \;;\; L_n|\Phi\rangle = 0$
  - $\{ |\Phi\rangle, L_{-n_1}|\Phi\rangle, L_{-n_1}L_{-n_2}|\Phi\rangle, \ldots\; L_{-n_1}\cdots L_{-n_r}|\Phi\rangle, \ldots \}$

- 7) Ward Id's for Primaries : $\langle T(\omega)\,\Phi_1(z_1\bar{z}_1) \cdots \Phi_n(z_n\bar{z}_n) \rangle =$

$$\sum_{j=1}^{n} \left(\frac{h_j}{(\omega - z_j)^2} + \frac{1}{\omega - z_j}\frac{\partial}{\partial w_j} \right) \langle \Phi_1(z_1\bar{z}_1) \cdots \Phi_n(z_n\bar{z}_n) \rangle$$

- 8) $T(z)\,T(\omega) = \dfrac{c/2}{(z-w)^4} + 2/(z-w)^2\, T(\omega) + 1/(z-w)\, \partial_\omega T + \cdots$

• WHY $N=2$ CFT?

- $N=2$ SUSY ON WORLDSHEET $\Rightarrow$
  $N=1$ SUSY IN SPACETIME

- NONRENORMALIZATION THEOREMS
  $\Rightarrow$ SOL'N TO ALL ORDERS IN STRING PERTURBATION THEORY
  &
  $\Rightarrow$ PHENOMENOLOGICAL RELEVANCE

II) N=2 SUPERCONFORMAL ALGEBRA

II.1) THE ALGEBRA

- USUAL (N=0) CONFORMAL ALGEBRA:
  GENERATED BY $T(z)$

① $$T(z)\,T(\omega) \sim \frac{c/2}{(z-\omega)^4} + \frac{2}{(z-\omega)^2} T(\omega) + \frac{1}{(z-\omega)} \partial_\omega T(\omega) + \cdots$$

($\cdots$ = NON-SINGULAR TERM $\sim \mathcal{L}_n T(\omega)$ = NEW FIELDS)
$= \oint [dz]\,(z-\omega)^{n+1} T(z) T(\omega)$

①′ $$[L_n, L_m] = (n-m) L_{n+m} + c/12\,(n^3-n)\,\delta_{n+m,0}$$

$$T(z) = \sum_{n\in\mathbb{Z}} z^{-n-2} L_n \Rightarrow L_n = \oint [dz]\, z^{n+1}\, T(z)$$

- N=1 SCA: ADD $G(z)$, dim $= 3/2$

① AS ABOVE

② $$T(z)\,G(\omega) \sim \frac{3}{2} \frac{G(\omega)}{(z-\omega)^2} + \frac{\partial G(\omega)}{(z-\omega)} + \cdots$$

③ $$G(z)\,G(\omega) \sim \frac{2c}{3(z-\omega)^3} + \frac{2\,T(\omega)}{(z-\omega)} + \cdots$$

In terms of modes: $G_n = \oint [dz]\, z^{n+1/2}\, G(z)$

(1') As above

(2') $[L_n, G_m] = \left(\frac{m}{2} - n\right) G_{n+m}$

(3') $\{G_m, G_n\} = 2L_{m+n} + \frac{c}{3}\left(m^2 - \frac{1}{4}\right)\delta_{m+n,0}$

• $N=2$ SCA: $G^1(z), G^2(z)$

(1) As above

(2) As above: $G \to G^1, G^2$

(3) $G^\alpha(z)\, G^\beta(\omega) \sim \left(\frac{2c}{3(z-\omega)^3} + \frac{2T(\omega)}{(z-\omega)}\right)\delta^{\alpha\beta} + i\left(\frac{2J(\omega)}{(z-\omega)^2} + \frac{\partial J(\omega)}{(z-\omega)}\right)\varepsilon^{\alpha\beta} + \cdots$

$J(\omega)$: dimension 1 current

(4) $J(z)\, G^\alpha(\omega) \sim i\, \varepsilon^{\alpha\beta}\, G^\beta(\omega) / (z-\omega) + \cdots$

(5) $T(z)\, J(\omega) \sim J(\omega)/(z-\omega)^2 + \partial J(\omega)/(z-\omega) + \cdots$

(6) $J(z)\, J(\omega) \sim \frac{c}{3}/(z-\omega)^2 + \cdots$

- USEFUL TO INTRODUCE : $G^{\pm}(z) = 1/\sqrt{2}\,(G^1(z) \pm i\,G^2(z))$

③ $\rightarrow$ $G^+(z)G^-(\omega) \sim \dfrac{2c}{3(z-\omega)^3} + \dfrac{2J(\omega)}{(z-\omega)^2} + \dfrac{2T(\omega)+\partial J(\omega)}{(z-\omega)} + \dots$

$$G^+(z)G^+(\omega) \sim G^-(z)G^-(\omega) \sim O(1)$$

④ $\rightarrow$ $J(z)\,G^{\pm}(\omega) \sim \pm\, G^{\pm}(\omega)/(z-\omega)$

- IN TERMS OF MODES :

$T(z) = \sum L_n z^{-n-2}$ $\qquad$ $J(z) = \sum J_n z^{-n-1}$

$G^{\pm}(z) = \sum G^{\pm}_{n\pm a}\, z^{-(n\pm a)-3/2}$ $\qquad$ $0 \leq a < 1$

$$[L_m, L_n] = (m-n)L_{m+n} + c/12\, m(m^2-1)\delta_{m+n,0}$$
$$[J_m, J_n] = c/3\, m\, \delta_{m+n,0}$$
$$[L_n, J_m] = -m\, J_{m+n}$$
$$[L_n, G^{\pm}_{m\pm a}] = (n/2 - (m\pm a))\, G^{\pm}_{m+n\pm a}$$
$$[J_n, G^{\pm}_{m\pm a}] = \pm\, G^{\pm}_{m+n\pm a}$$
$$\{G^+_{n+a}, G^-_{m-a}\} = 2L_{m+n} + (n-m+2a)J_{n+m} + c/3\left[(n+a)^2 - 1/4\right]\delta_{m+n,0}$$

RAMOND : $a \in \mathbb{Z}$ $(0) \Rightarrow G^{\pm}(z)$ PERIODIC IN $z \to e^{2\pi i} z$

NS : $a \in \mathbb{Z} + 1/2$ $(1/2) \Rightarrow G^{\pm}(z)$ ANTIPERIODIC "

II.2) Highest Weight Representations

• In N=0 case:

Primary field ϕ: $T(z)\,\phi(\omega) \sim \frac{h_\phi}{(z-\omega)^2}\,\phi(\omega) + \frac{1}{(z-\omega)}\,\partial_\omega \phi(\omega) + \cdots$

HWS: $|\phi\rangle = \phi(0)|0\rangle$ satisfies

$L_0|\phi\rangle = h_\phi|\phi\rangle$; $L_n|\phi\rangle = 0 \quad \forall\, n>0$

States in rep: $(\prod L_{-n_i})|\phi\rangle$

• In N=2 case:

Superconformal Primary Field ϕ

$T(z)\,\phi(\omega) \sim$ as above
$J(z)\,\phi(\omega) \sim (Q_\phi/(z-\omega))\,\phi(\omega) + \cdots$
$G^\pm(z)\,\phi(\omega) \sim (1/(z-\omega))\,\widehat{\phi}^\pm(\omega) + \cdots \quad = (1/(z-\omega))(G^\pm_{-1/2}\phi)(\omega)$
↳ superpartners of ϕ

$\Longleftrightarrow$ $|\phi\rangle = \phi(0)|0\rangle$ satisfies:
$L_0|\phi\rangle = h_\phi|\phi\rangle$; $J_0|\phi\rangle = Q_\phi|\phi\rangle$
$L_n|\phi\rangle = J_n|\phi\rangle = G^\pm_n|\phi\rangle = 0 \quad n>0$
States in rep: "$\prod L_{-n_i} J_{-m_i} G^\pm_{-l_k}$" $|\phi\rangle$

II.3) Chiral Primary Fields

L.V.W. (LERCHE VAFA WARNER)

- Special subset of NS primary fields "chiral primary":

$$G^+_{-1/2}|\phi\rangle = 0 \text{, i.e. } G^+(z)\phi(w) \sim \mathcal{O}(1)$$

- Also: "antichiral primary":

$$G^-_{-1/2}|\phi\rangle = 0 \text{, i.e. } G^-(z)\phi(w) \sim \mathcal{O}(1)$$

Either of these additional constraints eliminates half of the superpartners of $|\phi\rangle$.

- Recall: Holomorphic & Antiholomorphic Algebras
$\{T(z), J(z), G^{\pm}(z)\} \times \{\bar{T}(\bar{z}), \bar{J}(\bar{z}), \bar{G}^{\pm}(\bar{z})\} \Rightarrow$

(chiral, chiral) $\quad G^+_{-1/2}|\phi\rangle = \bar{G}^+_{-1/2}|\phi\rangle = 0$

(antichiral, chiral) $\quad G^-_{-1/2}|\phi\rangle = \bar{G}^+_{-1/2}|\phi\rangle = 0$

(chiral, antichiral) $\quad G^+_{-1/2}|\phi\rangle = \bar{G}^-_{-1/2}|\phi\rangle = 0$

(antichiral, antichiral) $\quad G^-_{-1/2}|\phi\rangle = \bar{G}^-_{-1/2}|\phi\rangle = 0$

- SPECIAL FEATURE OF SUCH CHIRAL (ANTI) FIELDS :

  - THERE ARE A FINITE NUMBER OF THEM
  - THEIR OPERATOR ALGEBRA IS CLOSED & NONSINGULAR

- LETS SEE HOW THIS GOES:

i) CONSIDER $\{G^-_{1/2}, G^+_{-1/2}\} = 2L_0 - J_0$ ACTING ON CHIRAL PRIMARY $|\phi\rangle$:

$$0 = \{G^-_{1/2}, G^+_{-1/2}\}|\phi\rangle = (2L_0 - J_0)|\phi\rangle = 2h_\phi - Q_\phi$$

$$\Rightarrow \quad h_\phi = Q_\phi/2$$

• IN FACT : USE $(G^+_{-1/2})^\dagger = G^-_{1/2} \Rightarrow \forall|\psi\rangle$:

$$\langle\psi|\{G^-_{1/2}, G^+_{-1/2}\}|\psi\rangle \geq 0 \Rightarrow h_\psi \geq Q_\psi/2$$

ii) ϕ, χ CHIRAL PRIMARY

$$\phi(z)\chi(\omega) \sim \sum_i (z-\omega)^{h_{\psi_i} - h_\phi - h_\chi} \psi_i(\omega)$$

$$Q_{\psi_i} = Q_\phi + Q_\chi \; ; \; h_{\psi_i} \geq \tfrac{1}{2} Q_{\psi_i} = \tfrac{1}{2} Q_\phi + \tfrac{1}{2} Q_\chi = h_\phi + h_\chi$$

$$\Rightarrow \lim_{z\to\omega} \phi(z)\chi(\omega) = (\phi\chi)(\omega) = \sum_{\text{CHIRAL PRIMARY}} \psi_j$$

$\Rightarrow$ CHIRAL PRIMARY FIELDS FORM A (NON-SINGULAR) RING.

iii) FINALLY:

CONSIDER $\{G^-_{3/2}, G^+_{-3/2}\} = 2L_0 - 3J_0 + 2c/3$.

LET ϕ BE CHIRAL PRIMARY:

$$\langle\phi| \{G^-_{3/2}, G^+_{-3/2}\} |\phi\rangle = \langle\phi| 2L_0 - 3J_0 + 2c/3 |\phi\rangle \geq 0$$

$$\Rightarrow \quad 2h_\phi - 3Q_\phi + 2c/3 \geq 0$$

AND $h_\phi = Q_\phi/2$

$$\Rightarrow \quad -4h_\phi + 2c/3 \geq 0 \Rightarrow \underline{\underline{h_\phi \leq c/6}}$$

$\Rightarrow$ FINITE # OF CHIRAL PRIMARY FIELDS

- $\Rightarrow$ CHIRAL PRIMARY FIELDS FORM A FINITE NON-SINGULAR RING

- MORE PRECISELY: A GENERAL $N=2$ SCFT HAS 4 SUCH RINGS

1) (c,c) $\quad$ 1*) (a,a)

2) (a,c) $\quad$ 2*) (c,a)

Spectral Flow

$$G^{\pm}(z) = \sum G^{\pm}_{n\pm a}\, z^{-(n\pm a)-3/2} \qquad 0 \le a < 1$$

Claim: Different choices for "a" (diff boundary conditions) yield isomorphic algebras

Proof: (Direct calculation) Let $\eta = a - 1/2$

Want to show: Algebra generated by $\{L_n, J_n, G^{\pm}_{n\pm(\eta+\frac{1}{2})}\}$ is isomorphic to that generated by $\{L_n, J_n, G^{\pm}_{n\pm 1/2}\}$.

Define:

$$L_n' = L_n + \eta J_n + \tfrac{1}{6}\eta^2 c\, \delta_{n,0}$$
$$J_n' = J_n + \tfrac{1}{3}\eta c\, \delta_{n,0}$$
$$G^{\pm\,\prime}_r = G^{\pm}_{r\pm\eta}$$

NB: $\mathrm{Alg}\{L_n', J_n', G^{\pm\,\prime}_r\}$ is trivially $\mathrm{Alg}\{L_n, J_n, G^{\pm}_{r\pm\eta}\}$

Now: $[L_n', L_m'] = [L_n + \eta J_n + \frac{1}{6}\eta^2 c\,\delta_{n,0},\ L_m + \eta J_m + \frac{1}{6}\eta^2 c\,\delta_{m,0}]$

$$\begin{aligned}
&= (n-m)L_{n+m} + c/12\, m(m^2-1)\,\delta_{m+n,0} \\
&\quad + \eta(-m)J_{m+n} - \eta(-n)J_{m+n} \\
&\quad + \eta^2 \tfrac{c}{3}\, n\, \delta_{m+n,0} \\
&= (n-m)\left(L_{n+m} + \eta J_{n+m} + \tfrac{\eta^2}{6} c\, \delta_{m+n,0}\right) + \tfrac{c}{12} m(m^2-1)\delta_{m+n,0}
\end{aligned}$$

$$[L_n', L_m'] = (n-m)L'_{n+m} + c/12\, m(m^2-1)\,\delta_{m+n,0} \quad ✓$$

- Similarly for all other parts of the algebra.

0.6
10.61
10.62

- WE CAN REALIZE THIS ISOMORPHISM BY A UNITARY OPERATOR:
  - $L_n' = U_\eta L_n U_\eta^{-1}$ • $J_n' = U_\eta J_n U_\eta^{-1}$ • $G_r^{\pm\,\prime} = U_\eta G_r^{\pm} U_\eta^{-1}$

- AND: GIVEN A REPRESENTATION $\{|S_j\rangle\}$ ON WHICH $\{L_n, J_n, G_r^{\pm}\}$ ACT, GET REP. $\{U_\eta |S_j\rangle\}$ ON WHICH $\{L_n', J_n', G_r^{\pm\,\prime}\}$ ACT.

- $\eta = 0$: SPACETIME BOSONS (NS)
  $\eta = 1/2$: SPACETIME FERMIONS (R)

"SPECTRAL FLOW OPERATOR" $U_{1/2} \leftrightarrow$ BOSONS $\leftrightarrow$ FERMIONS
$\leftrightarrow$ SUSY IN SPACETIME

EXPLICITLY: $J_n' = J_n + \frac{1}{3}\eta c\, \delta_{n,0} \Rightarrow J_0' = J_0 + \frac{1}{3}\eta c$

$\Rightarrow$ EXPECT U_η SHIFTS $U(1)$ CHARGE OF STATES BY $\frac{1}{3}\eta c$.

- GENERALLY: $U(1)$ CURRENT $= i\sqrt{\frac{c}{3}}\,\partial_z \phi$
- FIELD of CHARGE q: $f = \underset{\text{(NEUTRAL)}}{\hat{f}}\, e^{iq\sqrt{3/c}\,\phi}$
- $U_\eta \cdot f = \hat{f}\, e^{i\phi\left(q\sqrt{\frac{3}{c}} + \eta\sqrt{\frac{c}{3}}\right)}$

(CHECK: L_n' WORKS OUT)

$\Rightarrow U_\eta = e^{i\sqrt{\frac{c}{3}}\,\eta\,\phi} \Rightarrow U_{1/2} = e^{i\sqrt{\frac{c}{3}}\,\frac{1}{2}\,\phi}$ ([redacted])

(Left side only).

$N=2 \rightarrow$ SPECTRAL FLOW $\rightarrow$ SPACETIME SUSY.

II.4) FOUR EXAMPLES

A) ONE COMPLEX BOSON $X(z) = X^1(z) + iX^2(z)$
ONE COMPLEX FERMION $\psi(z) = \psi^1(z) + i\psi^2(z)$ $\Big\}$ $c=3$

$$S = \int d^2z \left(\partial X \bar{\partial} X^* + i\,\psi^* \bar{\partial}\psi + \psi\bar{\partial}\psi^*\right)$$

$$T(z) = -\partial X \partial X^* + \tfrac{1}{2}\psi^*\partial\psi + \tfrac{1}{2}\psi\partial\psi^*$$
$$G^+(z) = \psi^*\partial X$$
$$G^-(z) = \psi\partial X^*$$
$$J(z) = \psi^*\psi$$

EXCERCISE: USING $\langle X(z) X^*(w)\rangle \sim -\log(z-w)$
$\langle \psi(z)\psi^*(w)\rangle \sim -1/(z-w)$

SHOW THAT THE ABOVE GENERATE AN $N=2$ SCA.

• BY DIRECT CALCULATION:

$\psi^*(w)$ $h = 1/2$ $Q = -1$ ANTICHIRAL PRIMARY
$\psi(w)$ $h = 1/2$ $Q = +1$ CHIRAL PRIMARY

$\Rightarrow$ (c,c) : $\{1, \psi, \bar{\psi}, \psi\bar{\psi}\}$ $h \leq c/6$? YES: $\frac{1}{2} = \frac{3}{6}$.
$(a,c) = \{1, \psi^*, \bar{\psi}, \psi^*\bar{\psi}\}$
$(c,a) = \{1, \psi, \bar{\psi}^*, \psi\bar{\psi}^*\}$
$(a,a) = \{1, \psi^*, \bar{\psi}^*, \psi^*\bar{\psi}^*\}$

<u>NOTE</u>: FOR STRING THEORY IN LIGHT CONE GAUGE

$$M_4 \quad \times \quad \text{"INTERNAL"}$$

$$\left[\begin{array}{l}\text{2 REAL BOSONS} \\ \text{+ 2 REAL FERMIONS} \begin{pmatrix}\text{ON LEFT} \\ \text{\& RIGHT}\end{pmatrix}\end{array}\right] = \left.\begin{array}{l}\text{1 COMPLEX BOSON} \\ \text{1 COMPLEX FERMION}\end{array}\right\} \times \left\{\begin{array}{c}\text{"INTERNAL"} \\ C = 9 \quad N = 2 \\ \text{SCFT}\end{array}\right\}$$

$\downarrow$ EXAMPLE A

$\downarrow$ OTHER EXAMPLES

B) NONLINEAR SIGMA MODEL : (NLσM)

- MORE BOSONIC/FERMIONIC FIELDS
- REQUIRE BOSONIC FIELDS SPAN (ARE COORDS OF) SOME RIEMANNIAN MANIFOLD (FERMIONS ARE SECTIONS OF PULLBACK OF TANGENT BUNDLE TIMES SPIN BUNDLE)

- FIELDS : $X^i, X^{\bar{i}}$ COMPLEX BOSONIC FIELDS $i = 1, \dots, \dim_{\mathbb{C}} M$ (KÄHLER)

  $\psi^i\ \psi^{\bar{i}}$ RMOVING COMPLEX FERMIONS

  $\lambda^i, \lambda^{\bar{i}}$ LMOVING COMPLEX FERMIONS

- ACTION :

ANTISYMMETRIC TENSOR

$$S = \frac{i}{4\pi\alpha'}\int \Big\{ g^M_{i\bar{j}}(\partial_z X^i \bar{\partial}_{\bar{z}} X^{\bar{j}} + \bar{\partial}_{\bar{z}} X^i \partial_z X^{\bar{j}}) - i B_{i\bar{j}}(\partial X^i \bar{\partial} X^{\bar{j}} - \bar{\partial} X^i \partial X^{\bar{j}})$$
$$+ i\lambda^{\bar{i}} D_z \lambda^j g_{\bar{i}j} + i\psi^{\bar{i}} D_{\bar{z}}\psi^j g_{\bar{i}j}$$
$$+ R_{i\bar{i}j\bar{j}}\psi^i\psi^{\bar{i}}\lambda^j\lambda^{\bar{j}} \Big\}\, d^2z$$

$$\Big(D_{\bar{z}}\psi^i = \partial_{\bar{z}}\psi^i + \frac{\partial X^j}{\partial \bar{z}}\Gamma^i_{jk}\psi^k \Big)$$

$$\Big(S = \int d^2z\, d^4\theta\, K(\Phi^i, \Phi^{\bar{i}}) + B\ ;\ \Phi^i = X^i + \theta^-\psi^i + \bar{\theta}^-\lambda^i + \theta^-\bar{\theta}^- F^i \Big)$$

• $N=2$ SUSY : FORMULATED IN $N=2$ SUPERSPACE

IN COMPONENTS e.g.: $\delta X^i = i\bar{\alpha}_+ \psi^i + i\alpha_+ \lambda^i$
$$\delta \psi^i_+ = -\bar{\alpha}_- \partial_z X^i - i\alpha_+ \lambda^j \Gamma^i_{jm} \psi^m$$
$$\vdots$$

• HOW ABOUT CONFORMAL SYMMETRY?

• $\beta(g_{i\bar{j}}) = R_{i\bar{j}} \Rightarrow M$ MUST ADMIT RICCI FLAT METRIC

$\Rightarrow M$ IS COMPLEX, KÄHLER, ADMITTING METRIC WITH $R_{i\bar{j}}=0$

$\Rightarrow M$ IS A CALABI-YAU MANIFOLD.

(B_{ij} MUST BE CLOSED 2-FORM)

• Chiral (Anti-Chiral) Rings?

• **(c, c)**: Central charge $= 9 \Rightarrow h \leq 9/6 = 3/2$

$$\Rightarrow Q = 2h \leq 3$$
$$(\bar{Q} \leq 3)$$

$\uparrow \bar{Q}$; $Q \rightarrow$	0	1	2	3
3				
2				
1		$*$		
0				

$*$: $Q = \bar{Q} = 1 \quad h = \bar{h} = 1/2$

Geometrically: $\rightarrow$ harmonic (2,1) forms $h_{ij,\bar{k}}\, dx^i dx^j dx^{\bar{k}}$

via $\phi \sim (h_{ij,\bar{k}} \underbrace{\Omega^{ijk}}_{(3,0)}) \psi^{\bar{k}} \underbrace{\lambda_k}_{g_{k\bar{n}}\lambda^{\bar{n}}}$

Why harmonic? # depends on M $=$ " $h^{2,1}$ "

• **(a, c)**:

$\uparrow \bar{Q}$; $Q \rightarrow$	0	1	2	3
-3				
-2				
-1		$*$		
0				

Geometrically: Harmonic (1.1) forms:
$b_{i\bar{j}}\, dx^i \wedge dx^{\bar{j}}$

$\phi \sim b_{i\bar{j}} \psi^{\bar{j}} \lambda^i$

Ring structure?

15.5
15.9
15.6

COHOMOLOGY & CHIRAL RINGS

- LET $|\phi_R\rangle$ BE RAMOND GROUND STATE: $G_0^{\pm}|\phi_R\rangle = 0$
- SPECTRAL FLOW ON $|\phi_R\rangle$ YIELDS (c,c); (a,c).

- $G^+(z) = g_{i\bar{j}}\psi^i(z)\partial X^{\bar{j}}(z)$; $G^-(z) = g_{i\bar{j}}\psi^{\bar{j}}\partial X^i(z)$

  $G_0^+ = g_{i\bar{j}}\psi^i\partial X^{\bar{j}}$; $G_0^- = g_{i\bar{j}}\psi^{\bar{j}}\partial X^i$

- SPECTRUM OF RAMOND GROUND STATE:
  - TRUNCATE TO CONSTANT (IN σ) MODES
  - BUILD UP HILBERT SPACE FROM
    - $\{\psi^i, \psi^{\bar{j}}\} = g^{i\bar{j}}$ OTHERS VANISH
    - $\{\lambda^i, \lambda^{\bar{j}}\} = g^{i\bar{j}}$ " "
    - $\partial X^j = \pi^j \longrightarrow \partial/\partial X_j \implies$

    $G_0^+ = \psi^i\partial_i$; $G_0^- = \psi^{\bar{i}}\partial_{\bar{i}}$

    - FOCK-VACUUM: $\psi^i|0\rangle = \lambda^i|0\rangle = 0$
    - GENERAL STATE: $|\Phi\rangle = \sum_{r,s} b_{i_1\cdots i_r\bar{j}_1\cdots\bar{j}_s}\lambda^{i_1}\cdots\lambda^{i_r}\psi^{\bar{j}_1}\cdots\psi^{\bar{j}_s}|0\rangle$

    - DIFFERENTIAL FORMS; BUNDLE-VALUED FORMS

    - $G_0^+|\Phi\rangle = G_0^-|\Phi\rangle \Rightarrow b \in H_{\bar{\partial}}^{0,s}(M_{CY}, \Lambda^r T^*)$
    - ALSO: $b \in H_{\bar{\partial}}^{0,s}(M_{CY}, \Lambda^r T)$

C) N=2 Landau-Ginsburg Theories

Kastor, Martinec, Shenker; Vafa, Warner

Let $\Phi(z,\bar{z},\theta^+,\theta^-,\bar{\theta}^+,\bar{\theta}^-)$ be a complex $N=2$ scalar superfield which is chiral:

$$D_+\Phi = \bar{D}_+\Phi = 0 \qquad \left(D_\pm = \partial/\partial\theta^\pm + \theta^\mp \partial/\partial z\right)$$
$$\left(\bar{D}_\pm = \partial/\partial\bar{\theta}^\pm + \bar{\theta}^\mp \partial/\partial\bar{z}\right)$$

$$S_{LG} = \int K(\Phi^i, \bar{\Phi}^i)\, d^2z\, d^4\theta + \int W(\Phi^i)\, d^2\theta^-\, d^2z + \int W(\bar{\Phi}^i)\, d^2\theta^+\, d^2z$$

$(d^2\theta^\pm = d\theta^\pm d\bar{\theta}^\pm)$

- K: e.g. $K = \sum \bar{\Phi}^i \Phi^i$
- W holomorphic polynomial
    - $V(\phi) \sim |\nabla W(\phi)|^2$ take W: $\partial W/\partial\Phi^i|_{\Phi=0} = 0$
    - $\det(\partial^2 W/\partial\Phi^i\partial\Phi^j)|_0 = 0$ (degenerate crit. pts)
    - $\partial W/\partial\Phi^i = 0 \Rightarrow \Phi^i = 0$
- Allow RG to flow theory to IR - assume $\exists$ fixed point $\Rightarrow$ superconformal theory
- Key fact: $K_{\text{fixed point}}$ in general $\neq K$
  $W_{\text{fixed point}} = W$

- A LITTLE MORE *PRECISELY*:

NONRENORMALIZATION THEOREM $\Rightarrow$

- F TERM NOT RENORMALIZED
- $W(\Phi^i)$ INVARIANT UP TO WAVE FCN RENORMALIZATION

NOW: UNDER CHANGE OF SCALE: $z \to \lambda^{-1} z$, $\theta \to \lambda^{-1/2}\theta$

$\Rightarrow$
$$\int d^2z\, d^2\theta \;\longrightarrow\; \lambda^{-1}\int d^2z\, d^2\theta$$

$\Rightarrow$ $\quad W(\Phi^i)$ MUST $\to$ $\lambda W(\Phi^i)$ UNDER

WAVE FCN RENORMALIZATION $\Phi^i \to \lambda^{\omega_i}\Phi^i$.

- ω_i = ANOMALOUS DIM OF Φ^i; $\Phi^i_{\text{SCALAR}} \Rightarrow h_i = \bar{h}_i = \omega_i/2$
- AT FIXED POINT: $W(\Phi^i)$:
$$W(\lambda^{\omega_i}\Phi^i) = \lambda W(\Phi^i) \quad \text{QUASIHOMOGENEOUS}$$

HEURISTICALLY:

LG THEORY AT IR FIXED POINT: UNIQUE CLASSICAL VACUUM STATE, DEGENERATE CRITICAL POINT, (MASSLESS FLUCTUATIONS).

D) Minimal Models

• N=0 CFT : Unitary $c<1$ theories

$$c = 1 - 6/m(m+1) \;;\; h_{p,q}(m) = \frac{[(m+1)p - mq]^2 - 1}{4m(m+1)}$$

$$1 \le p \le m-1 \;;\; 1 \le q \le p$$

• N=1, N=2 Analogous minimal models

• N=1 $\quad c = 3/2\,(1 - 8/m(m+2)) \le 3/2 \quad m=3,\ldots$

• N=2 $\quad c = 3\,(1 - 2/m+2) \le 3 \quad m=1,2,\ldots$

$$= 3m/m+2 = \underline{3p/p+2}$$

As in N=0 case, finite # superconformal primary fields.

$$h_{\ell,m} = \frac{\ell(\ell+2)}{4(p+2)} - \frac{m^2}{4(p+2)} \;,\; Q_{\ell m} = m/p+2$$

$$(\ell = 0,1,\ldots,p \;;\quad m = -\ell, -\ell+2, \ldots, \ell)$$

(NS sector)

0.3) Conformal Field Theory & String Theory

• Recall: Motivated CFT discussion by realizing (classical) conformal symmetry of string action

$$\bullet S = \frac{1}{4\pi\alpha'} \int d\sigma\, d\tau \sqrt{h}\, h^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu G^M_{\mu\nu}(x)$$

• **Key Points:**

1) For general $G^M_{\mu\nu}$ classical conformal symmetry is violated far more severly than via central charge C. (Will return). For now: take $G^M_{\mu\nu} = \eta_{\mu\nu}$ (flat).

2) The value of C in this case is # X's. (Will presently see).

3) For strings: need $C = \bar{C} = 0$

4) Ghosts for gauge fixing reparametrization invariance contribute $C_g = \bar{C}_g = -26$
$\Rightarrow$ need $\underline{26}$ X's.

5) Supersymmetric case: $C_{sg} = \bar{C}_{sg} = +11 \Rightarrow$
$C_{g\,\text{total}} = \bar{C}_{g\,\text{total}} = -15$. Thus need $\underline{10}(\underset{1}{X} + \underset{1/2}{\Psi}) \Rightarrow c = 15$.

EXAMPLE: FREE BOSON

- $S = \int \partial X \bar{\partial} X$. EOM $\Rightarrow X(z,\bar{z}) = \chi(z) + \chi(\bar{z})$.

- PROPAGATOR: $\langle \chi(z)\chi(\omega)\rangle = -\log(z-\omega)$; $\langle \chi(\bar{z})\chi(\bar{\omega})\rangle = -\log(\bar{z}-\bar{\omega})$
- OPERATOR PRODUCT: $\partial\chi(z)\cdot\partial\chi(\omega) = -\frac{1}{(z-\omega)^2} + \cdots$

- $T(\omega) = -\frac{1}{2} :\partial\chi(z)\partial\chi(\omega): \quad \Rightarrow$

a) $T(\omega)\,\partial\chi(z) = -\frac{1}{2} :\partial\chi(z)\partial\chi(z): \partial\chi(\omega)$

$$= -\frac{1}{2}\cdot 2\cdot \partial\chi(z)\langle\partial\chi(z)\partial\chi(\omega)\rangle + \cdots$$

$$= \partial\chi(z)\cdot\frac{1}{(z-\omega)^2} + \cdots$$

$$= \left(\partial\chi(\omega) + (z-\omega)\partial^2\chi(z)\right) 1/(z-\omega)^2 + \cdots$$

$$= \frac{1\cdot\partial\chi(\omega)}{(z-\omega)^2} + \frac{1}{(z-\omega)}\partial(\partial\chi(z)) + \cdots$$

b) $T(\omega)T(z) = -\frac{1}{2}:(\partial\chi(\omega)\partial\chi(\omega)): -\frac{1}{2}:(\partial\chi(z)\partial\chi(z)):$

$$= \cdots = \frac{\boxed{1}/2}{(\omega-z)^4} + \frac{2}{(\omega-z)^2}T(z) + \frac{1}{(\omega-z)}\partial T(z) + \cdots$$

$\Rightarrow C = 1$.

$\Rightarrow$ FOR d BOSONS $\Rightarrow C = d$.

More General Formulation of Strings

• **Original:** $S = \int \partial X^\mu \bar\partial X^\nu \eta^{26}_{\mu\nu}$ $\quad \mu,\nu = 1,\dots,26$

Bosonic

$$= \underline{\int \partial X^\mu \bar\partial X^\nu \eta^{4}_{\mu\nu}} + \underline{\int \partial X^{\mu'} \bar\partial X^{\nu'} \delta^{22}_{\mu'\nu'}}$$

• **General:** $S = \int \partial X^\mu \bar\partial X^\nu \eta^{4}_{\mu\nu}$ + "Internal" CFT with $c = \bar c = 22$

• **Original Superstring**

$$S = \int \partial X^\mu \bar\partial X^\nu \eta^{10}_{\mu\nu} + \psi^\mu \partial \psi^\nu \eta^{10}_{\mu\nu} + \lambda^\mu \bar\partial \lambda^\nu \eta_{\mu\nu}$$

$\mu,\nu = 1,\dots,10$

• **General:**

$$S = \int \left(\partial X^\mu \bar\partial X^\nu \eta^{4}_{\mu\nu} + \psi^\mu \partial \psi^\nu \eta^{4}_{\mu\nu} + \lambda^\mu \bar\partial \lambda^\nu \eta^{4}_{\mu\nu}\right)$$

+ "Internal" $c = \bar c = 9$ CFT

with $N = 2$ World Sheet Supersymmetry