INTERNATIONAL ATOMIC ENERGY AGENCY
UNITED NATIONS EDUCATIONAL, SCIENTIFIC AND CULTURAL ORGANIZATION
INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS
I.C.T.P., P.O. BOX 586, 34100 TRIESTE, ITALY, CABLE: CENTRATOM TRIESTE

SMR.858 - 7

Lecture III

SUMMER SCHOOL IN HIGH ENERGY PHYSICS AND COSMOLOGY

12 June - 28 July 1995

MONOPOLE SOLUTIONS IN SUPERSYMMETRIC THEORIES

J. HARVEY
Enrico Fermi Institute
University of Chicago
Chicago, Il 60637
USA

Please note: These are preliminary notes intended for internal distribution only.

MAIN BUILDING STRADA COSTIERA, 11 TEL. 2240111 TELEFAX 224163 TELEX 460392 ADRIATICO GUEST HOUSE VIA GRIGNANO, 9 TEL. 224241 TELEFAX 224531 TELEX 460449
MICROPROCESSOR LAB. VIA BEIRUT, 31 TEL. 224471 TELEFAX 224600 TELEX 460392 GALILEO GUEST HOUSE VIA BEIRUT, 7 TEL. 22401 TELEFAX 2240310 TELEX 460392

Small g.t. $\left. g \xrightarrow[r\to\infty]{} 1 \right\}$ identify points in $\mathcal{A} = \{A, \Phi\}$ to give physical config space
$\mathcal{C} = \mathcal{A}/\mathcal{G}$

$\therefore$ physical states inv. under small g.t.

large g.t. $\left. g \not\xrightarrow[r\to\infty]{} 1 \right\}$ related different points in $\mathcal{C}$, ~~pt~~

physical states not inv. under large g.t.

Dyon Start in $A_0 = 0$ gauge w/ BPS monopole
A_{0i}, Φ_i w/ $B_i = D_i \Phi$

A deformation
$$A_i \to A_i + \delta A_i(\vec{x},t) \quad \text{tangent to } \mathcal{M}_1$$
$$\Phi \to \Phi + \delta\Phi(\vec{x},t)$$

must satisfy
1) Gauss law $D_i \delta\dot{A}_i + [\Phi, \delta\dot{\Phi}] = 0$
2) linearized Bog eqn $\delta B_i = \delta D_i \Phi$ or
$$\epsilon_{ijk} D_j \delta A_k = D_i \delta\Phi + [\delta A_i, \Phi]$$

and remain in $A_0 = 0$ gauge. Such a deformation is
$$\delta A_i = D_i(\chi\Phi) \qquad \chi(t)$$
$$\delta\Phi = 0$$
$$\delta A_0 = 0 = D_0(\chi\Phi) - \dot{\chi}\Phi$$

Note this obeys

1) using $\delta \dot{A}_i = \dot{\chi} D_i \Phi = \dot{\chi} B_i$ and $D_i B_i = 0$

2) ~~obtain~~ Exercise 6

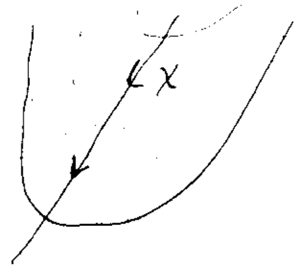
and also note

- for $\dot{\chi} = 0$ this deformation is a <u>large</u> g.t. by
$$g = e^{\chi \Phi} \underset{r \to \infty}{\longrightarrow} e^{\chi v \hat{r}^a T^a} \neq 1$$

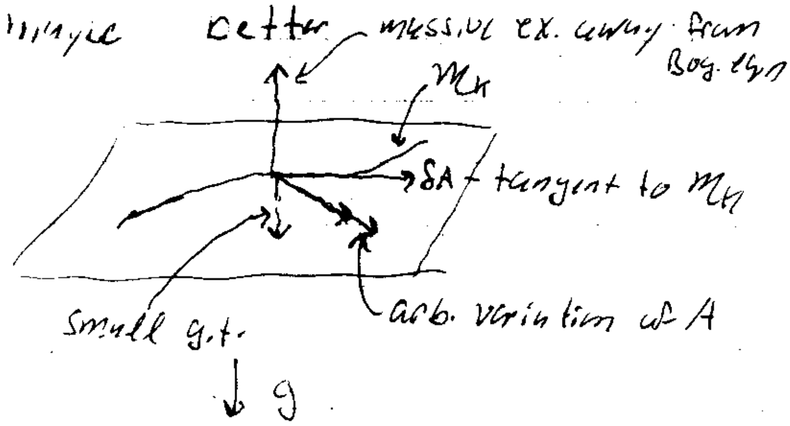
$\therefore$ χ is a physical zero mode

- for $\dot{\chi} \neq 0$ it is not pure gauge, linearized Bog eqn is still satisfied so no change in <u>potential</u> energy ($\vec{B}^2 + (D^i \Phi)^2$) but there is an increase in kinetic energy ($\vec{E}^2 = F_{0i} F_{0i}$)

Just as we expect by physical analogy with motion down a flat valley in $\mathcal{C}$



Or imagine better



~~We will lose some of this when we discuss susy theories, but first let me discuss one other topic - Witten effect.~~

~~This~~

In gauge theory there is a famous term - "Θ term" which can be added to the Lagrangian:

$$\mathcal{L}_\Theta = -\frac{\Theta e^2}{32\pi^2} F^a_{\mu\nu} {}^*F^{a\mu\nu}$$

this is a surface term and does not affect classical eqn's of motion but effects of Θ show up in instanton effects. Witten discovered Θ also effects the electric charge of monopoles.

I will give 2 arguments, one from Coleman, one from Witten.

First consider QED. then $\mathcal{L}_\Theta = \frac{\Theta e^2}{8\pi^2} \vec{E}\cdot\vec{B}$

now look at $\vec{E},\vec{B}$ in presence of mag. monopole (Dirac say)

$$\vec{E} = \vec{\nabla} A_0$$
$$\vec{B} = \vec{\nabla}\times\vec{A} + \frac{g}{4\pi}\frac{\vec{r}}{r^2}$$

Substituting in

$$L_\Theta = \int d^3\vec{r}\, \mathcal{L}_\Theta = \frac{\Theta e^2}{8\pi^2}\int d^3r\, \vec{\nabla}A_0\cdot\left(\vec{\nabla}\times\vec{A} + \frac{g}{4\pi}\frac{\vec{r}}{r^2}\right)$$

$$= -\frac{\Theta e^2 g}{8\pi^2\cdot 4\pi}\int d^3r\, A_0\, \vec{\nabla}\cdot\frac{\vec{r}}{r^2}$$

$$= -\frac{\Theta e^2 g}{8\pi^2}\int d^3r\, A_0\, \delta^3(\vec{r})$$

this is coupling of scalar potential A_0 to elec. charge of magnitude $-\frac{\Theta e^2 g}{8\pi^2}$ located at origin

So monopole picks up charge

$$e\left(-\frac{eg}{4\pi}\frac{\theta}{2\pi}\right)$$ for $eg = 4\pi$, as in $SO(3)$ gauge theory,

$$e\,(-\theta/2\pi).$$

~~Axmuld~~

Witten's orig. argument is as follows. The op. that measures electric charge of a state is generator of $U(1)$ g.t. that are constant at infinity. In Higgs theory $U(1)$ is picked out by Higgs field Φ^a. $U(1)$ g.t. are g.t. around direction of $\hat{\Phi}^a = \frac{\Phi^a}{|\Phi|}$

i.e. generator of $\delta A_\mu^a = \frac{1}{ev}(D_\mu \Phi)^a$ if we rotate by 2π must get 1, so

$$e^{2\pi i N} = 1 \quad \text{w/ } N \text{ the generator.}$$

Can compute N using Noether:

$$N = \frac{\partial \mathcal{L}}{\partial \partial_0 \vec{A}_\mu^a}\,\delta \vec{A}_\mu^a \qquad \delta \vec{A}_\mu^a = \frac{1}{ev}(D_\mu \Phi)^a$$

Including $\theta F\tilde{F}$ term find

$$N = \frac{Q}{e} + \frac{\theta e}{8\pi^2}\, g \quad \text{w/ } g = \frac{1}{v}\int d^3x\, D\Phi^a B^a \text{ - mag. charge}$$

$$Q = \frac{1}{v}\int d^3x\, D\Phi^a E^a \text{ - elec. charge.}$$

So have statement $e^{2\pi i N} = 1$

$$\Rightarrow \frac{Q}{e} + \frac{\theta e g}{8\pi^2} = \text{integer } n \quad \text{and with } eg = 4\pi$$

$$Q = ne - \frac{e\theta}{2\pi}$$ i.e. allowed electric charges shifted by θ/π and can be fractional.

Ex. 7 a) Carry out computation of N using Noether method

b) Verify that having monopole w/ elec. charge $Q = ne - e\theta/2\pi$ satisfies DSZ condition and that $\mathcal{L}_\theta$ is not CP invariant.

Now we have an explicit monopole sol'n w/ calculable mass, spin, etc.

What about duality which was original motivation? We have in BPS limit recall)

	Mass	(Q_e, Q_m)	Spin
Higgs	0	$(0,0)$	0
photon	0	$(0,0)$	1
$W^\pm$	ve	$(e,0)$	1
M	vg	$(0,g)$	?

This is classical result, but at classical level all these states obey Bog. bound

$$M = \cancel{v\sqrt{q^2+g^2}}\; v\sqrt{Q_e^2 + Q_m^2}$$

Now at weak coupling where we can believe semi-classical analysis

$$M_W = ev \ll v$$
$$M_M = gv = \frac{4\pi}{e} v \gg v$$

so although we have electric + magnetic charges, monopoles are much heavier than W's

However if we enlarge our view of duality we can say that duality should take

$$\vec{E} \to \vec{B}\ , \quad \vec{B} \to -\vec{E}$$

and $\quad e \leftrightarrow g \;\; (= 4\pi/e)$

or $\quad e \to 4\pi/e \qquad$ leaves spectrum inv. then

Based on classical spectrum and some other arguments Montonen - Olive proposed above as exact duality of SO(3) gauge theory. But when proposed there were many problems

1. quantum corr. will spoil $V(\Phi) = 0$ and must formulae
2. No reason to think M has spin 1 like W does
3. Hard to test since need to calculate at strong coupling to compare theory and its dual.

1, 2, and part of 3 solved by embedding in $N = 4$ SUSY (or $N = 2$ in special cases).

From Witten effect we can see a beautiful extension of duality.

Including θ term the Lagrangian in BPS limit is determined by 2 real parameters, θ, e.

$$\mathcal{L} = -\frac{1}{4}F^2 - \frac{\theta e^2}{32\pi^2} F{}^*F - \frac{1}{2} D_\mu \Phi D^\mu \Phi \qquad F^2 = F^a_{\mu\nu} F^{a\mu\nu} \text{ etc}$$

Now remembering $** = -1$ in Minkowski space, note

$$(F + i{}^*F)^2 = 2F^2 + 2i F{}^*F \quad \text{so}$$

$$\operatorname{Im}\left(\frac{\theta}{2\pi} + \frac{4\pi i}{e^2}\right)(F + i{}^*F)^2 = \frac{\theta}{\pi} F{}^*F + \frac{8\pi^2}{e^2} F^2$$

and

$$\mathcal{L} = \cancel{-\frac{1}{16}} - \frac{1}{32\pi} \operatorname{Im}\left(\frac{\theta}{2\pi} + \frac{4\pi i}{e^2}\right)(F + i{}^*F)^2 - \frac{1}{2} D_\mu \Phi D^\mu \Phi$$

so physics depends only on combination $\tau \equiv \frac{\theta}{2\pi} + \frac{4\pi i}{e^2}$

Now physics is periodic in θ w/ period 2π

$\theta \to \theta + 2\pi$ is $\tau \to \tau + 1$ should leave physics inv.

and at $\theta = 0$, duality is

$$e \to \frac{4\pi}{e} \quad \text{or} \quad \frac{4\pi}{e^2} \to \frac{4\pi}{(4\pi/e)^2} = \frac{e^2}{4\pi}$$

which is $\tau \to -1/\tau$.

This suggests that at arbitrary θ if duality is correct the full duality group should be the one generated by

$$\tau \to \tau + 1$$
$$\tau \to -1/\tau.$$

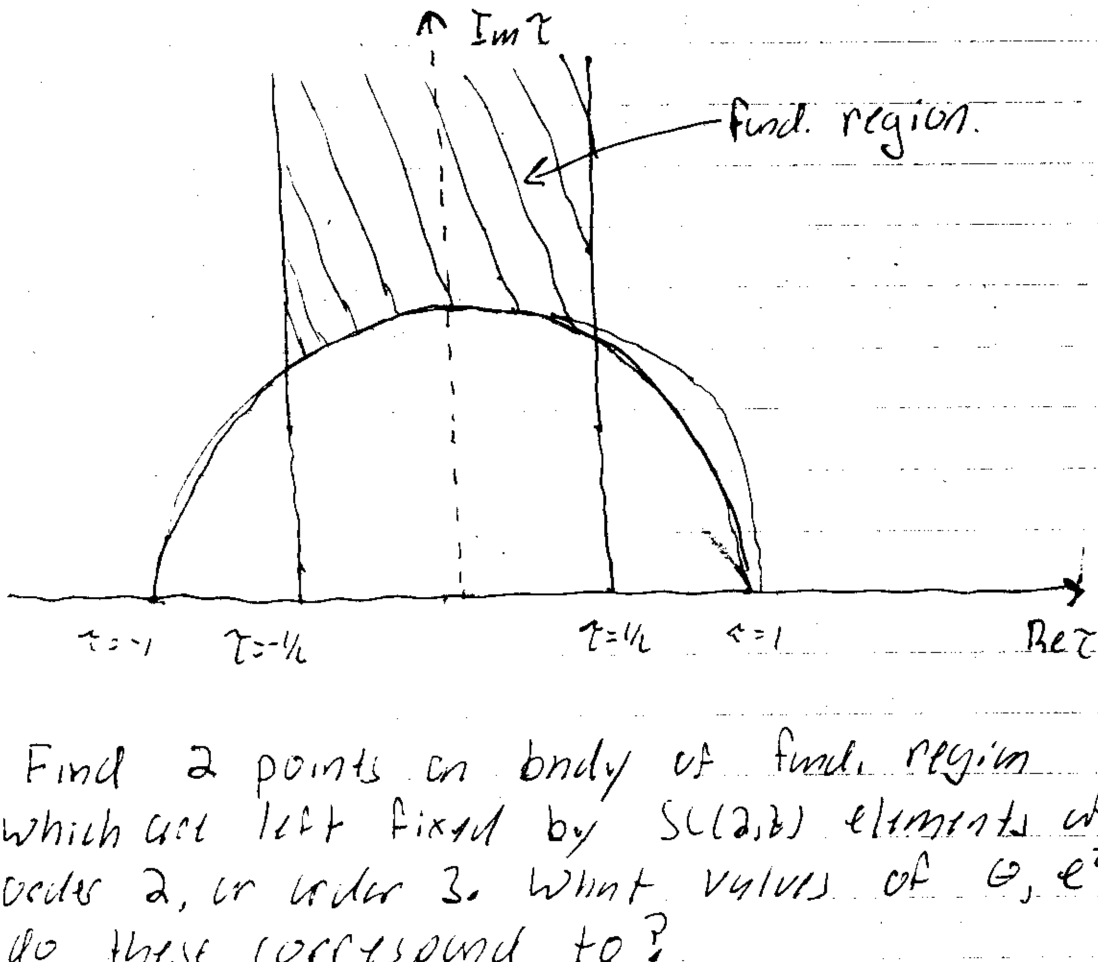
It is a well known fact that this group is $SL(2,\mathbb{Z})$ consisting of transformations

$$\tau \to \frac{a\tau + b}{c\tau + d} \qquad \begin{array}{l} a,b,c,d \in \mathbb{Z} \\ ad - bc = 1 \end{array}$$

i.e. $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$.

Note since $e^2 > 0$ τ naturally lives on the upper half plane $\mathrm{Im}\,\tau > 0$.

Note hard to show that by $SL(2,\mathbb{Z})$ transf. can always map τ to fundamental region shown below



Ex. 8 Find 2 points on bndy of fund. region which are left fixed by $SL(2,\mathbb{Z})$ elements of order 2, or order 3. What values of θ, e^2 do these correspond to?

Finally consider spectrum of states obeying BPS bound.

$$M = v\sqrt{Q_e^2 + Q_m^2} \quad \text{or} \quad M^2 = v^2 (Q_e^2 + Q_m^2)$$

We know allowed values of Q_e, Q_m are

$$Q_m = \frac{4\pi}{e} n_m$$

$$Q_e = n_e e - e n_m \theta / 2\pi$$

by writing in terms of τ we find

$$M^2 = 4\pi v^2 (n_e, n_m) \frac{1}{\operatorname{Im}\tau} \begin{pmatrix} 1 & -\operatorname{Re}\tau \\ -\operatorname{Re}\tau & |\tau|^2 \end{pmatrix} \begin{pmatrix} n_e \\ n_m \end{pmatrix}$$

Ex. 9 Show that M^2 is left-inv. by the $SL(2,\mathbb{Z})$ transformation

$$\tau \to \frac{a\tau + b}{c\tau + d}$$

$$\begin{pmatrix} n_e \\ n_m \end{pmatrix} \to \begin{pmatrix} a & -b \\ -c & d \end{pmatrix} \begin{pmatrix} n_e \\ n_m \end{pmatrix} \qquad \text{(check signs!)}$$

To make sense of duality in true quantum theory we have to embed this in a SUSY theory so that $V(\Phi) = 0$ enforced by susy, monopoles can have spin one etc.

This involves adding in coupling to fermion fields. First lets discuss this w/o worrying about susy.

We will want to add to $\mathcal{L}$ terms involving fermion fields Ψ. Need to choose
- rep. of $SU(2)$ - will consider both $\underset{\sim}{2}$ fundamental
  $\underset{\sim}{3}$ adjoint

~~- (Weyl, Majorana / or Dirac)~~

First consider a Dirac fermion Ψ, we can couple this to previous fields by adding to $\mathcal{L}$

$$\mathcal{L}_\Psi = i\bar{\Psi}_n \gamma^\mu (D_\mu \Psi)_n - if\, \bar{\Psi}_n T^a_{nm} \Phi^a \Psi_m$$

here $(T^a)_{nm}$ is anti-herm. generator of $SU(2)$ in rep. carried by Ψ, $n = 1 \ldots \dim R$ for Ψ in rep R. Since we will consider only $\underset{\sim}{2}$, $\underset{\sim}{3}$ we either ~~can~~ take

$$\underset{\sim}{2} \quad n,m = 1,2 \qquad T^a_{nm} = -\tfrac{i}{2}\tau^a_{nm}$$

$$\underset{\sim}{3} \quad n,m = 1,2,3 \qquad T^a_{nm} = \epsilon^a{}_{nm}$$

We now analyze this in a monopole bkgnd. We will first proceed concretely, then mention general results.

Concretely, consider a charge 1 monopole with

$$\Phi^a = \frac{\hat{r}^a}{er} H(ver)$$

$$A_i^a = -\epsilon^a{}_{ij} \frac{\hat{r}^j}{er} (1 - K(ver))$$

When H, K have form in previous lecture this is BPS soln w/ $V(\Phi) \equiv 0$, but even if $V(\Phi) \neq 0$ we can find a soln of this form with same b.c.

It will be convenient to use Dirac rep. for γ matrices with

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \gamma^i = \begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix} \qquad \alpha^i = \gamma^0\gamma^i = \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix}$$

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} = 2\,\mathrm{diag}(-1,1,1,1)$$

Then writing the Dirac eqn is

$$(i\gamma^\mu D_\mu - f\Phi)\psi = 0 \qquad \text{w/ group indices suppressed.}$$

Using $A_0 = 0$, writing $\psi = e^{iEt}\psi(\vec{x})$ we have

$$(i\alpha^i D_i + \gamma^0 f \Phi)\psi = E\psi$$

and writing $\psi = \begin{pmatrix} \chi^+ \\ \chi^- \end{pmatrix}$ this gives

$$(i\sigma^i D_i + f\Phi)\chi^- = E\chi^+$$

$$(i\sigma^i D_i - f\Phi)\chi^+ = E\chi^-$$

~~Now within we quantize~~
Now if we know spectrum of Dirac op. in monopole bkgnd we can quantize by writing

$$\psi = \underbrace{\sum a_0 \psi_0}_{\text{zero energy}} + \underbrace{\sum_{n>0} b_n \psi_n^+ + d_n^+ \psi_n^-}_{\text{non-zero energy}}$$

and treating coeff. as creation, ann. op's. Only modes assoc. w/ ψ_0 will ~~give solns that render~~ leave monopole energy unchanged. They are "fermionic collective coordinates"

So to study monopole gnd. state we are interested in zero-energy soln's.

Let $\quad \not{D} = i\sigma^i D_i - f\Phi$

$\quad \not{D}^+ = i\sigma^i D_i + f\Phi \quad$ (since Φ anti-hermitian)

We then want soln's of $\quad \not{D}\chi^+ = 0$

$\quad \not{D}^+\chi^- = 0$

i.e. want $\ker \not{D}$, $\ker \not{D}^+$

Ex. ~~8~~10 Show that $\not{D}\not{D}^{+} = (i\vec{D})\cdot(i\vec{D}) + (i\Phi)(i\Phi) \geq 0$

pos. def. op.

hint: use Bog eqn in form $[D_1, D_2] = [D_3, \Phi]$

+ cyclic

Conclude that

$\ker \not{D}^{+} \subset \ker \not{D}\not{D}^{+} = \{0\}$

Now there is an index Thm due to Callias (CMP 62 (1978) 213) which computes

$$\text{index of Dirac op} \equiv \dim\ker\not{D} - \dim\ker\not{D}^{+} = k$$

in a k-monopole background.

$\therefore$ we expect to find a unique z.m. of $\not{D}$ for $k=1$.

This is not hard to do using the explicit $k=1$ ~~BPS~~ PS soln