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IV

SUMMER SCHOOL IN HIGH ENERGY PHYSICS AND COSMOLOGY

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BASICS OF CFT, STRING COMPACTIFICATIONS, MIRROR SYMMETRY AND SPACETIME TOPOLOGY CHANGE

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Please note: These are preliminary notes intended for internal distribution only.

SUMMARY OF LECTURE II

I) INTRODUCTION & MOTIVATION
II) THE $N=2$ SUPERCONFORMAL ALGEBRA
- GENERAL PROPERTIES (HIGHEST WT REPS; (ANTI) CHIRAL PRIMARIES ...)
- EXAMPLES (FREE THEORY; CY σ-MODEL; L-G MODELS, MIN MODELS)

III) FAMILIES OF $N=2$ THEORIES

- TRULY MARGINAL OPERATORS:

$$S_{CFT} \to S'_{CFT} = S_{CFT} + \varepsilon \int d^2 z \, \mathcal{O}(z,\bar{z}) \quad \left\{ \begin{array}{l} h = \bar{h} = 1 \\ \text{(NECESSARY)} \end{array} \right.$$

$\underline{N=2}$: • $\phi \in (c,c)$, $\Phi_{c,c} \equiv (G^-_{-1/2} \bar{G}^-_{-1/2} \phi)$

• $\phi \in (a,c)$, $\Phi_{a,c} \equiv (G^+_{-1/2} \bar{G}^-_{-1/2} \phi)$

WHY? • MACROSCOPIC : $S_{SPACETIME} = \cdots + \int d^4x \, d^2\theta \, W_{SPACETIME}$

• MICROSCOPIC : WARD IDENTITIES

GEOMETRIC : S_{CFT} = NLσM ON M

• $\Phi_{c,c}$, $\Phi_{a,c}$: DEFORMATIONS OF COMPLEX & KAHLER STR

~ HARMONIC (2,1), (1,1) diff forms on M

~ e.g. $h_{i\bar{j}} \, \partial x^i \bar{\partial} x^{\bar{j}}$

• MODULI SPACES : A FIRST LOOK

• LOCAL STRUCTURE : SPECIAL GEOMETRY

• OUR AIM : GLOBAL STRUCTURE

IV) INTERRELATIONS BETWEEN REALIZATIONS OF N=2 SCA

• LANDAU-GINSBURG / MINIMAL MODELS

• LG : $S = \int d^2z d^4\theta\, K(\Psi, \bar{\Psi}) + (\int d^2z d^2\theta\, \Psi^{P+2} + c.c.)$

(1 CHIRAL SUPERFIELD; $W(\Psi) = \Psi^{P+2}$; TAKE IR LIMIT)

• MIN MODEL : LEVEL P

(NO COMPLETE PROOF; STRONG EVIDENCE)

• LANDAU-GINSBURG / CY σ-MODEL

• LG : $S = \int d^2z d^4\theta\, K(\Psi_i, \bar{\Psi}_i) + (\int d^2z d^2\theta \sum_{i=1}^{5} \Psi_i^{P_i+2} + c.c.)$

ORBIFOLD

(5 CHIRAL SUPERFIELDS; $W(\Psi_1, \ldots, \Psi_5) = \sum \Psi_i^{P_i+2}$; TAKE IR LIMIT)

• CY σ-MODEL: $S = \int d^2z\, g^M_{\mu\bar{\nu}}\, \partial X^\mu \bar{\partial} X^{\bar{\nu}} + \ldots$ (SEE LECTURE I)

$M : \sum z_i^{P_i+2} = 0$ IN $WCP^4(\omega_1, \ldots \omega_5)$: $d = (P_i+2)\omega_i$; $d = \sum \omega_i$

Remarks

1) Need not restrict to $W = \sum_i \Psi_i^{p_i+2}$. Rather, W can be any homogeneous (quasi) polynomial in $(\Psi_1, \ldots, \Psi_5)$ of degree d.
   - Same argument $\rightarrow$ CY σ-model on M:
     $(z_1, \ldots, z_5)$: $W(z_1, \ldots, z_5) = 0$ in $W\mathbb{CP}^4_{(\omega_1, \ldots, \omega_5)}$
   - Different choices of $W \leftrightarrow$ Different complex structures on M.

2) Need not restrict to 5 superfields / CY 3-folds.

3) This argument picks out a particular Kähler structure on CY – (in fact, it lies outside range of σ-model perturbation theory) – one of enhanced discrete symmetries.

4) Witten's argument lets us more clearly see family of fixed point theories:

35.3
35.4

2) ARGUMENT EASILY EXTENDED TO OTHER CY/LG THEORIES. SOMETIMES CY $\leftrightarrow$ CONTINUOUS GAUGED LG THEORY.

3) (ASPINWALL, BRG, MORRISON; WITTEN) IN SLIGHTLY MORE COMPLICATED EXAMPLES CAN HAVE NOT TWO BUT MANY PHASES. SOMETIMES DIFFERENT PHASES INTERPRETABLE AS σ-MODELS ON TOPOLOGICALLY DISTINCT CY MANIFOLDS. CAN WE MOVE FROM PHASE TO PHASE, USING MARGINAL OPERATORS, WITHOUT ENCOUNTERING A PHYSICAL SINGULARITY? NEED MIRROR MANIFOLDS TO ANSWER THIS QUESTION.

(4) RECALL: IF $W_{LG} = \sum_i \psi_i^{p_i+2}$ WE HAVE $LG/ORB \cong \otimes MM_{p_i}/ORB$)

LG P.T. GOOD — σ-MODEL P.T. GOOD

"LG PHASE" — "CY-σ MODEL PHASE"

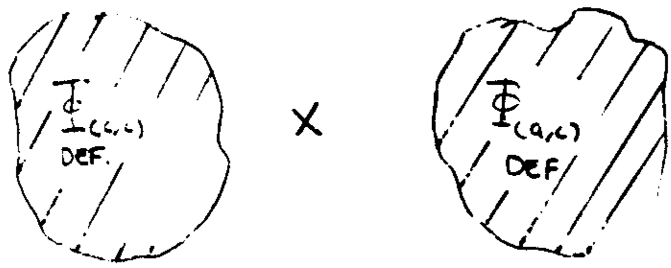
- OR: IF DISREGARD P.T.: ALL CY's, for INSTANCE.

$\downarrow$

POINT OF GVW ARGUMENT

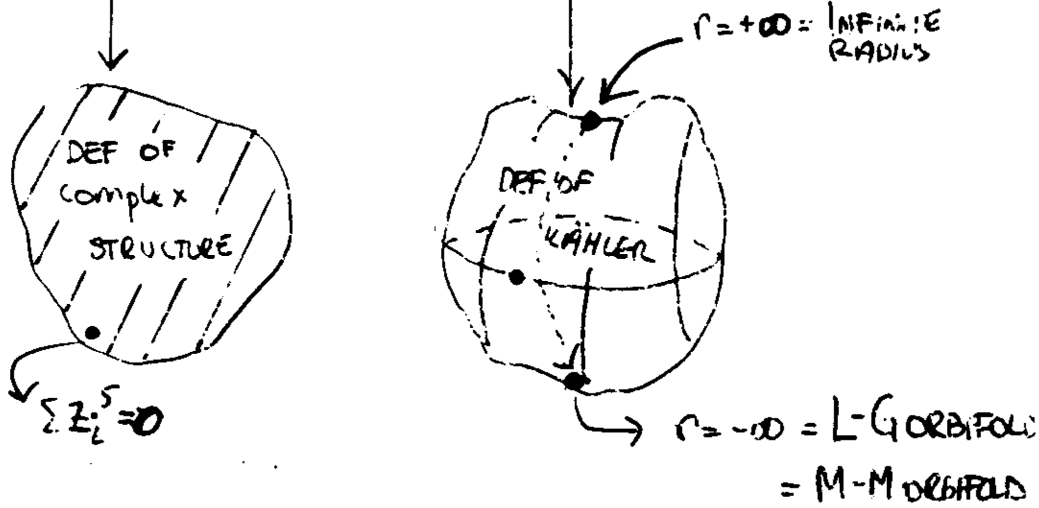
Moduli Space: II (for definiteness: Quintic)

- Abstract $N=2$ moduli space:



- Geometric Interpretation

Quintic: $h^{2,1} = 101$, $h^{1,1} = 1$ (complex dim.)



TWO "PHASES":
UPPER HEMISPHERE: KÄHLER MOD OF QUINTIC

LOWER HEMISPHERE: MOD SPACE OF LG ORBIFOLD

7) MIRROR MANIFOLDS

7.1) MOTIVATION AND STRATEGY

• **HAVE SEEN:** CORRESPONDENCE BETWEEN

- ABSTRACT PROPERTIES OF $N=2$ SCFT'S
- GEOMETRIC PROPERTIES OF CY'S USED TO REALISE $N=2$ VIA NLσ MODELS

• **IN PARTICULAR:**

WILL RETURN TO RING STRUCTURE

• ABSTRACT SCFT: (c,c) RING $\quad$ (a,c) RING

$[h=\bar{h}=1/2, Q=\bar{Q}=1]$ $\quad$ $[h=\bar{h}=\frac{1}{2}, -Q=\bar{Q}=1]$

$\to$ MARGINAL OPERATORS $\Phi_{c,c}$; $\Phi_{a,c}$

• GEOMETRIC:

(2,1) FORM $\leftrightarrow$ HARMONIC FORMS $\quad$ HARMONIC FORMS

$$h_{\bar{i}}{}^{j}\, dx^{\bar{i}} \otimes \frac{\partial}{\partial x^{j}} ; \; (H^1(M,T)) \qquad h_{\bar{i}j}\, dx^{\bar{i}} dx^{j} \; (H^1(M,T^*))$$

$$\to (g_{\bar{j}j}\, h_{\bar{i}}{}^{j}) \psi^{\bar{i}} \lambda^{\bar{j}} \qquad h_{\bar{i}j} \psi^{\bar{i}} \lambda^{j}$$

$\to$ MARGINAL OPS: (TO LOWEST ORDER)

$$(g_{\bar{j}j}\, h_{\bar{i}}{}^{j}) \bar{\partial} x^{\bar{i}} \partial x^{\bar{j}} ; \quad h_{\bar{i}j} \bar{\partial} x^{\bar{i}} \partial x^{j}$$

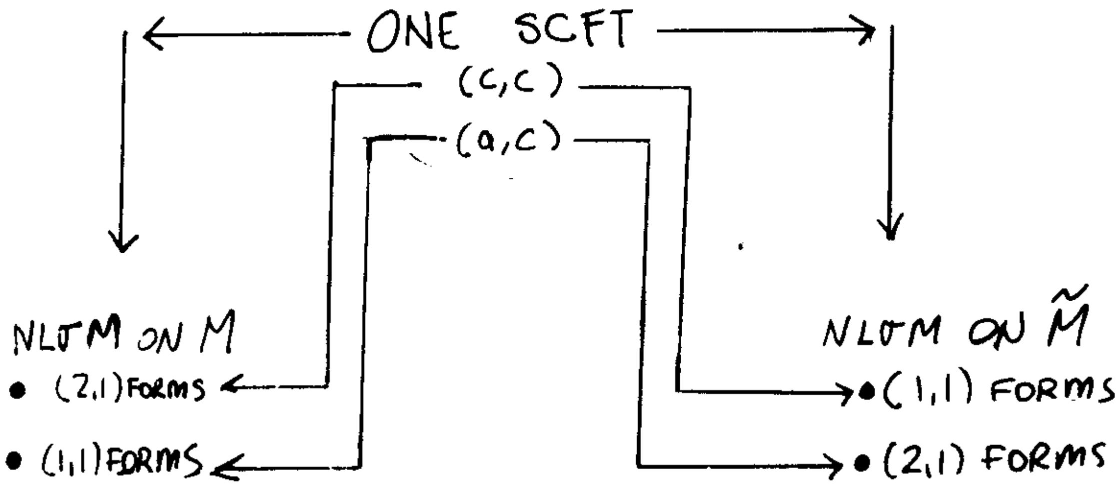
• ASYMMETRY: CFT: $Q \leftrightarrow -Q$

GEOMETRY: MORE PRONOUNCED DIFFERENCE

• RESOLUTIONS:

1) THAT'S JUST THE WAY IT IS

2) (LVW, DIXON) CONJECTURE: MAYBE THERE IS ANOTHER HALF TO THE PICTURE



• CAN THIS PICTURE BE REALIZED?

• YES: (BRG, PLESSER) FOR ∃ PARTICULAR CLASS OF $N=2$ THEORIES / CY σ-MODELS, CAN REALIZE BY DIRECT CONSTRUCTION

• NB: CANDELAS, LYNKER, SCHIMMRIGK; BATYREV
BERGLUND, HUBSCH;

STRATEGY OF CONSTRUCTION

$$W = \sum \varphi_i^{P_i+2}$$

$$\begin{tikzcd}[column sep=large, row sep=huge]
\begin{array}{c}\text{FERMAT LG ORBIFOLD} \cong \otimes\,\text{MIN MODELS ORBIFOLD} \\ (\text{"LG POINT"})\end{array} \arrow[r, "\mathcal{O}"] \arrow[d, "\begin{array}{c}\text{GEPNER} \\ *\text{GVW} \\ \text{MARTINEC} \\ *\text{WITTEN}\end{array}"'] \arrow[dr, "Q \to -Q"] & \mathcal{O}\left(\frac{LG}{ORB} \cong \frac{\otimes MM_i}{ORB}\right) \overset{Q \leftrightarrow -Q}{\cong} \frac{LG}{ORB'} \cong \frac{\otimes MM_i}{ORB'} \arrow[d] \\
\sigma\text{-MODEL ON CY } M \arrow[r, "\mathcal{O}"] & \begin{array}{c}\sigma\text{-MODEL ON CY} \\ \mathcal{O}(M) \equiv \tilde{M}\end{array}
\end{tikzcd}$$

- CAN SUCH AN OPERATION $\mathcal{O}$ BE FOUND?

  YES: $\mathcal{O}$ = MORE ORBIFOLDING

- IN PARTICULAR:

$$\begin{tikzcd}[column sep=small]
 & (c,c) \arrow[dl] \arrow[dr] & & & (a,c) \arrow[dl] \arrow[dr] & \\
\begin{array}{c}(2,1)\text{ FORMS} \\ M\end{array} & & \begin{array}{c}(1,1)\text{ FORMS} \\ \tilde{M}\end{array} & \begin{array}{c}(1,1)\text{ FORMS} \\ M\end{array} & & \begin{array}{c}(2,1)\text{ FORMS} \\ \tilde{M}\end{array}
\end{tikzcd}$$

$$\Rightarrow \chi_M = 2\left(\#\,(1,1)\text{ HARM. FORMS} = h^{1,1}_M - \#\text{ HARM. }(2,1)\text{ FORMS} = h^{2,1}_M\right)$$

$$\chi_{\tilde{M}} = 2\left(h^{1,1}_{\tilde{M}} - h^{2,1}_{\tilde{M}}\right) = 2\left(h^{2,1}_M - h^{1,1}_M\right) = -\chi_M$$

$\Rightarrow$ "MIRRORS"

• ROUGH ARGUMENT

1. MM_p HAS $\mathbb{Z}_{p+2} \times \bar{\mathbb{Z}}_{p+2}$ DISCRETE SYMMETRY

$$\mathbb{Z}^{DIAG}_{p+2} \subset \mathbb{Z}_{p+2} \times \bar{\mathbb{Z}}_{p+2}$$

$$MM_p / \mathbb{Z}^{DIAG}_{p+2} \overset{Q \to -Q}{\cong} MM_p$$

2. $$\left(\bigotimes_{i=1}^{5} MM_{p_i} \right) / \mathbb{Z}^{DIAG}_{p_i+2} \overset{Q_{TOT} \to -Q_{TOT}}{\cong} \bigotimes_{i=1}^{5} MM_{p_i}$$

3. HAVE SEEN: CY $\sum z_i^{p_i+2} = 0 \longleftrightarrow$ LG : $W = \sum \Psi_i^{p_i+2}$ ORBIFOLDED $\longleftrightarrow \otimes MM_{p_i}$ ORBIFOLDED $\longleftrightarrow$ PROJECTION TO INTEGRAL $U(1)$ CHARGES

NOTATION : $(P_1, P_2, \ldots, P_5) = \otimes MM_{P_i}$ ORBIFOLDED $\longleftrightarrow$ PROJECTION TO INTEGRAL $U(1)$ CHARGES

$$+ \left[\bigotimes_{i=1}^{5} MM_{P_i} / \mathbb{Z}_{P_1+2} \times \cdots \times \mathbb{Z}_{P_5+2} \right]_{U(1)\ \text{PROJECTION}} \overset{Q_{TOT} \to -Q_{TOT}}{\not\cong} (P_1, \ldots, P_5)$$

$$\sum z_i^{P_i+2} = 0 \ : \ M$$

$$= (P_1, \ldots, P_5) / H$$

(H = MAXIMAL SUBGROUP of $\mathbb{Z}_{P_1+2} \times \cdots \mathbb{Z}_{P_5+2}$ PRESERVING (C,C) FIELD OF $h = 3/2$ $Q = 3$, $\bar{h} = 0$ $\bar{Q} = 0$)
(HOLO (3,0) FORM ON M

$$\updownarrow$$

$$\left(\sum z_i^{P_i+2} = 0 \right) / H \ : \ \tilde{M}$$

V.2) DETAILS

1. MINIMAL MODEL AUTOMORPHISM

A) CHARACTERS: $c = 3p/p+2 \qquad p = 1, 2, \ldots$

HW STATES:

• NS: $\Phi_{\ell,m}$ $\qquad h_{\ell,m} = \frac{\ell(\ell+2)}{4(p+2)} - \frac{m^2}{4(p+2)}, \quad Q_{\ell,m} = m/p+2$

$\ell = 0, 1, \ldots, p$

$m = -\ell, -\ell+2, \ldots, \ell$

• SIMILARLY FOR ANTIHOL. SECTOR

• $m = \ell \Rightarrow h_{\ell,\ell} = \ell/2(p+2) = \frac{1}{2} Q_{\ell,\ell}$

$\Rightarrow \Phi_{\ell,\ell}$ CHIRAL PRIMARY

• $m = -\ell \quad \Rightarrow \Phi_{\ell,-\ell}$ ANTICHIRAL PRIMARY

• R: $G_0^+ \Psi^+_{\ell,m} = 0 \qquad G_0^- \Psi^-_{\ell,m} = 0$

$\Psi^{\pm}_{\ell,m}$ $\qquad h^{\pm}_{\ell,m} = \frac{\ell(\ell+2)}{4(p+2)} - \frac{(m\pm1)^2}{4(p+2)} + \frac{1}{8}$

$\ell = 0, 1, \ldots, p$

$m = -\ell, -\ell+2, \ldots, \ell$ $\qquad Q^{\pm}_{\ell,m} = \frac{m \pm 1}{p+2} \mp \frac{1}{2}$

• SIMILARLY FOR ANTIHOL. SECTOR

(• RAMOND GROUND STATES: $h = c/24 = p/8(p+2)$

$\Psi^+_{\ell,\ell}$; $\Psi^-_{\ell,-\ell}$ WITH $\Psi^-_{\ell-p,-(\ell-p)} \simeq \Psi^+_{\ell,\ell}$)

- PROVES USEFUL: SEPARATE REPRESENTATIONS INTO TWO PIECES VIA "FERMION # MOD 2" (GEPNER)
  $G^{\pm}$ FERMION # = 1
  → DIVIDE STATES IN THE REP BY # of $G^{\pm}$ ACTIONS : EVEN vs ODD : INDEX "S"

- $$\left.\begin{matrix} \Phi_{\ell,m} \\ \psi^{\pm}_{\ell,m} \end{matrix}\right\} \Phi_{\ell,m,s} \begin{cases} s=0,2 & NS \\ s=\pm 1 & R \\ s \equiv s \ (\text{MOD } 4) \end{cases}$$

- $$\chi^{\ell}_{ms} = \operatorname{Tr}_{\mathcal{H}^{\ell}_{ms}} e^{2\pi i \tau (L_0 - c/24)} = \sum (\text{STRING FCNS})(\theta\text{-FCNS}) \quad (\text{Qiu})$$
  $$\left(\chi^{\ell}_{ms} \text{ INV UNDER } \begin{matrix} m \to m+2(p+2) \\ s \to s+4 \end{matrix}\right)$$

- MODULAR TRANSFORMATION PROPERTIES (Qiu)
  $$\chi^{\ell}_{ms}(\tau+1) = e^{i\pi \frac{\ell(\ell+2)}{2(p+2)}} e^{i\pi\left(\frac{-m^2}{2(p+2)} + \frac{s^2}{4}\right)} \chi^{\ell}_{ms}(\tau)$$
  $$\chi^{\ell}_{ms}(-1/\tau) = \frac{1}{\sqrt{2}(p+2)} \sum_{\substack{\ell', m', s' \\ \ell'+m'+s' = 0 \text{ MOD } 2}} \sin\left(\pi \frac{(\ell+1)(\ell'+1)}{p+2}\right) e^{i\pi\left(\frac{mm'}{p+2} - \frac{ss'}{2}\right)} \chi^{\ell'}_{m's'}(\tau)$$

FACTORIZATION : $\ell \to$ LEVEL p AFFINE $SU(2)$; $m \to$ LEVEL $p+2$ θFCNS; $s \to$ LEVEL 2 θFCN

$$\Rightarrow Z = \tfrac{1}{2} \sum A_{\ell,\bar\ell}\, B^{m\bar m}\, C^{s\bar s}\, \chi^{\ell}_{ms}\, \chi^{*\,\bar\ell}_{\bar m \bar s}$$

$$= \tfrac{1}{2} \sum A_{\ell\bar\ell}\, \chi^{\ell}_{ms}\, \chi^{*\bar\ell}_{ms}$$

- "A-INVARIANT" : $A_{\ell\bar\ell} = \delta_{\ell\bar\ell}$

$$Z = \tfrac{1}{2} \sum \chi^{\ell}_{ms}\, \chi^{*\ell}_{ms}$$

NB: EQUIVALENTLY: $Z = \tfrac{1}{2} \sum \chi^{\ell}_{ms}\, \chi^{*\ell}_{-m-s}$ (CHANGE U(1) SIGN CONVENTION IN ANTIHOL. SECTOR)

B) SYMMETRIES

$N=2$ MINIMAL MODEL @LEVEL P: $\mathbb{Z}_2 \times \mathbb{Z}_{P+2}$ SYM.

$$\Phi^{\ell}_{ms} \mapsto e^{-2\pi i\, m/P+2}\, \Phi^{\ell}_{ms}\ ; \quad \Phi^{\ell}_{ms} \mapsto e^{2\pi i s/2}\, \Phi^{\ell}_{ms}$$

NOTE: $Q = -m/_{P+2} + s/_2 \Rightarrow$ COMBINED ACTION IS $e^{2\pi i J_0}$

C) ORBIFOLD : BY DIAGONAL $\mathbb{Z}_{P+2} \subset \mathbb{Z}_{P+2} \times \bar{\mathbb{Z}}_{P+2}$

$$Z[x,y] = e^{\frac{-2\pi i x (m-\bar m)}{2(P+2)}} \boxed{}\; e^{-2\pi i y \frac{(m+\bar m)}{2(P+2)}}$$

$x, y \in \mathbb{Z}$ MOD $(P+2)$

- HOW GET $Z[x,y]$?
  - $Z[x,0] = \sum e^{-2\pi i x m/(P+2)} \chi^{\ell}_{ms} \chi^{*\ell}_{ms}$ (INSERTION)
  - $Z[0,x] = (\tau \to -1/\tau)$

$$\sum \chi^{\ell}_{ms} \chi^{*\ell}_{m-2x,s}$$

  - $Z[x,y] = \sum e^{-2\pi i x(m+\tilde{m})/2(P+2)} \chi^{\ell}_{ms} \chi^{*\ell}_{\tilde{m}s}$

$\tilde{m} \equiv m-2y$ (INSERTION)

- NOW: BUILD NEW THEORY $Z^{NEW} \equiv Z/\mathbb{Z}_{P+2}$

$$Z^{NEW} = \frac{1}{P+2} \sum_{x,y \in 0,\ldots,P+1} Z[x,y] = \frac{1}{P+2} \sum_{x,y} \sum_{m} e^{-2\pi i \frac{x(m+\tilde{m})}{2(P+2)}} \chi^{\ell}_{ms} \chi^{*\ell}_{\tilde{m}s}$$

$$\tilde{m} = m-2y$$

  - SUM ON x: $m+\tilde{m} \equiv 0 \; (2(P+2)) \Rightarrow 2m-2y \equiv 0 \; (2(P+2))$

$$\Rightarrow \tilde{m} = m-2y \equiv -m \; (2(P+2))$$

  - $Z^{NEW} = \sum \chi^{\ell}_{ms} \chi^{*\ell}_{-ms}$

- PERFORM SAME PROCEDURE FOR $(\underline{\mathbb{Z}_2} \times \mathbb{Z}_{P+2})^{DIAG} = e^{2\pi i (J_0+\bar{J}_0)/2}$
  - $Z^{NEW} = Z/(\mathbb{Z}_2 \times \mathbb{Z}_{P+2}) = \sum \chi^{\ell}_{ms} \chi^{*\ell}_{-m-s} = Z$

$$Z^{NEW} \overset{Q \to -Q}{\cong} Z^{ORIGINAL}$$

An Important Property of Orbifolding

- If a manifold M respects a group of discrete symmetries G, then can form M/G.

  The symmetry G is "gone" in M/G.

- If a CFT respects G, can form CFT/G.

  The symmetry G acts non-trivially on CFT/G as a **Quantum Symmetry.**

- Example: $G = \mathbb{Z}_n$, $CFT = K$.

$$Z_{K/\mathbb{Z}_n} = \frac{1}{|G|} \sum_{g,h \in G} g \, \underset{h}{\square} \, .$$

Then: h-twisted sectors states $|h\rangle \longrightarrow e^{2\pi i m/n} |h\rangle$ if $h = e^{2\pi i m/n}$.

Call this $\mathbb{Z}_n$ action $G^Q = \mathbb{Z}_n^Q$

- Then: $\underline{\underline{Z_{(K/\mathbb{Z}_n)/\mathbb{Z}_n^Q} = Z_K}}$ "unorbifolding"

• EXAMPLE: $\mathbb{Z}_2 = \{+,-\}$.

$$Z_{K/\mathbb{Z}_2} = \frac{1}{2}\left({}^{+}\underset{+}{\boxed{K}} + {}^{-}\underset{+}{\boxed{K}} + {}^{+}\underset{-}{\boxed{K}} + {}^{-}\underset{-}{\boxed{K}} \right) \equiv Z_{K'}$$

$$Z_{K'/\mathbb{Z}_2^Q} = \frac{1}{2}\left({}^{+}\underset{+}{\boxed{K'}} + {}^{-}\underset{+}{\boxed{K'}} + {}^{+}\underset{-}{\boxed{K'}} + {}^{-}\underset{-}{\boxed{K'}} \right)$$

• $${}^{+}\underset{+}{\boxed{K'}} = \frac{1}{2}\left({}^{+}\underset{+}{\boxed{K}} + {}^{-}\underset{+}{\boxed{K}} + {}^{+}\underset{-}{\boxed{K}} + {}^{-}\underset{-}{\boxed{K}} \right) \quad (Z_{K'})$$

• $${}^{-}\underset{+}{\boxed{K'}} = \frac{1}{2}\left({}^{+}\underset{+}{\boxed{K}} + {}^{-}\underset{+}{\boxed{K}} - {}^{+}\underset{-}{\boxed{K}} - {}^{-}\underset{-}{\boxed{K}} \right) \quad (\text{INSERTION})$$

• $${}^{+}\underset{-}{\boxed{K'}} = \frac{1}{2}\left({}^{+}\underset{+}{\boxed{K}} - {}^{-}\underset{+}{\boxed{K}} + {}^{+}\underset{-}{\boxed{K}} - {}^{-}\underset{-}{\boxed{K}} \right) \quad (\tau \to -1/\tau)$$

• $${}^{-}\underset{-}{\boxed{K'}} = \frac{1}{2}\left({}^{+}\underset{+}{\boxed{K}} - {}^{-}\underset{+}{\boxed{K}} - {}^{+}\underset{-}{\boxed{K}} + {}^{-}\underset{-}{\boxed{K}} \right) \quad (\text{INSERTION})$$

$$\Rightarrow Z_{K'/\mathbb{Z}_2^Q} = {}^{+}\underset{+}{\boxed{K}} = Z_K$$

• GENERALIZATION: SAME IDEA

- $G = \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_r}$. K/G HAS SECTORS ${}^{(a_1,\dots,a_r)}\underset{(b_1,\dots,b_r)}{\boxed{}}$
- $(b_1,\dots,b_r) \equiv g_1^{b_1} \cdots g_r^{b_r}$
- CONSIDER $\mathbb{Z}_n^Q \equiv$ ACTION VIA $g_1^{c_1 b_1} g_2^{c_2 b_2} \cdots g_r^{c_r b_r}$; $n \cdot c_i \equiv 0 (n_i$
- THEN: $(K/G)/\mathbb{Z}_n^Q = K/H$ $\quad H \subset G$ s.t. $h = (a_1,\dots,a_r)$
  $\Rightarrow \vec{c}\cdot\vec{a}/n \in \mathbb{Z}$

Mirror Construction (G:P)

• Notation: 1) $\chi^{\vec{\ell}}_{\vec{\mu}} \equiv \chi^{\ell_1}_{q_1 s_1} \chi^{\ell_2}_{q_2 s_2} \cdots \chi^{\ell_r}_{q_r s_r}$; $\vec{\ell} = (\ell_1, \ldots, \ell_r)$, $\vec{\mu} = (q_1, \ldots q_r ; s_1, \ldots s_r)$

2) Group Actions: $\vec{\beta} = (\beta_1, \ldots, \beta_r ; \beta_{r+1}, \ldots, \beta_{2r}) \equiv 2\vec{\gamma}$

$$\chi^{\ell_j}_{q_j s_j} \mapsto e^{2\pi i \frac{\beta_j}{2} \frac{q_j}{(p_j+2)}} e^{-2\pi i \frac{\beta_{r+j}}{2} \frac{s_j}{2}} \chi^{\ell_j}_{q_j s_j}$$

• $Z^{PROD} \equiv \sum \chi^{\vec{\ell}}_{\vec{\mu}} \chi^{*\vec{\ell}}_{\vec{\mu}}$; • $G = \prod_{i=1}^{r} \mathbb{Z}_2 \times \mathbb{Z}_{(p_i+2)}$

$$\left\{ \begin{array}{l} \beta_{(j)} = (0, \ldots, 0, \overset{j^{th}}{2}, 0 \ldots 0 ; 0 \cdots 0) \\ \beta'_{(j)} = (0, \ldots 0, \ldots 0 ; 0 \cdots \underset{j^{th}}{2} \cdots 0) \end{array} \right\}$$

• $$Z^{PROD}/G = \sum_{\substack{\vec{\mu} - \vec{\bar{\mu}} \in \Lambda_{\vec{\beta}_{(1)}, \ldots, \vec{\beta}_{(2r)}} \\ \vec{\beta}_i \cdot (\vec{\mu} + \vec{\bar{\mu}}) \in 2\mathbb{Z}}} \chi^{\vec{\ell}}_{\vec{\mu}} \chi^{*\vec{\ell}}_{\vec{\bar{\mu}}}$$

Note: $\vec{\mu} - \vec{\bar{\mu}}$ Labels Twisted Sectors.

• So: Define $\mathbb{Z}^{Q}_{n}$ By $\vec{\beta} = 2\vec{\gamma} = (2, 2, \ldots ; 2, 2, \ldots 2)$, i.e. Action

$$e^{2\pi i \frac{\vec{\beta}}{2} \cdot (\vec{\mu} - \vec{\bar{\mu}})} \qquad \left(\vec{A} \cdot \vec{B} \equiv \sum_{i=1}^{r} -\frac{A_i B_i}{2(p_i+2)} + \frac{A_{r+i} B_{r+i}}{4} \right)$$

• Then: $(Z^{PROD}/G)/\mathbb{Z}^{Q}_{n} \cong Z^{PROD}/H$ with $H = \{ \tilde{\beta} \} : \frac{\vec{\beta}}{2} \cdot \tilde{\beta} \in \mathbb{Z}$

i.e. $\sum_{j=1}^{r} \tilde{\beta}_j / 2(p_j+2) \in \mathbb{Z}$.

• Now:

• $\text{LHS} = \bullet \sum \chi^{\vec{\ell}}_{\vec{\mu}} \chi^{*\vec{\ell}}_{-\vec{\mu}} / \mathbb{Z}_n^Q$ BY MINMODEL PROPERTY

• $\mathbb{Z}_n^Q$ ACTS VIA $e^{2\pi i \frac{\vec{\beta}}{2} \cdot (\vec{\mu} - \vec{\bar{\mu}})} = e^{2\pi i \vec{\beta} \cdot \vec{\mu}}$

$$= e^{2\pi i \left[\sum_{j=1}^{r} -\mu_j/(p_j+2) + \frac{\mu_{j+r}}{2} \right]}$$

$$= e^{2\pi i J_0}$$

$\Rightarrow \text{LHS} \cong \mathbb{Z}^{\text{PROD}} \Big|_{\substack{U(1)\ \text{CHARGE} \\ \text{PROJECTION}}} \leftrightarrow \text{CY } M : \sum z_i^{p_i+2} = 0$

$\overset{\varphi}{\underset{-Q}{\updownarrow}} \;|||$

• $\text{RHS} = \mathbb{Z}^{\text{PROD}}/H = \left[\mathbb{Z}^{\text{PROD}} \Big|_{U(1)\ \text{PROJ}} \right] \Big/ \left\{ \begin{array}{l} \text{MAX GROUP} \subset G \\ \text{PRESERVING} \\ \text{HIGHEST WT}(C,C) \\ \text{FIELD} \end{array} \right\}$

$\updownarrow$

$\text{CY}: M/H' = \tilde{M}$

• Now:

• $\mathrm{LHS} = \bullet \sum \chi^{\vec{\ell}}_{\vec{\mu}} \chi^{*\vec{\ell}}_{-\vec{\mu}} \Big/ \mathbb{Z}_n^Q$ BY MINMODEL PROPERTY

$$\bullet\ \mathbb{Z}_n^Q \text{ ACTS VIA } e^{2\pi i \frac{\vec{\beta}}{2}\cdot(\vec{\mu}-\vec{\bar{\mu}})} = e^{2\pi i \vec{\beta}\cdot\vec{\mu}}$$

$$= e^{2\pi i \left[\sum_{j=1}^{r} -\mu_j/(p_j+2) + \frac{\mu_{j+r}}{2} \right]}$$

$$= e^{2\pi i J_0}$$

$$\Rightarrow \mathrm{LHS} \cong \mathbb{Z}^{\mathrm{PROD}} \Big|_{\substack{U(1)\ \text{CHARGE} \\ \text{PROJECTION}}} \leftrightarrow \mathrm{CY}\ M: \sum z_i^{p_i+2} = 0$$

$$\overset{\varphi}{\underset{-Q}{\updownarrow}} \ \|\wr$$

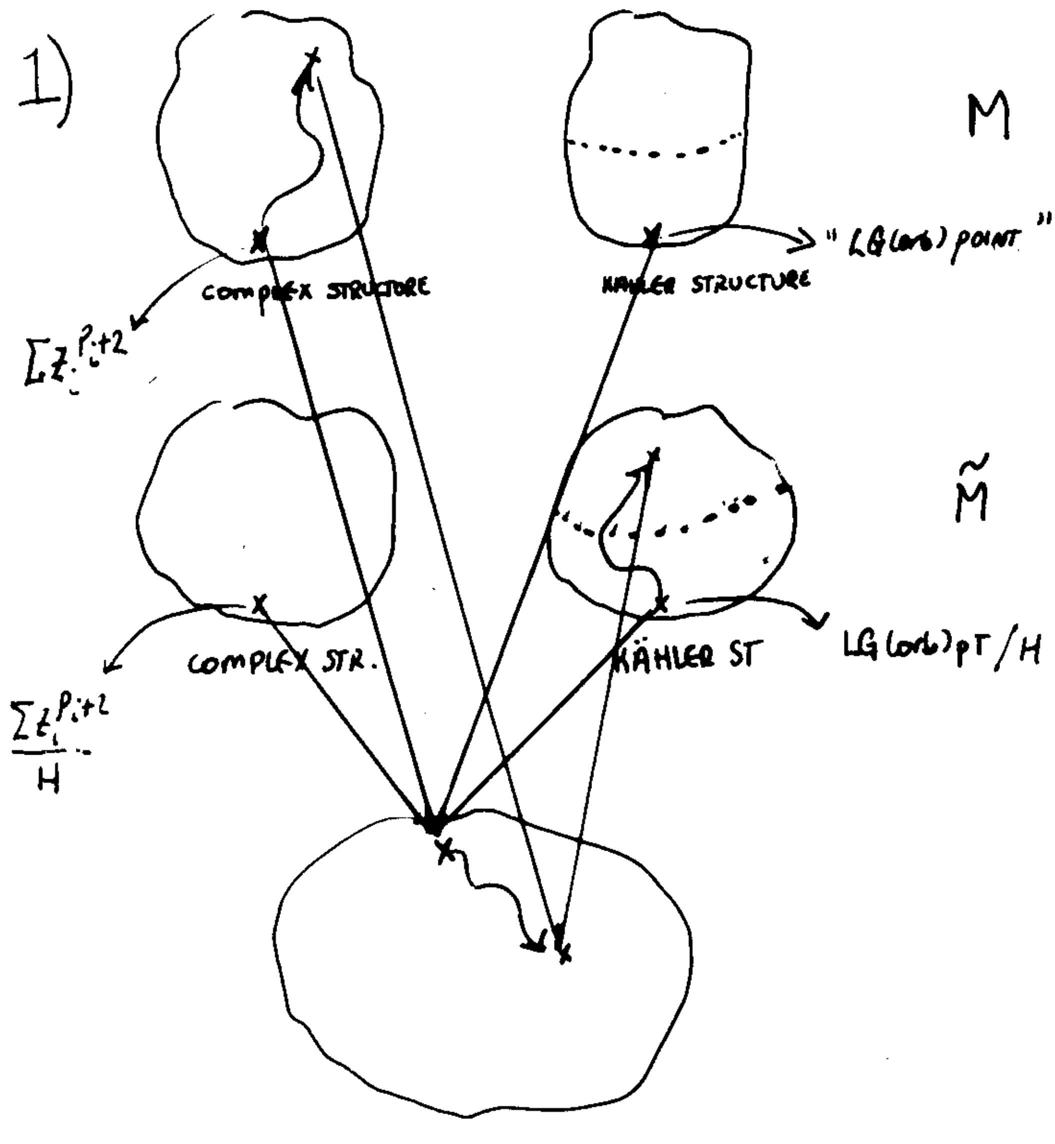
$$\bullet\ \mathrm{RHS} = \mathbb{Z}^{\mathrm{PROD}}/H = \left[\mathbb{Z}^{\mathrm{PROD}} \Big|_{U(1)\ \text{PROJ}} \right] \Big/ \left\{ \begin{matrix} \text{MAX GROUP} \subset G \\ \text{PRESERVING} \\ \text{HIGHEST WT } (C,C) \\ \text{FIELD} \end{matrix} \right\}$$

$$\updownarrow$$

$$\mathrm{CY}: M/H' = \tilde{M}$$

V.3) REMARKS

1)



CFT

2) CONSTRUCTION WORKS
- IN ANY # DIMENSIONS
- USING ANY SU(2) AFFINE INV
- NON HYPERSURFACE EXAMPLES

3) MIRROR SYMMETRY IS NOT THE TRIVIAL CFT STATEMENT THAT WE CAN REPACE $Q \to -Q$.
THIS TRIVIAL CFT STATEMENT HAS A TRIVIAL GEOMETRIC COUNTERPART: $\underline{\underline{(M, T_M) \leftrightarrow (M, T_M^*)}}$

- RATHER: MIRROR SYMMETRY SAYS THAT $\exists$ 2 CY's $M, \tilde{M}$ S.T. CORRESPONDING CFT's DIFFER BY $Q \to -Q$.

THEN THE TRIVIAL CFT STATEMENT ABOVE IS USEFUL TO CONCLUDE $M, \tilde{M} \to$ SAME $N=2$ SCFT.

4) OTHER CONSTRUCTIONS?

V.4) EXAMPLES

• QUINTIC IN $\mathbb{CP}^4$: $M = \sum Z_i^5 = 0$ in $\mathbb{CP}^4$ $\begin{cases} h_M^{1,1} = 1 \\ h_M^{2,1} = 101 \end{cases}$

$H \subset (\mathbb{Z}_5)^5$; $H = (\mathbb{Z}_5)^3$

$\tilde{M} = \sum z_i^5 = 0 / (\mathbb{Z}_5)^3$ $\begin{cases} h_{\tilde{M}}^{1,1} = 101 \\ h_{\tilde{M}}^{2,1} = 1 \end{cases}$

Symmetries	$h^{2,1}$	$h^{1,1}$	χ
	101	1	-200
[0,0,0,1,4]	49	5	-88
[0,1,2,3,4]	21	1	-40
[0,1,1,4,4] [0,1,2,3,4]	21	17	-8
[0,1,1,4,4]	17	21	8
[0,1,3,1,0] [0,1,1,0,3]	1	21	40
[0,1,4,0,0] [0,3,0,1,1]	5	49	88
[0,1,0,0,4] [0,0,1,0,4] [0,0,0,1,4]	1	101	200

Table 2

Orbifolds of $z_1^5 + z_2^5 + z_3^5 + z_4^5 + z_5^5 = 0$ in $\mathbb{CP}^4$

NOTATION:

[0,0,0,1,4] MEANS: $(z_1, \dots, z_5) \longmapsto (z_1, z_2, z_3, \alpha z_4, \alpha^4 z_5)$

$\alpha^5 = 1$

M	$h^{2,1}$	$h^{1,1}$	S	M'
$z_1^5 + z_2^5 + z_3^5 + z_4^5 + z_5^5 = 0$	101	1	$\mathbb{Z}_5^5$	$M/(\mathbb{Z}_5^3)$
$z_1^5 + z_2^{10} + z_3^{10} + z_4^{10} + z_5^2 = 0$	145	1	$\mathbb{Z}_5 \times \mathbb{Z}_{10}^3 \times \mathbb{Z}_2$	$M/(\mathbb{Z}_{10}^2)$
$z_1^3 + z_2^3 + z_3^3 + z_4^3 = 0$ $z_1 x_1^3 + z_2 x_2^3 + z_3 x_3^3 = 0$	35	8	$\mathbb{Z}_3 \times \mathbb{Z}_9^3$	$M/(\mathbb{Z}_3 \times \mathbb{Z}_9)$
$z_1^3 + z_2^4 + z_3^5 + z_4^6 + z_5^{20} = 0$	47	23	$\mathbb{Z}_3 \times \mathbb{Z}_4 \times \mathbb{Z}_6 \times \mathbb{Z}_{20}$	$M/(\mathbb{Z}_2)$
$z_1^3 + z_2^4 + z_3^4 + z_4^{12} + z_5^{12} = 0$	89	5	$\mathbb{Z}_3 \times \mathbb{Z}_4^2 \times \mathbb{Z}_{12}^2$	$M/(\mathbb{Z}_3 \times \mathbb{Z}_4^2)$
$z_1^4 + z_2^6 + z_3^{21} + z_4^{28} + z_5^2 = 0$	52	28	$\mathbb{Z}_4 \times \mathbb{Z}_6 \times \mathbb{Z}_{21} \times \mathbb{Z}_{28}$	$M/(\mathbb{Z}_2)$
$z_1^5 + z_2^6 + z_3^{10} + z_4^{30} + z_5^2 = 0$	91	7	$\mathbb{Z}_5 \times \mathbb{Z}_6 \times \mathbb{Z}_{10} \times \mathbb{Z}_{30}$	$M/(\mathbb{Z}_2 \times \mathbb{Z}_{10})$

Table 1

Mirrors M' of Some Minimal Model Compactifications M

4

12.4 12.3

QUICK MIRROR REVIEW

- WHY MIGHT ONE EXPECT EXISTENCE OF MIRROR MANIFOLDS?

CFT		GEOMETRY
$\left\{ \begin{array}{l} (Q=1, \bar{Q}=-1) \\ (Q=1, \bar{Q}=1) \end{array} \right\}$	$\longleftrightarrow$	$\left\{ \begin{array}{l} \text{HARMONIC } (1,1) \text{ FORMS} \\ \text{HARMONIC } (2,1) \text{ FORMS} \end{array} \right\}$

TRIVIAL DIFFERENCE: SIGN OF U(1) CHARGE $\xleftarrow{\text{STRANGE}}$ $\xrightarrow{\text{ASYMMETRY}}$ MAJOR DIFFERENCE: DISTINCT MATHEMATICAL OBJECTS

- FURTHERMORE: HOW DETERMINE 'RHS FROM LHS?

- **POSSIBLE RESOLUTION**: (DIXON; LERCHE, VAFA, WARNER)

CFT

$A_{(\alpha)}$ $(Q=1, \bar{Q}=1)$

$B_{(\beta)}$ $(Q=1, \bar{Q}=-1)$

"QUANTUM" GEOMETRY

("STRINGY" GEOMETRY)

M

$C_{(\alpha)}$ HARMONIC (2,1) FORMS

$K_{(\beta)}$ HARMONIC (1,1) FORMS

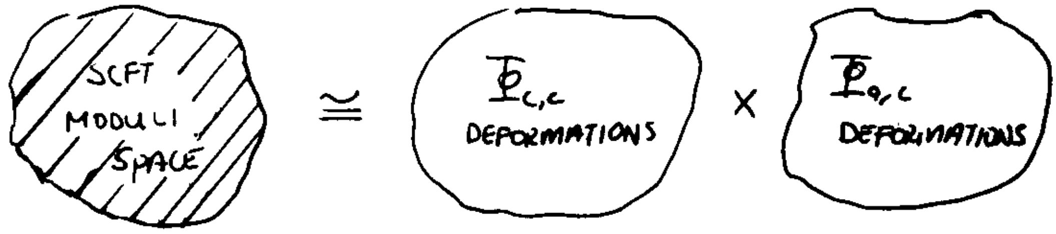
$\check{M}$

$\tilde{K}_{(\alpha)}$ HARMONIC (1,1) FORMS

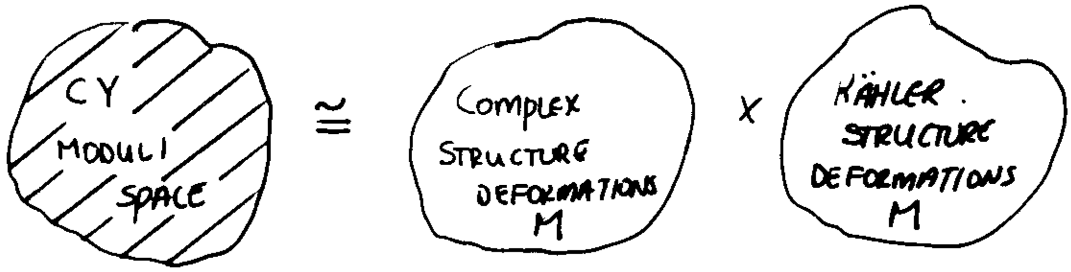
$\tilde{C}_{(\beta)}$ HARMONIC (2,1) FORMS

Moduli Spaces I:

SCFT:

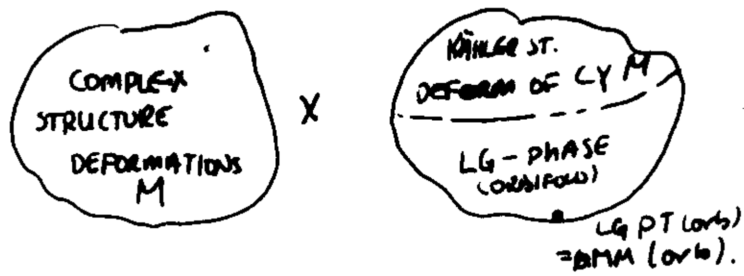


GEOMETRY:



(• THIS PICTURE IS DEFICIENT)

Moduli Spaces II:



(• THIS PICTURE IS DEFICIENT)

Moduli Spaces III:

(• THIS PICTURE IS DEFICIENT)

Using Mirror Symmetry:

$$\begin{tikzcd} & \begin{array}{c}\text{CFT}\\ \text{Any Observable}\end{array} \arrow[dl, "\begin{array}{c}\text{Formulation}\\ \#1\end{array}"] \arrow[dr, "\begin{array}{c}\text{Formulation}\\ \#2\end{array}"'] & \\ M & & \tilde{M} \end{tikzcd}$$

- Lets study this

OBSERVABLES

- $\langle \mathcal{O}_{i_1} \cdots \mathcal{O}_{i_n} \rangle \equiv \int [\mathcal{D}X] \mathcal{O}_{i_1} \cdots \mathcal{O}_{i_n} e^{i S_{CFT}}$

- FOR CFT BUILT AS $N=2$ NLσM ON CY M:

$$= \int ([\mathcal{D}X] \cdots) \mathcal{O}_{i_1} \cdots \mathcal{O}_{i_n} e^{i S_{NL\sigma M}(M)}$$

- EXPANSION PARAMETER(S): $\alpha' / \text{"}R_{CY}^2\text{"}$ ← FROM KÄHLER FORM J ON M

- IF J CHOSEN TO BE IN REGION OF MODULI SPACE SUCH THAT PERTURBATION THEORY CONVERGES (i.e. "C.Y PHASE" AS DISCUSSED)

$$\langle \mathcal{O}_{i_1} \cdots \mathcal{O}_{i_n} \rangle \overset{\text{(TYPICALLY)}}{=} \int_M \text{INTEGRAND}(\mathcal{O}_{i_1} \cdots \mathcal{O}_{i_n}) + \sum_{\substack{\text{INSTANTONS} \\ \text{i.e. RATIONAL CURVES} \\ \text{ON } M}} e^{-\frac{1}{\alpha'} \int_{I_n} X^*(J)} \text{(INTEGRALS)}$$

(AT ZERO MOMENTUM)

- WHY? STATIONARY PHASE: "POINT PARTICLE CONTRIBUTION" + "STRINGY CONTRIBUTIONS"

$=$ "CLASSICAL GEOMETRY" + "QUANTUM" (i.e. STRINGY) CORRECTIONS

IMPORTANT CASE: (c,c) & (a,c) RINGS

CFT

$$\langle \mathcal{O}_{(\alpha)}\,\mathcal{O}_{(\beta)}\,\mathcal{O}_{(\gamma)} \rangle$$

(a,c) (a,c) (c,a)

M

$$\langle K_{(\alpha)} K_{(\beta)} K_{(\gamma)} \rangle = \underbrace{\int_M b_{(\alpha)} \wedge b_{(\beta)} \wedge b_{(\gamma)}}_{\text{(1,1) FORMS}} + \sum_{\substack{\text{ALL RATIONAL}\\ \text{CURVES } I_n}} e^{-\frac{1}{\alpha'}\int_{I_n} X^*(J)} \int_{I_n} b_{(\alpha)} \int_{I_n} b_{(\beta)} \int_{I_n} b_{(\gamma)}$$

KÄHLER FORM ON M

$\tilde{M}$

$$\langle C_{(\alpha)} C_{(\beta)} C_{(\gamma)} \rangle = \int_{\tilde{M}} \Omega \wedge h^m_{(\alpha)} \wedge h^n_{(\beta)} \wedge h^p_{(\gamma)}\, \Omega_{mnp} + 0$$

• EXPLICITLY:

$$M = \sum z_i^5 = 0 \text{ IN } \mathbb{CP}^4 \;;\quad \tilde{M} = M/(\mathbb{Z}_5)^3$$

$$h^{1,1}_{M} = 1 \qquad\qquad h^{2,1}_{\tilde{M}} = 1$$

$$LHS = 5 + \sum_{d=1}^{\infty} n_d \; e^{2\pi i (B+iR) d} \big/ 1 - e^{2\pi i (B+iR) d}$$

n_d ← # RATIONAL CURVES IN M OF DEGREE d

$$\|$$

$$RHS = \int_{\tilde{M}} \Omega \wedge h^m \wedge h^n \wedge h^p \, \Omega_{mnp} \quad \text{(CALCULABLE)}$$

• MATHEMATICS: USE RHS TO CALCULATE n_d
MIRROR MAP

¡ CANDELAS, de la OSSA, [redacted] → EVALUATE n_d $\forall d$!

$N[1] = 2875$

$N[2] = 609250$

$N[3] = 317206375$
$N[4] = 242467530000$

$N[5] = 229305888887625$

$N[6] = 248249742118022000$

$N[7] = 1012162303458000611227 50$

$N[8] = 192323666399988021017600000$

$N[9] = 3672997320939822426172824 56250$

$N[10] = 70428816497845468611348824 97500$

$N[11] = 1354842473951260627644461070753$
2625

$N[12] = 2613295702542192770450894804480$
4464000

$N[13] = 505197638419537782637037675018466$
7397147125

$N[14] = 978499212206555629383944696833124$
7634048766000

$N[15] = 189832167832561310503557582920040$
$8166891095896875 0$

$N[16] = 368803989089118431757579698597536$
$2027667736824217 6000$

$N[17] = 717399930727759234257569473137100$
04388338109828244715250

$N[18] = 139702324572802672116486725323870$
$36642403149195242252302 4000$

$N[19] = 272313733853214419087881850261769$
$9220045019254001137273017158 75$

$N[20] = 531268826499235771139178144834727$
$1406692226792386647145193600000 00$

N[21] = 103728927847051492396380402799008
604532292763825498664000030293037062 5

N[22] = 202671878211989920305509683314638
7700347323659962437116715734670993358000

N[23] = 396247786628512727684920719916328
3567073477542806145226422135415424327750 25

N[24] = 775167105238701766770454626557939
0655289204606015247102789434740891188029 4
40000

N[25] = 151725084642484037758085240878522
6102148852200164615668664257115460366912 5
50781250

N[26] = 297121466846797930826083705852862
1102398131978227247486393997328498860762 6
32723580000

N[27] = 582114569121021145760737952497829
3915325805479753656718257360379617697621 7
27655113181875

N[28] = 114094780815226988137684405525841
58542798682481166805354273750693422717627
2632875273228800000

N[29] = 223713728155276341673221527582384
24014518762633064814122475032120104110952
5031571037498871846 25

N[30] = 438810588514451090792854649487758
06537115971495405339472374540808969269978
19497163028893963925 0000

N[31] = 8610104569211904147712090870715 78
68966940976641592090592658920885614256433
9923985285005562181886 61875

N[32] = 168996192605304399834509858257491
98550034155612528561890625919366340225415
99075883770341714551281418 24000

N[33] = 331798026905466792184470345065598
96878371567056822074166939594149004558156
98438289734881549007175984708 71125