INTERNATIONAL ATOMIC ENERGY AGENCY
UNITED NATIONS EDUCATIONAL, SCIENTIFIC AND CULTURAL ORGANIZATION
INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS
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SMR.858 - 8

Lecture IV

SUMMER SCHOOL IN HIGH ENERGY PHYSICS AND COSMOLOGY

12 June - 28 July 1995

MONOPOLE SOLUTIONS IN SUPERSYMMETRIC THEORIES

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Lecture 4

To make sense of duality in true quantum theory we have to embed this in a SUSY theory so that $V(\Phi) = 0$ enforced by susy, monopoles can have spin one etc.

This involves adding in coupling to fermion fields. First lets discuss this w/o worrying about susy.

We will want to add to $\mathcal{L}$ terms involving fermion fields Ψ. Need to choose
- rep. of $SU(2)$ – will consider both $\underset{\sim}{2}$ fundamental, $\underset{\sim}{3}$ adjoint.

~~- Weyl, Majorana, or Dirac~~

First consider a Dirac fermion Ψ, we can couple this to previous fields by adding to $\mathcal{L}$

$$\mathcal{L}_\Psi = i\bar{\Psi}_n \gamma^\mu (D_\mu \Psi)_n - if \bar{\Psi}_n T^a_{nm} \Phi^a \Psi_m$$

here $(T^a)_{nm}$ is anti-herm. generator of $SU(2)$ in rep. carried by Ψ, $n = 1 \ldots \dim R$ for Ψ in rep R. Since we will consider only $\underset{\sim}{2}$, $\underset{\sim}{3}$ we either ~~can~~ take

$\underset{\sim}{2}$ $\quad n,m = 1,2 \quad T^a_{nm} = -\frac{i}{2}\tau^a_{nm}$

$\underset{\sim}{3}$ $\quad n,m = 1,2,3 \quad T^a_{nm} = \epsilon^a{}_{nm}$

~~Fix a monopole bkg~~
~~Fermi simplicity~~
We now analyze this in a monopole bkgnd. We will first proceed concretely, then mention general results.

Concretely, consider a charge 1 monopole with

$$\Phi^a = \frac{\hat{r}^a}{er} H(ver)$$

$$A_i^a = -\epsilon^a{}_{ij} \frac{\hat{r}^j}{er} (1 - K(ver))$$

when H, K have form in previous lecture this is BPS soln w/ $V(\Phi) \equiv 0$, but even if $V(\Phi) \neq 0$ we can find a soln of this form with same b.c.

It will be convenient to use Dirac rep. for γ matrices with

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \gamma^i = \begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix} \qquad \alpha^i = \gamma^0\gamma^i = \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix}$$

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} = 2\,\mathrm{diag}(-1,1,1,1)$$

Then writing the Dirac eqn is

$$(i\gamma^\mu D_\mu - f\Phi^a)\Psi = 0 \qquad \text{w/ group indices suppressed.}$$

Using $A_0 = 0$, writing $\Psi = e^{iEt}\psi(\vec{x})$ we have

$$(i\alpha^i D_i + \gamma^0 f \Phi)\psi = E\psi$$

and writing $\psi = \begin{pmatrix} \chi^+ \\ \chi^- \end{pmatrix}$ this gives

$$(i\sigma^i D_i + f\Phi)\chi^- = E\chi^+$$

$$(i\sigma^i D_i - f\Phi)\chi^+ = E\chi^-$$

~~Now with we quantize~~
Now if we know spectrum of Dirac op. in monopole bkgnd we can quantize by writing

$$\psi = \underbrace{\sum a_0 \psi_0}_{\text{zero energy}} + \underbrace{\sum_{n>0} b_n \psi_n^+ + d_n^+ \psi_n^-}_{\text{non-zero energy}}$$

and treating coeff. as creation, ann. op's. Only modes assoc. w/ ψ_0 will ~~give solutions that~~ leave monopole energy unchanged. They are "fermionic collective coordinates"

So to study monopole gnd. state we are interested in zero-energy soln's.

Let $\quad \not{D} = i\sigma^i D_i - f\Phi$

$\qquad \not{D}^+ = i\sigma^i D_i + f\Phi \qquad$ (since Φ anti-hermitian)

We then want soln's of $\quad \not{D}\chi^+ = 0$

$\qquad \not{D}^+\chi^- = 0$

i.e. want $\ker \not{D}$, $\ker \not{D}^+$

Ex. ~~8~~10 Show that $\not{D}\not{D}^+ = (i\vec{D})\cdot(i\vec{D}) + (i\Phi)(i\Phi) \geq 0$ (32)
pos. def. op.

hint: use Bog eq'n in form $[D_1, D_2] = [D_3, \Phi]$
+ cyclic.

Conclude that
$\ker \not{D}^+ \subset \ker \not{D}\not{D}^+ = \{0\}$

Now there is an index Thm due to Callias (CMP 62 (1978) 213)
which computes

$$\text{index of Dirac op} \equiv \dim \ker \not{D} - \dim \ker \not{D}^+ = k \times \begin{cases} 1 & \underset{\sim}{2} \\ 2 & \underset{\sim}{3} \\ \vdots & \vdots \end{cases}$$

in a k-monopole background.

∴ We expect to find a unique z.m. of $\not{D}$ for $k=1$, ~~2~~ $\underset{\sim}{2}$
and two for $k=1$ $\underset{\sim}{3}$ rep. for fermions
This is not hard to do using the explicit $k=1$
~~BPS~~ PS sol'n - but explicit form is not really needed.

What are the consequences?

First consider fermions in $\underset{\sim}{2}$. Then in expansion
of fermion about monopole we have

$$\psi = a_0 \underset{\uparrow \text{z.m.}}{\psi_0} + \text{non-zero modes}$$

and anti-comm. rel'ns of $\psi \Rightarrow \{a_0^+, a_0\} = 1$ $\{a_0, a_0\} = \{a_0^+ a_0^+\} = 0$

The spectrum of states w. semi classical monopole is thus

$$|\Omega\rangle \quad \text{with } a_0|\Omega\rangle = 0$$
$$a_0^+ |\Omega\rangle$$

i.e. monopole gnd state is 2-fold degenerate.
Now this theory has a global symmetry - fermion #
corresponding to $\psi \to e^{i\alpha}\psi$. If ψ carries fermion number one then so does a_0 and a_0^+ has fermion number -1 so we have

State	F (fermion #)
$\|\Omega\rangle$	F
$a_0^+\|\Omega\rangle$	$F-1$

What is F? Theory has fermion # conj. symmetry which takes $F \to -F$, for this to be a symmetry of monopole sector must take $F = 1/2$ so spectrum is

$|\Omega\rangle \quad F = 1/2 \qquad$ Jackiw + Rebbi,
$a_0^+|\Omega\rangle \quad F = -1/2 \qquad$ PRD 13 (1976) 3398.

So even though all states in zero monopole sector have integer fermion #, monopoles carry 1/2 integer fermion #. For a more careful discussion of this and Θ-dep w/ massless fermions see C. Callan, PRD 26 (1982) 2058.

An even more interesting example of this fractionalization occurs if we take N_f 2 of Dirac fermions.

$$\mathcal{L}_\psi = \sum_{I=1}^{N_f} i\bar{\psi}^I \not{D}\psi^I - if\bar{\psi}^I \Phi \psi^I$$

in this theory the $U(1) \simeq O(2)$ fermion number symmetry is extended to $O(2N_f)$. There are $N_f\,\psi$

which is $2N_f$ Weyl fermions - these transform as a vector of $O(2N_f)$. We will now have a zero mode for each so

$$\psi^I = a_0^I \psi_0 + \cdots$$

Now to see the spectrum go back to $N_f = 1$ and write

$$a_0 = \tfrac{1}{\sqrt{2}}(b_0^1 + i b_0^2)$$

$$a_0^+ = \tfrac{1}{\sqrt{2}}(b_0^1 - i b_0^2)$$

then b_0 satisfy $\{b_0^i, b_0^j\} = \delta^{ij} \quad i,j = 1,2$

Clifford algebra of dim. $2^{D/2} = 2^{2/2} = 2$ so 2 states. i.e. vacuum state must furnish representation of zero mode algebra.

Now with $I = 1..N_f$ we can replace a_0^I, a_0^{+I} by $b_0^a \quad a = 1 \ldots 2N_f$ w/

$$\{b_0^a, b_0^b\} = \delta^{ab}$$

And rep's of this Clifford algebra have dim $2^{2N_f/2} = 2^{N_f}$

- that is they are <u>spinors</u> of $SO(2N_f)$!

<u>Ex. 11.</u> Show counting of 2^{N_f} states agrees w/ # states constructed using a_0^I, a_0^{+I} and that this corresponds to the embedding

$$SO(2N_f) \supset SU(N_f) \quad \text{given by}$$

$$2N_f \to N_f + \bar{N}_f$$

and
$$2^{N_f} \to \sum_{M=0}^{N_f} \binom{N_f}{M}$$

Sum over antisymmetric tensor reps of $SU(N_f)$.

One further interesting aspect has to do with a correlation between dyon charge and "chirality" in $SO(2N_f)$ (or better $Spin(2N_f)$).

Namely, we derived the condition

$$e^{2\pi i N} = e^{2\pi i(Q/e + \frac{\theta e}{8\pi^2} g)} = e^{2\pi i(Q/e + \frac{\theta n_m}{2\pi})} = 1$$

by considering 2π rotations about $\hat{\Phi}$ and thus deduced

$$Q = ne - \frac{e\theta}{2\pi} n_m$$

but this was in $SO(3)$ before we added fermions in $\underset{\sim}{2}$ and made gauge group $SU(2)$. In $SU(2)$ a 2π rotation gives $\begin{pmatrix} -1 & \\ & -1 \end{pmatrix}$ act on $\underset{\sim}{2}$ and I on $\underset{\sim}{3}$

So correct statement is now

$$e^{2\pi i(Q/e + \frac{\theta n_m}{2\pi})} = \underset{\substack{\uparrow \\ \text{center of } SU(2)}}{(-1)^H}$$

or $e^{2\pi i Q/e} = e^{-i n_m \theta}(-1)^H$

if $Q = n_e e - \frac{e n_m \theta}{2\pi}$ then this says

n_e integer $\leftrightarrow (-1)^H = +1$

n_e $\frac{1}{2}$ odd int $\leftrightarrow (-1)^H = -1$

Note this is true for $n_m = 0$ states, fermions in $\underset{\sim}{2}$ have $(-1)^H = -1$, $n_e = \pm 1/2$, $W^\pm$ have $(-1)^H = 1$ $n_e = \pm 1$.

Now in monopole sector $(-1)^H$ is an op anti-commuting with fermion fields so

$$\{(-1)^H, b_0^I\} = 0$$

Since $\{b_0^I, b_0^J\} = \delta^{IJ}$ Cliff algebra

(b_0^I like gamma ~~matrices~~ γ^μ)

$\Rightarrow (-1)^F$ is "γ_5" that is "chirality" operator in spinor of $SO(2N_F)$ with

$$2^{N_F} \to 2^{N_F - 1} + 2^{N_F - 1}$$
$$(-1)^F \quad +1 \qquad -1$$

Consider the case $N_F = 4$. Then $2^4 = 16 \to 8_s + 8_c$ of $SO(8)$. This says we have

State	(n_e, n_m)	Spin (8)
Fermion	$(\pm 1/2, 0)$	8_v
Monopole	$(0, 1)$	8_s
Dyon	$(\pm 1/2, 1)$	8_c

In fact this theory, when supersymmetrized is a finite $N=2$ gauge theory and SW have conjectured it has an exact $SL(2,\mathbb{Z})$ duality which also acts as triality permuting $8_v, 8_s, 8_c$ ... but this would take us too far afield.

Now consider adjoint fermions. Then the index theorem predicts two zero modes

$$\psi = a_{0,1/2}\, \psi_0^{1/2} + a_{0,-1/2}\, \psi_0^{-1/2} + \text{non z-modes}$$

and the spectrum is now 4-fold degenerate with states

$$
\begin{array}{lcl}
 & S_z & \\
|\Omega\rangle & 0 & a_{0\,1/2}|\Omega\rangle = a_{0\,-1/2}|\Omega\rangle = 0 \\
a^+_{0\,1/2}|\Omega\rangle & 1/2 & \\
a^+_{0\,-1/2}|\Omega\rangle & -1/2 & \\
a^+_{0\,1/2}\,a^+_{0\,-1/2}|\Omega\rangle & 0 &
\end{array}
$$

Now the reason for the labelling $\pm 1/2$ is that $a_{0\pm 1/2}$ creates a state with spin $S_z = \pm 1/2$
so we get 4 states - 2 w/ spin 0, 2 w/ spin 1/2.

Heuristically, this occurs for the following reason.
The monopole is spherically symmetric under

$$\vec{J} = \underbrace{\vec{L}}_{\text{orbital}} + \underbrace{\vec{S}}_{\text{spin}} + \underbrace{\vec{T}}_{SU(2)}$$

i.e. a diagonal subgroup of $SU(2)_{\text{rotation}} \otimes SU(2)_{\text{gauge}}$

Fermions in $\underline{2}$ of $SU(2)_G$ and $SU(2)_R$ can have $J=0$ since $\underline{2}\times\underline{2} = \underline{1} + \underline{3}$

But fermions in $\underline{3}$ of $SU(2)_G$ and $\underline{2}$ of $SU(2)_R$ have J values $\underline{3}\times\underline{2} = \underline{2} + \underline{4}$
i.e. carry 1/2 integer "spin"

<u>Ex. 12</u> Construct explicit z.m. for fermions in $\underline{3}$ for $k=1$ BPS monopole. Construct ~~the~~ the operator $\vec{J}$ and verify the above analysis

Ref. H. Osborn Phys. Lett 83B (1979) 321.

- Question - Can we interpret fermion z.m. as due to a symmetry broken by monopole sol'n? - yes - <u>supersymmetry</u>.

We are now ready to incorporate all this structure into SUSY theories.

We will consider $N=2$, $N=4$ SYM. This can be motivated as follows

1) Bog. bound follows from SUSY algebra - requires central terms in algebra which occur for $N=2,4$ but not $N=1$

2) Want SUSY to relate all the fields in the theory

$$\begin{matrix} N=1 \left\{ \begin{matrix} A_\mu \\ \psi \end{matrix} \right. \\ N=1 \left\{ \Phi \right. \end{matrix} \quad \left. \begin{matrix} \\ \\ \end{matrix} \right\} N=2,4$$

I will not use superfields or superspace - you can translate into that language if it is familiar

First consider $N=2$. We get the Lagrangian from

$$\mathcal{L}(F^2 + (D\Phi)^2) + \mathcal{L}_\psi(\bar\psi D\psi + \bar\psi\Phi\psi)$$

in adjoint rep.

by adding one more scalar field and a potential terms. Field content will be

$$\begin{aligned} A_\mu &= A_\mu^a T^a && \text{gauge field} \\ \psi &= \psi^a T^a && \text{Dirac fermion} \\ S, P &= S^a T^a,\ P^a T^a && \text{2 real scalars} \end{aligned} \Bigg\} \text{ all in adj. rep. of } SU(2)$$

With

$$\mathcal{L} = \mathrm{Tr}\Big\{ -\tfrac{1}{4} F_{\mu\nu}F^{\mu\nu} - \tfrac{1}{2}(D_\mu P)^2 - \tfrac{1}{2}(D_\mu S)^2 - \tfrac{e^2}{2}[S,P]^2$$
$$+ i\bar\Psi\gamma^\mu D_\mu\Psi - g\bar\Psi[S,\Psi] - g\bar\Psi\gamma_5[P,\Psi]\Big\}$$

This is inv under SUSY transf.

$$\delta A_\mu = i\bar\alpha\gamma_\mu\chi - i\bar\chi\gamma_\mu\alpha \qquad \text{spinor } \alpha$$
$$\delta P = \bar\alpha\gamma_5\chi - \bar\chi\gamma_5\alpha$$
$$\delta S = i\bar\alpha\chi - i\bar\chi\alpha$$
$$\delta\chi = (\sigma^{\mu\nu}F_{\mu\nu} - \not{D}S + i\not{D}P\gamma_5 - i[P,S]\gamma_5)\alpha$$

Now we do have a potential, but it has a flat direction when $[S,P]=0$. So minimum with
$\langle S\rangle \propto \langle P\rangle$
In fact by a chiral rotation can set $\langle P\rangle = 0$, then $\langle S\rangle$ undetermined classically.

$\langle S\rangle$ breaks $SU(2)\to U(1)$ and have a BPS monopole with $\Phi^a \equiv S^a$, A_i^a as before. Impose $S^aS^a \xrightarrow[r\to\infty]{} v^2$ as b.c.

Now what about SUSY? Classical sol'n has $A_i, S \neq 0$, $\psi = 0$ so SUSY transf is
$\delta A_\mu = \delta P = \delta S = 0$

$\delta\chi = (\sigma^{\mu\nu}F_{\mu\nu} - \not{D}S)\alpha$ now using $B_i = D_i\Phi$ can write in terms of $\Gamma_5 = \gamma_0\gamma_5 = \begin{pmatrix}1&0\\0&-1\end{pmatrix}$ as

$$\delta\chi = \gamma^i B_i(1-\Gamma_5)\alpha \qquad \text{so if } \alpha_\pm = \tfrac{1}{2}(1\pm\Gamma_5)\alpha$$

We have

$\delta_\alpha \chi \neq 0$ broken susy.

$\delta_{\alpha_+} \chi = 0$ un-broken susy.

Since α started out with 8 real comp. (1 Dirac or 2 Majorana) there are 4 real choices of α_- w/ $\delta_{\alpha_-} \chi \neq 0$. These configs.

$\delta_{\alpha_-}\chi = \gamma^i B_i \alpha_-$ are deformations about monopole config. of zero energy - that is they are fermion z.m. (4 since $a_{0\,1/2}$ $a_{0\,-1/2}$ $a^\dagger_{0\,1/2}$ $a^\dagger_{0\,-1/2}$)

Ex. 13 Verify that $\delta_{\alpha_-} \chi = \gamma^i B_i \alpha_-$ is a solution of Dirac eq'n for fermion in presence of BPS monopole of charge 1.

Now in addition to new insight on origin of fermion z.m., susy all gives new insight into Bog. bound
E. Witten + D. Olive, Phys. Lett. 78 B (1978) 97.

Recall general $N=2$ susy algebra, here written in terms of 2 Majorana susy's $Q_{\alpha i}$ $i=1,2$

$$\{Q_{\alpha i}, \bar{Q}_{\beta j}\} = \delta_{ij}\gamma^\mu_{\alpha\beta} P_\mu + \delta_{\alpha\beta} U_{ij} + (\gamma_5)_{\alpha\beta} V_{ij}$$

$U_{ij} = -U_{ji}$ $V_{ij} = -V_{ji}$ are central charges

for $N=2$ ~~[illegible]~~ there are 2 real charges.