



SMR.858 - 7

Lecture III

SUMMER SCHOOL IN HIGH ENERGY PHYSICS AND COSMOLOGY

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MONOPOLE SOLUTIONS IN SUPERSYMMETRIC THEORIES

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Please note: These are preliminary notes intended for internal distribution only.

Small g.t. $\left. \begin{matrix} g \rightarrow 1 \\ r \rightarrow \infty \end{matrix} \right\}$ identify points in $\mathcal{Q} = \{A, \Phi\}$
to give physical config space
 $\mathcal{C} = \mathcal{Q}/\mathcal{G}$

$\therefore$ physical states inv. under small g.t.

large g.t. $\left. \begin{matrix} g \rightarrow 1 \\ r \rightarrow \infty \end{matrix} \right\}$ related different points in $\mathcal{C}$,
~~for~~

physical states not inv. under large g.t.

Dyon Start in $A_0 = 0$ gauge w/ BPS monopole
 $A_i, \Phi; \quad w/ \quad B_i = D_i \Phi$

A deformation $\begin{matrix} A_i \rightarrow A_i + \delta A_i(\vec{x}, t) \\ \Phi \rightarrow \Phi + \delta \Phi(\vec{x}, t) \end{matrix}$ tangent to $\mathcal{M}$

must satisfy

- 1) Gauss Law $D_i \delta A_i + [\Phi, \delta \Phi] = 0$
- 2) linearized
Bog eqn $\epsilon_{ijk} D_j \delta A_k = D_i \delta \Phi + [\delta A_i, \Phi]$

and remain in $A_0 = 0$ gauge. Such a deformation is

$$\begin{aligned} \delta A_i &= D_i(\chi \Phi) & \chi(t) \\ \delta \Phi &= 0 \\ \delta A_0 &= 0 = D_0(\chi \Phi) - \dot{\chi} \Phi \end{aligned}$$

Note this obeys

1) using $\delta A_i = \dot{\chi} D_i \Phi = \dot{\chi} B_i$ and $D_i B_i = 0$

2) - ~~obey~~ Exercise 6

and also note

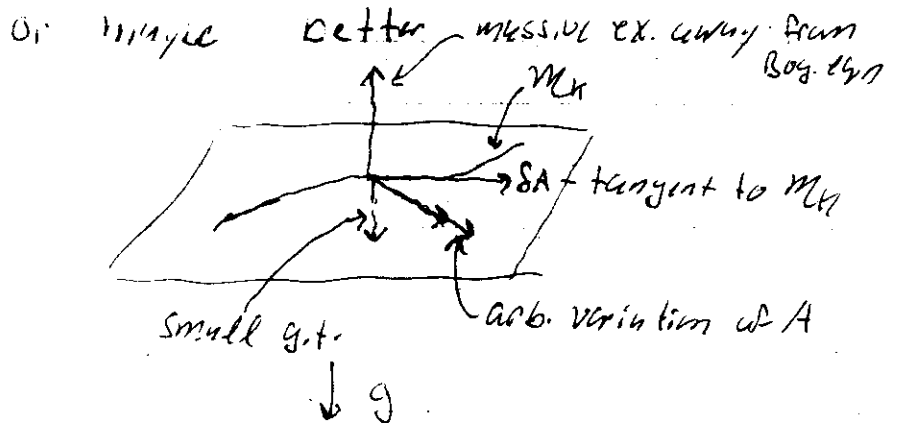
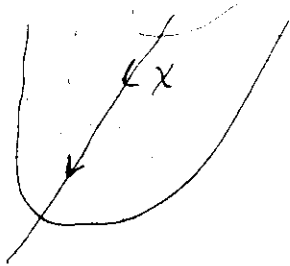
- for $\dot{\chi} = 0$ this deformation is a large g.t. by

$$g = e^{\chi \Phi} \xrightarrow{r \rightarrow \infty} e^{\chi V \hat{r}^a T^a} \neq 1$$

$\therefore \chi$ is a physical zero mode

- for $\dot{\chi} \neq 0$ it is not pure gauge, linearized Bog eqn is still satisfied so no change in potential energy ($\vec{B}^2 + (D^i \Phi)^2$) but there is an increase in kinetic energy ($\vec{E}^2 = F_{0i} F_{0i}$)

Just as we expect by physical analogy with motion down a flat valley in $\mathcal{E}$



~~We will see some of this when we discuss
 susy theories, but first let me discuss the other
 topic - the Witten effect.~~

~~Witten effect~~

This

In gauge theory there is a famous term - "θ term" which
 can be added to the Lagrangian:

$$\mathcal{L}_\theta = -\frac{\theta e^2}{32\pi^2} F_{\mu\nu}^a * F^{a\mu\nu}$$

this is a surface term and does not affect classical
 eqns of motion but effects of θ show up in instanton
 effects. Witten discovered θ also effects the electric
 charge of monopoles.

I will give 2 arguments, one from Coleman, one from
 Witten.

First consider QED, then $\mathcal{L}_\theta = \frac{\theta e^2}{8\pi^2} \vec{E} \cdot \vec{B}$

Now look at $\vec{E}, \vec{B}$ in presence of mag. monopole (Dirac say)

$$\vec{E} = -\vec{\nabla} A_0$$

$$\vec{B} = \vec{\nabla} \times \vec{A} + \frac{g}{4\pi} \frac{\vec{r}}{r^2}$$

Substituting in

$$\begin{aligned} \mathcal{L}_\theta &= \int d^3r \mathcal{L}_\theta = \int d^3r \frac{\theta e^2}{8\pi^2} \int d^3r \vec{\nabla} A_0 \cdot (\vec{\nabla} \times \vec{A} + \frac{g}{4\pi} \frac{\vec{r}}{r^2}) \\ &= -\frac{\theta e^2 g}{8\pi^2 \cdot 4\pi} \int d^3r A_0 \vec{\nabla} \cdot \frac{\vec{r}}{r^2} \\ &= -\frac{\theta e^2 g}{8\pi^2} \int d^3r A_0 \delta^3(\vec{r}) \end{aligned}$$

this is coupling of scalar potential A_0 to elec. charge of
 magnitude $-\frac{\theta e^2 g}{8\pi^2}$ located at origin.

So magnetic picks up charge

$$e \left(-\frac{eg}{4\pi} \frac{\theta}{2\pi} \right)$$

for $eg = 4\pi$, as in
SO(3) gauge theory,

$$e \left(-\theta/2\pi \right).$$

~~Answer~~

Witten's orig. argument is as follows. The op. that measures electric charge of a state Φ is generator of $U(1)$ g.t. that we constant at infinity. In Higgs theory $U(1)$ is picked out by Higgs field Φ^a . $U(1)$ g.t. are g.t. around direction of $\hat{\Phi}^a = \frac{\Phi^a}{|\Phi|}$

i.e. generator of $\delta A_\mu^a = \frac{1}{eV} (D_\mu \Phi)^a$ if we rotate by 2π must get 1, so

$$e^{2\pi i N} = 1 \quad \text{w/ } N \text{ the generator.}$$

Can compute N using Noether:

$$N = \frac{\partial \mathcal{L}}{\partial \partial_0 \hat{A}_\mu^a} \delta \hat{A}_\mu^a$$

$$\delta \hat{A}_\mu^a = \frac{1}{eV} (D_\mu \Phi)^a$$

Including G.F.P. term find

$$N = \frac{Q}{e} + \frac{6e}{8\pi^2} g \quad \text{w/ } g = \frac{1}{V} \int d^3x D\Phi^a B^a \quad \text{-mag. charge}$$

$$Q = \frac{1}{V} \int d^3x D\Phi^a E^a \quad \text{-elec. charge.}$$

so have statement $e^{2\pi i N} = 1$

$$\Rightarrow \frac{Q}{e} + \frac{6eg}{8\pi^2} = \text{integer } N \quad \text{and with } eg = 4\pi$$

$$Q = ne - \frac{e\theta}{2\pi}$$

i.e. allowed electric charges shifted by $6/\pi$ and can be fractional.

Ex. 7 a) Carry out computation of N using Noether method

b) Verify that having monopole w/ elec charge $Q = ne - e\theta/4\pi$ satisfies DSE condition and that $\mathcal{L}_0$ is not CP invariant.

Now we have an explicit monopole sol'n w/ calculable mass, spin, etc.

What about duality which was original motivation? We have in BPS limit recall

	Mass	(Q_e, Q_m)	Spin
Higgs	0	(0, 0)	0
photon	0	(0, 0)	1
$W^\pm$	ve	(e, 0)	1
M	vg	(0, g)	?

This is classical result, but at classical level all these states obey Bog. bound

$$M = \cancel{\sqrt{Q_e^2 + Q_m^2}} \sim \sqrt{Q_e^2 + Q_m^2}$$

Now at weak coupling where we can believe semi-classical analysis

$$M_W = ev \ll v$$

$$M_M = gv = \frac{4\pi}{e} v \gg v$$

so although we have electric + magnetic charges, monopoles are much heavier than W 's

However if we enlarge our view of duality we
 can say that duality should take

$$\vec{E} \rightarrow \vec{B}, \quad \vec{B} \rightarrow -\vec{E}$$

and $e \leftrightarrow g \quad (e = 4\pi/g)$

or $e \rightarrow 4\pi/g$ leaves spectrum inv. then

Based on classical spectrum and some other arguments Montonen-Olive proposed above as exact duality of $SO(3)$ gauge theory. But when proposed there were many problems.

1. quantum corr. will spoil $V(\Phi) = 0$ and must formulate
2. No reason to think M has spin 1 like W does
3. Hard to test since need to calculate at strong coupling to compare theory and its dual.

1, 2, and part of 3 solved by embedding in $N=4$ SUSY (or $N=2$ in special cases).

From Witten effect we can see a beautiful extension of duality.

Including θ term the Lagrangian in BPS limit is determined by 2 real parameters, θ, e .

$$\mathcal{L} = -\frac{1}{4} F^2 - \frac{\theta e^2}{32\pi^2} F^* F - \frac{1}{2} D_\mu \Phi D^\mu \Phi \quad F^2 = F_{\mu\nu}^a F^{a\mu\nu} \text{ etc}$$

Now remembering $** = -1$ in Minkowski space,
note

$$(F + i^* F)^2 = 2F^2 + 2i F^* F \quad \text{so}$$

$$\text{Im} \left(\frac{\theta}{2\pi} + \frac{4\pi i}{e^2} \right) (F + i^* F)^2 = \frac{\theta}{\pi} F^* F + \frac{8\pi^2}{e^2} F^2$$

$$\text{and} \quad \mathcal{L} = \cancel{\frac{1}{4} F^2} - \frac{1}{32\pi} \text{Im} \left(\frac{\theta}{2\pi} + \frac{4\pi i}{e^2} \right) (F + i^* F)^2 - \frac{1}{2} D_\mu \Phi D^\mu \Phi$$

so physics depends only on combination $\tau \equiv \frac{\theta}{2\pi} + \frac{4\pi i}{e^2}$

Now physics is periodic in θ w/ period 2π

$\theta \rightarrow \theta + 2\pi$ is $\tau \rightarrow \tau + 1$ should have physics inv.

and at $\theta = 0$, duality is

$$e \rightarrow \frac{4\pi}{e} \quad \text{or} \quad \frac{4\pi}{e^2} \rightarrow \frac{4\pi}{(4\pi/e)^2} = \frac{e^2}{4\pi}$$

which is $\tau \rightarrow -1/\tau$.

This suggests that at arbitrary θ if duality is correct the full duality group should be the one generated by

$$\tau \rightarrow \tau + 1$$

$$\tau \rightarrow -1/\tau$$

It is a well known fact that this group is $SL(2, \mathbb{Z})$ consisting of transformations

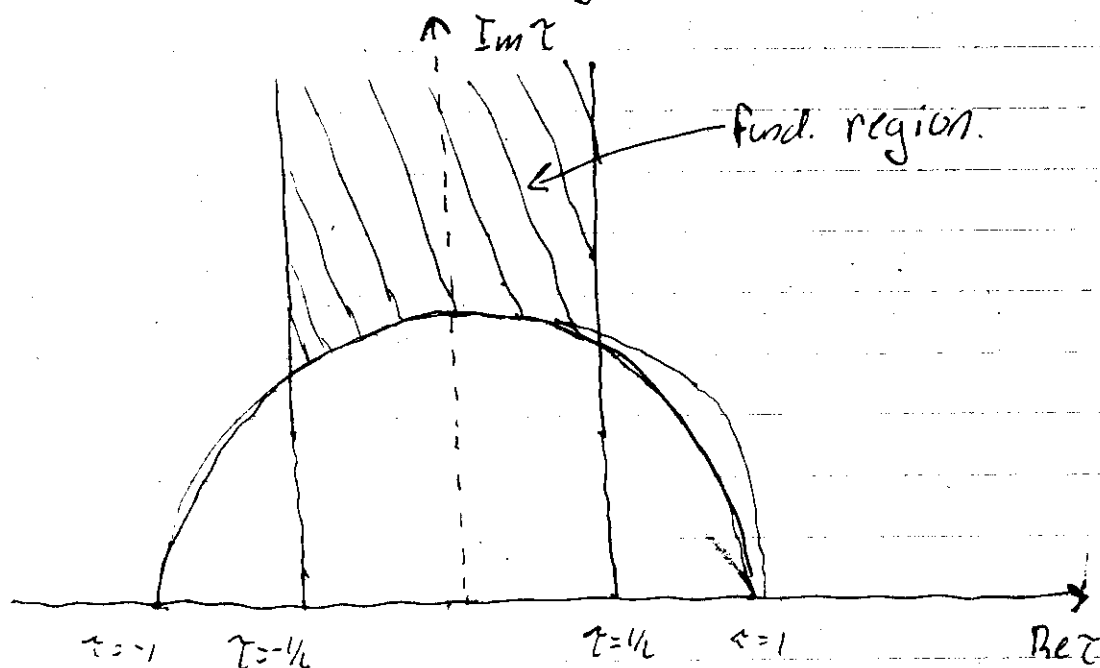
$$\tau \rightarrow \frac{a\tau + b}{c\tau + d} \quad \begin{matrix} a, b, c, d \in \mathbb{Z} \\ ad - bc = 1 \end{matrix}$$

i.e. $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})$.

~~Ex. 8~~

Note since $e^2 > 0$ τ naturally lies on the upper half plane $\text{Im } \tau > 0$.

Note hard to show that by $SL(2, \mathbb{Z})$ trans. can always map τ to fundamental region shown below



Ex. 8 Find 2 points on bndy of fund. region which are left fixed by $SL(2, \mathbb{Z})$ elements of order 2, or order 3. What values of θ, e^2 do these correspond to?

(28)

Abstract Finally, consider spectrum of states obeying BPS bound.

$$M = v \sqrt{Q_e^2 + Q_m^2} \quad \text{or} \quad M^2 = v^2 (Q_e^2 + Q_m^2)$$

We know allowed values of Q_e, Q_m are

$$Q_m = \frac{4\pi}{e} n_m$$

$$Q_e = n_e e - e n_m \theta / 2\pi$$

by writing in terms of τ we find

$$M^2 = 4\pi v^2 (n_e, n_m) \frac{1}{\text{Im}\tau} \begin{pmatrix} 1 & -\text{Re}\tau \\ -\text{Re}\tau & |\tau|^2 \end{pmatrix} \begin{pmatrix} n_e \\ n_m \end{pmatrix}$$

Ex. 9 Show that M^2 is left-inv. by the $SL(2, \mathbb{R})$ transformation

$$\tau \rightarrow \frac{a\tau + b}{c\tau + d}$$

$$\begin{pmatrix} n_e \\ n_m \end{pmatrix} \rightarrow \begin{pmatrix} a & -b \\ -c & d \end{pmatrix} \begin{pmatrix} n_e \\ n_m \end{pmatrix} \quad (\text{check signs!})$$

To make sense of duality in true quantum theory we have to embed this in a susy theory so that $V(\Phi) = 0$ enforced by susy, monopoles can have spin one etc.

This involves adding in coupling to fermion fields. First lets discuss this w/o worrying about susy.

We will want to add to $\mathcal{L}$ terms involving fermion fields ψ . Need to choose

- rep. of $SU(2)$
- will consider both 2 fundamental
3 adjoint

~~- 1/2, 3/2, 5/2, 7/2, 9/2, 11/2, 13/2, 15/2, 17/2, 19/2, 21/2, 23/2, 25/2, 27/2, 29/2, 31/2, 33/2, 35/2, 37/2, 39/2, 41/2, 43/2, 45/2, 47/2, 49/2, 51/2, 53/2, 55/2, 57/2, 59/2, 61/2, 63/2, 65/2, 67/2, 69/2, 71/2, 73/2, 75/2, 77/2, 79/2, 81/2, 83/2, 85/2, 87/2, 89/2, 91/2, 93/2, 95/2, 97/2, 99/2, 101/2, 103/2, 105/2, 107/2, 109/2, 111/2, 113/2, 115/2, 117/2, 119/2, 121/2, 123/2, 125/2, 127/2, 129/2, 131/2, 133/2, 135/2, 137/2, 139/2, 141/2, 143/2, 145/2, 147/2, 149/2, 151/2, 153/2, 155/2, 157/2, 159/2, 161/2, 163/2, 165/2, 167/2, 169/2, 171/2, 173/2, 175/2, 177/2, 179/2, 181/2, 183/2, 185/2, 187/2, 189/2, 191/2, 193/2, 195/2, 197/2, 199/2, 201/2, 203/2, 205/2, 207/2, 209/2, 211/2, 213/2, 215/2, 217/2, 219/2, 221/2, 223/2, 225/2, 227/2, 229/2, 231/2, 233/2, 235/2, 237/2, 239/2, 241/2, 243/2, 245/2, 247/2, 249/2, 251/2, 253/2, 255/2, 257/2, 259/2, 261/2, 263/2, 265/2, 267/2, 269/2, 271/2, 273/2, 275/2, 277/2, 279/2, 281/2, 283/2, 285/2, 287/2, 289/2, 291/2, 293/2, 295/2, 297/2, 299/2, 301/2, 303/2, 305/2, 307/2, 309/2, 311/2, 313/2, 315/2, 317/2, 319/2, 321/2, 323/2, 325/2, 327/2, 329/2, 331/2, 333/2, 335/2, 337/2, 339/2, 341/2, 343/2, 345/2, 347/2, 349/2, 351/2, 353/2, 355/2, 357/2, 359/2, 361/2, 363/2, 365/2, 367/2, 369/2, 371/2, 373/2, 375/2, 377/2, 379/2, 381/2, 383/2, 385/2, 387/2, 389/2, 391/2, 393/2, 395/2, 397/2, 399/2, 401/2, 403/2, 405/2, 407/2, 409/2, 411/2, 413/2, 415/2, 417/2, 419/2, 421/2, 423/2, 425/2, 427/2, 429/2, 431/2, 433/2, 435/2, 437/2, 439/2, 441/2, 443/2, 445/2, 447/2, 449/2, 451/2, 453/2, 455/2, 457/2, 459/2, 461/2, 463/2, 465/2, 467/2, 469/2, 471/2, 473/2, 475/2, 477/2, 479/2, 481/2, 483/2, 485/2, 487/2, 489/2, 491/2, 493/2, 495/2, 497/2, 499/2, 501/2, 503/2, 505/2, 507/2, 509/2, 511/2, 513/2, 515/2, 517/2, 519/2, 521/2, 523/2, 525/2, 527/2, 529/2, 531/2, 533/2, 535/2, 537/2, 539/2, 541/2, 543/2, 545/2, 547/2, 549/2, 551/2, 553/2, 555/2, 557/2, 559/2, 561/2, 563/2, 565/2, 567/2, 569/2, 571/2, 573/2, 575/2, 577/2, 579/2, 581/2, 583/2, 585/2, 587/2, 589/2, 591/2, 593/2, 595/2, 597/2, 599/2, 601/2, 603/2, 605/2, 607/2, 609/2, 611/2, 613/2, 615/2, 617/2, 619/2, 621/2, 623/2, 625/2, 627/2, 629/2, 631/2, 633/2, 635/2, 637/2, 639/2, 641/2, 643/2, 645/2, 647/2, 649/2, 651/2, 653/2, 655/2, 657/2, 659/2, 661/2, 663/2, 665/2, 667/2, 669/2, 671/2, 673/2, 675/2, 677/2, 679/2, 681/2, 683/2, 685/2, 687/2, 689/2, 691/2, 693/2, 695/2, 697/2, 699/2, 701/2, 703/2, 705/2, 707/2, 709/2, 711/2, 713/2, 715/2, 717/2, 719/2, 721/2, 723/2, 725/2, 727/2, 729/2, 731/2, 733/2, 735/2, 737/2, 739/2, 741/2, 743/2, 745/2, 747/2, 749/2, 751/2, 753/2, 755/2, 757/2, 759/2, 761/2, 763/2, 765/2, 767/2, 769/2, 771/2, 773/2, 775/2, 777/2, 779/2, 781/2, 783/2, 785/2, 787/2, 789/2, 791/2, 793/2, 795/2, 797/2, 799/2, 801/2, 803/2, 805/2, 807/2, 809/2, 811/2, 813/2, 815/2, 817/2, 819/2, 821/2, 823/2, 825/2, 827/2, 829/2, 831/2, 833/2, 835/2, 837/2, 839/2, 841/2, 843/2, 845/2, 847/2, 849/2, 851/2, 853/2, 855/2, 857/2, 859/2, 861/2, 863/2, 865/2, 867/2, 869/2, 871/2, 873/2, 875/2, 877/2, 879/2, 881/2, 883/2, 885/2, 887/2, 889/2, 891/2, 893/2, 895/2, 897/2, 899/2, 901/2, 903/2, 905/2, 907/2, 909/2, 911/2, 913/2, 915/2, 917/2, 919/2, 921/2, 923/2, 925/2, 927/2, 929/2, 931/2, 933/2, 935/2, 937/2, 939/2, 941/2, 943/2, 945/2, 947/2, 949/2, 951/2, 953/2, 955/2, 957/2, 959/2, 961/2, 963/2, 965/2, 967/2, 969/2, 971/2, 973/2, 975/2, 977/2, 979/2, 981/2, 983/2, 985/2, 987/2, 989/2, 991/2, 993/2, 995/2, 997/2, 999/2~~

First consider a Dirac fermion ψ , we can couple this to previous fields by adding to $\mathcal{L}$

$$\mathcal{L}_\psi = i \bar{\psi}_n \gamma^\mu (D_\mu \psi)_n - i \bar{\psi}_n T_{nm}^a \Phi^a \psi_m$$

here $(T^a)_{nm}$ is anti-herm. generator of $SU(2)$ in rep. carried by ψ , $n = 1 \dots \dim \mathcal{C}$ for ψ in rep $\mathcal{C}$

Since we will consider only 2, 3 we either ~~or~~ take

$$2 \quad n, m = 1, 2 \quad T_{nm}^a = -\frac{i}{2} \tau_{nm}^a$$

$$3 \quad n, m = 1, 2, 3 \quad T_{nm}^a = \epsilon^{a nm}$$

~~Not a monopole~~

~~monopole~~

We now analyze this in a monopole bkgnd. ~~With~~ We will first proceed concretely, then mention general results.

Concretely, consider a charge 1 monopole with

$$\Phi^a = \frac{\vec{r}^a}{r} H(r)$$

$$A_i^a = -\epsilon^{aij} \frac{\vec{r}^j}{r} (1 - K(r))$$

When H, K have form in previous lecture this is BPS soln w/ $V(\Phi) \equiv 0$, but even if $V(\Phi) \neq 0$ we can find a soln of this form with same b.c.

It will be convenient to use Dirac rep. for γ matrices with

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \gamma^i = \begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix} \quad \alpha^i = \gamma^0 \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix}$$

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} = 2\text{diag}(-1, 1, 1, 1)$$

Then writing the Dirac eqn is

$$(i\gamma^\mu D_\mu - f\Phi^a) \psi = 0 \quad \text{w/ group indices suppressed.}$$

Using $A_0 = 0$, writing $\psi = e^{iEt} \psi(\vec{x})$ we have

$$\cancel{(i\alpha^i D_i + \gamma^0 f\Phi)} (i\alpha^i D_i + \gamma^0 f\Phi) \psi = E\psi$$

and writing $\psi = \begin{pmatrix} \chi^+ \\ \chi^- \end{pmatrix}$ this gives

$$(i\sigma^i D_i + f\Phi)X^- = EX^+$$

$$(i\sigma^i D_i - f\Phi)X^+ = EX^-$$

Now when we quantize

Now if we know spectrum of Dirac eq. in monopole
bgnd we can quantize by writing

$$\psi = \underbrace{\sum a_0 \psi_0}_{\text{zero energy}} + \underbrace{\sum_{n>0} b_n \psi_n^+ + d_n^+ \psi_n^-}_{\text{non-zero energy}}$$

and treating coeff. as creation, ann. op's. Only modes
assoc. w/ ψ_0 will ~~exist~~ ~~remain~~ have monopole
energy unchanged. They are "fermionic collective
coordinates"

So to study monopole gnd. state we are interested
in zero-energy soln's.

Let $D = i\sigma^i D_i - f\Phi$

$$D^+ = i\sigma^i D_i + f\Phi \quad (\text{since } \Phi \text{ anti-hermitian})$$

We then want soln's of

$$DX^+ = 0$$

$$D^+X^- = 0$$

i.e. want $\text{Ker } D$, $\text{Ker } D^+$

Ex. 810 Show that $D D^+ = (i\vec{D}) \cdot (i\vec{D}) + (i\vec{E})(i\vec{E})$ is ≥ 0 (32)
pos. def. op.

hint: use Jac eq'n in form $[D_1, D_2] = [D_3, \Phi]$
+ cyclic

conclude that

$$\ker D^+ \subset \ker D D^+ = \{0\}$$

Now there is an index Thm due to Callias (CMP 62 (1978) 213)
which computes

$$\text{Index of Dirac op} \equiv \dim \ker D - \dim \ker D^+ = K$$

in a K -monopole background.

$\therefore$ We expect to find a unique z.m. of D for $K=1$.

This is not hard to do using the explicit $K=1$
~~Dirac~~ PS sol'n