Primer in Electromagnetics

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Vector Algebra

Definitions of Vectors

Electric and magnetic fields are vectors which are defined by direction and magnitude in space $E(x,y,z)$ and $B(x,y,z)$, where we use a Cartesian coordinate system (x,y,z). The distribution of such vectors is called a vector field in contrast to a scalar field such as the distribution of temperature $T(x,y,z)$. In component form such vectors can be written as

$$\mathbf{E} = E_x\mathbf{x} + E_y\mathbf{y} + E_z\mathbf{z}.$$

Vectors can be added by adding their components:

$$\mathbf{E} + \mathbf{B} = (E_x + B_x)\,\mathbf{x} + (E_y + B_y)\,\mathbf{y} + (E_z + B_z)\,\mathbf{z}$$

or multiplied in two different ways:
scalar product, resulting in a scalar:

$$\mathbf{E} \cdot \mathbf{B} = E_xB_x + E_yB_y + E_zB_z = |E||B|\cos\theta$$

where θ is the angle between the vectors;
and the vector product resulting in a vector :

$$\mathbf{E} \times \mathbf{B} = (E_yB_z - E_zB_y, E_zB_x - E_xB_z, E_xB_y - E_yB_x),$$

where $|\mathbf{E} \times \mathbf{B}| = |E||B|\sin\theta$. The resulting vector is orthogonal to both vectors E and B.

Differential Vector Expressions

To describe the variation of scalar and vector fields we define a gradient for scalars

$$\nabla T = \mathrm{grad}T = (\frac{\partial T}{\partial x}, \frac{\partial T}{\partial y}, \frac{\partial T}{\partial z}),$$

which is a vector.
For vectors we define two differential expressions. The first is the divergence of a vector field:

$$\nabla \cdot \mathbf{E} = \mathrm{div}\mathbf{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z},$$

which is a scalar.
Geometrically, the divergence of a vector is the outward flux of this vector per unit volume. As

an example consider a small cube with dimensions $dx, dy. dz$. Put this cube in a uniform vector field and you get zero divergence, because the flux into the cube is equal to the flux out. Now, put the cube into a field free area and place a positive charge into the cube. The flux of fields is all outwards and the divergence is nonzero.

The divergence can be evaluated by integrating over all volume and we get with Gauss's theorem

$$\int_V \nabla\mathbf{E}\, dV = \oint \mathbf{E} \cdot \mathbf{n}\, da,$$

where n is a unit vector normal to the surface and da a surface element. The volume integral becomes an integral over the outer surface.

The second differential expression is the "curl" of a vector:

$$\nabla \times \mathbf{B} = \left(\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z}, \frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x}, \frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} \right)$$

The "curl" of a vector per unit area is the circulation about the direction of the vector. Again we may evaluate the "curl" with the help of Stokes' theorem

$$\int_S (\nabla \times \mathbf{B})_n \cdot d\mathbf{a} = \oint \mathbf{B}\, d\mathbf{s},$$

where da is a surface element with a direction normal to the surface and ds an element of the surface boundary with a direction parallel to the tangent of the boundary line.

Knowledge of the divergence and curl as a function of position defines completely the vector field.

Cylindrical and Polar Coordinates

Cylindrical coordinates (ρ, φ, ζ)

$$\begin{aligned}
\nabla\phi &= \left[\frac{\partial\phi}{\partial\rho}, \frac{1}{\rho}\frac{\partial\phi}{\partial\varphi}, \frac{\partial\phi}{\partial\zeta} \right] \\
\nabla\mathbf{a} &= \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho a_\rho) + \frac{1}{\rho}\frac{\partial a_\varphi}{\partial\varphi} + \frac{\partial a_\zeta}{\partial\zeta} \\
\nabla \times \mathbf{a} &= \left[\frac{1}{\rho}\frac{\partial a_\zeta}{\partial\varphi} - \frac{\partial a_\varphi}{\partial\zeta}, \frac{\partial a_\rho}{\partial\zeta} - \frac{\partial a_\zeta}{\partial\rho}, \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho a_\varphi) - \frac{1}{\rho}\frac{\partial a_\rho}{\partial\varphi} \right] \\
\Delta\phi &= \frac{\partial^2\phi}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial\phi}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2\phi}{\partial\varphi^2} + \frac{\partial^2\phi}{\partial\zeta^2}
\end{aligned}$$

Transformation to cylindrical coordinates (ρ, φ, ζ)

$$\begin{aligned}
(x, y, z) &= (\rho \cos\varphi, \rho \sin\varphi, \zeta) \\
ds^2 &= d\rho^2 + \rho^2 d\varphi^2 + d\zeta^2 \\
dV &= \rho\, d\rho\, d\varphi\, d\zeta
\end{aligned}$$

Polar coordinates (r, φ, θ)

$$\nabla\phi = \left[\frac{\partial\phi}{\partial r}, \frac{1}{r}\frac{\partial\phi}{\partial\varphi}, \frac{1}{r\sin\theta}\frac{\partial\phi}{\partial\theta}, \right]$$

$$\nabla\mathbf{a} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 a_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}(\sin\varphi\, a_\varphi) + \frac{1}{r\sin\theta}\frac{\partial a_\theta}{\partial\theta}$$

$$\nabla\times\mathbf{a} = \left[\begin{array}{c} \frac{1}{r\sin\theta}\left(\frac{\partial(\sin\theta a_\zeta)}{\partial\varphi} - \frac{\partial a_\varphi}{\partial\theta}\right), \\ \frac{1}{r\sin\theta}\left(\frac{\partial a_r}{\partial\theta} - \sin\theta\frac{\partial(r a_\theta)}{\partial r}\right), \\ \frac{1}{r}\left(\frac{\partial}{\partial r}(r a_\varphi) - \frac{\partial a_r}{\partial\varphi}\right). \end{array} \right]$$

$$\Delta\phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\phi}{\partial\varphi^2} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\phi}{\partial\theta}\right)$$

Transformation to polar coordinates (r, φ, θ)

$$(x,y,z) = (r\cos\varphi\sin\theta, r\sin\varphi\sin\theta, r\cos\theta)$$
$$\mathrm{d}s^2 = \mathrm{d}r^2 + r^2\sin^2\theta\,\mathrm{d}\varphi^2 + r^2\,\mathrm{d}\theta^2$$
$$\mathrm{d}V = r^2\sin\theta\,\mathrm{d}r\,\mathrm{d}\varphi\,\mathrm{d}\theta$$

Maxwell's Equations

Coulomb's law	∇E	$=$	$\frac{\rho}{\epsilon_0\epsilon_r}$
	∇B	$=$	0
Faraday's law	$\nabla\times E$	$=$	$-\frac{\partial}{\partial t}B$
Ampere's law	$\nabla\times B$	$=$	$\mu_0\mu_r j + \epsilon_0\epsilon_r\mu_0\mu_r\frac{\partial}{\partial t}E$

The dielectric constant or permittivity of free space is

$$\epsilon_0 = \frac{10^7}{4\pi c^2}\frac{\mathrm{C}}{\mathrm{V\,m}} = 8.854187817\times 10^{-12}\frac{\mathrm{C}}{\mathrm{V\,m}},$$

and the magnetic permeability:

$$\mu_0 = 4\pi\times 10^{-7}\frac{\mathrm{V\,s}}{\mathrm{A\,m}} = 1.256637061\times 10^{-6}\frac{\mathrm{V\,s}}{\mathrm{A\,m}}$$

.

Both constants are related to the speed of light v by

$$\epsilon_0\epsilon_r\mu_0\mu_r v^2 = 1,$$

or in vacuum by

$$\epsilon_0\mu_0 c^2 = 1.$$

Field of Point Charge

We apply Gauss's theorem on a point charge q at rest. The natural coordinate system is the polar system because in this system the only dependence is on the radius and we have therefore for the l.h.s. with $\mathrm{d}V = 4\pi r^2 \mathrm{d}r$ the integral becomes $\int \nabla E\,\mathrm{d}V = \int \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 E_r)\mathrm{d}V = 4\pi R^2 E_r$, where R is the radial distance from the charge. On the r.h.s. we get $\int \frac{\rho}{\epsilon_0 \epsilon_r}\mathrm{d}V = \frac{q}{\epsilon_0 \epsilon_r}$, where we have integrate over the whole charge q. From this the electric field at distance R is $E_r = \frac{1}{4\pi\epsilon_0\epsilon_r}\frac{q}{R^2}$, which is Coulomb's law.

Magnetic Charge

If we perform the same integration to the second of Maxwell's equations we get $B_r = 0$ indicating that no magnetic charges exist.

Induction

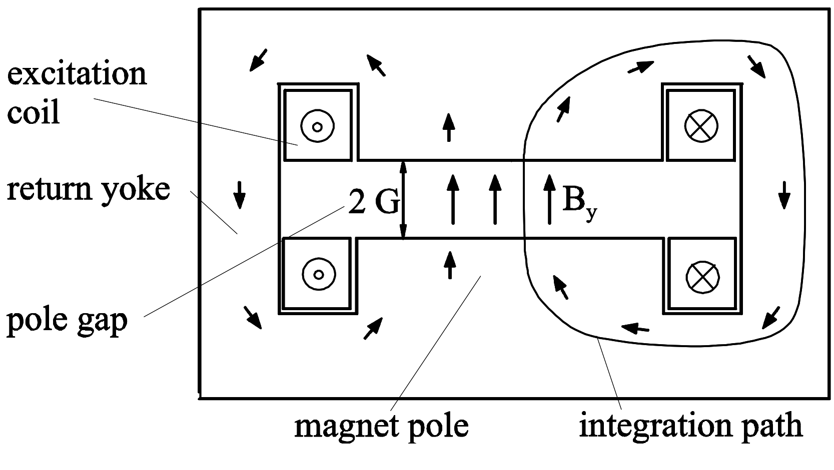
Applying Stoke's theorem to the third of maxwell's equations we get on the left side a line integral along the boundaries of the surface area S which is equivalent to a voltage. On the right hand side we integrate all the magnetic flux traversing the surface S.

$\int_S \nabla \times E\,\mathrm{d}a = \oint E\,\mathrm{d}s = -\int_S \frac{\partial \mathbf{B}}{\partial t}\mathrm{d}a = -\frac{\partial \Phi}{\partial t}$.

A magnetic flux varying in time generates an electromotive force around the area of the flux. Similarly, from Maxwell's fourth equation and the second term on the right hand side we get a magnetic induction from a time varying electric field. This is the principle of induction or that of a transformer.

Magnets

In this last example, we use Maxwell's fourth equation with only the first term on the right hand side. Charged particle beams are deflected in the uniform field of bending magnets. Such fields are called dipole fields and can be generated, for example, between the poles of an electromagnetic bending magnet with a cross section as shown schematically in the following Figure.



Cross section of bending magnet

The magnetic field B is generated by an electrical current I in current carrying coils surrounding the magnet poles. A ferromagnetic return yoke provides an efficient return path for the magnetic flux. The magnetic field is determined by Maxwell's fourth equation with only the first term on the r.h.s. Applying Stokes' theorem, we get $\int_S \nabla \times B\,\mathrm{d}a = \oint B\,\mathrm{d}s = \mu_0\mu_r \int_S j\,\mathrm{d}a = \mu_0\mu_r 2I_{\mathrm{tot}}$, where μ_r is the relative permeability of the ferromagnetic material, j is the current density in the excitation coils, and the integration path as shown. The integration on the r.h.s. is just the total current in both excitation coils. The l.h.s. must be evaluated along an integration path surrounding the excitation coils. We choose an integration path which is convenient for analytical evaluation. Starting in the middle of the lower magnet pole and integrating straight to the middle of the upper pole, we know from symmetry that the magnetic field along this path has only a vertical nonvanishing component $B_y \neq 0$, which is actually the desired field in the magnet gap and $\mu_r = 1$. Within the iron the contribution to the integral vanishes, since we assume no saturation effects and set $\mu_r \Rightarrow \infty$. The total path integral becomes therefore $G B_y = \left[\frac{\mu_0 c}{4\pi}\right] \frac{4\pi}{c} I_{tot}$. In more practical units $I_{tot}\,(\mathrm{Amp}) = \frac{1}{\mu_0} B_y[\mathrm{T}]\, G[\mathrm{m}]$ or $I_{tot}\,(\mathrm{Amp}) = 7957.7\, B_y[\mathrm{T}]\, G[\mathrm{cm}]$.

Lorentz Force

The trajectory of charged particles can be influenced only by electric and magnetic fields through the Lorentz force

$$\mathbf{F} = q\mathbf{E} + q\,(\mathbf{v} \times \mathbf{B}).$$

Guiding particles through appropriate electric and/or magnetic fields is called particle beam optics or beam dynamics.

Energy Conservation

The rate of work done in a charged particle-field environment is defined by the Lorentz force and the particle velocity

$$\mathbf{F}_L \mathbf{v} = (e\mathbf{E} + e[\mathbf{v} \times \mathbf{B}])\mathbf{v}.$$

Noting that $[\mathbf{v} \times \mathbf{B}] \cdot v = 0$ we set $eE \cdot v = j \cdot E$ and the total rate of work done by all particles and fields is the integral over all particles and fields

$$\int \mathbf{j}\mathbf{E}\,\mathrm{d}V = \epsilon_0 \int (c^2 \nabla \times \mathbf{B} - \dot{\mathbf{E}}) \cdot \mathbf{E}\,\mathrm{d}V.$$

With the vector relation $\nabla(a \times b) = b(\nabla \times a) - a(\nabla \times b)$ we get

$$\int \mathbf{j} \cdot \mathbf{E}\,\mathrm{d}V = \epsilon_0 \int \left[c^2 \mathbf{B} \underbrace{\nabla \times \mathbf{E}}_{=-\dot{\mathbf{B}}} - c^2 \nabla(\mathbf{E} \times \mathbf{B}) - \dot{\mathbf{E}}\mathbf{E} \right] \mathrm{d}V = \int \left[\frac{\mathrm{d}u}{\mathrm{d}t} + \mathbf{S} \right] \mathrm{d}V,$$

where $u = \frac{\epsilon_0}{2}(E^2 + [c^2]B^2)$ is the field energy density and the Poynting vector is defined by

$$\mathbf{S} = c^2 \epsilon_0 (\mathbf{E} \times \mathbf{B}).$$

. Applying Gauss's theorem to the vector product we get an expression for the energy conservation of the complete particle-field system

$$\underbrace{\frac{\mathrm{d}}{\mathrm{d}t} \int u\,\mathrm{d}V}_{\text{change of field energy}} + \underbrace{\int \mathbf{j} \cdot \mathbf{E}\,\mathrm{d}V}_{\text{particle energy loss or gain}} + \underbrace{\oint \mathbf{S} \cdot \mathbf{n}\,\mathrm{d}s}_{\text{radiation loss through closed surface } s} = 0.$$

Equation exhibits characteristic features of electromagnetic radiation. Both electric and magnetic fields are orthogonal to each other ($\mathbf{E}\perp\mathbf{B}$), orthogonal to the direction of propagation ($\mathbf{E}\perp\mathbf{n}$, $\mathbf{B}\perp\mathbf{n}$), and the vectors E, B, S form a right handed orthogonal system. For plane waves $n \times E = cB$ and

$$\mathbf{S} = c\epsilon_0 \mathbf{E}^2 \mathbf{n}.$$

The Poynting vector is defined as the radiation energy flow through a unit surface area in the direction n and scales proportionally to the square of the electric radiation field.

Vector and Scalar Potential

Both electric and magnetic fields can be derived from a scalar ϕ and vector A potential:

$$\mathbf{B} = \nabla \times \mathbf{A},$$

$$\mathbf{E} = -\frac{\partial}{\partial t}\mathbf{A} - \nabla\phi.$$

We choose the scalar potential such that $c\nabla A + \frac{1}{c}\frac{\partial}{\partial t}\varphi = 0$, a condition known as the Lorentz gauge. With $\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla\mathbf{A}) - \Delta A$ and inserting the definitions for the potential into the last two Maxwell's equations we get the wave equations

$$\nabla^2\mathbf{A} - \frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} = \frac{\mathbf{j}}{c^2\epsilon_0}$$

for the vector equation and the scalar potential

$$\nabla^2\varphi - \frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2} = -\frac{\rho}{\epsilon_0}.$$

These are the well-known wave equations with the solutions

$$\mathbf{A}(t) = \frac{1}{4\pi c^2\epsilon_0}\int \frac{\mathbf{j}(x,y,z)}{R}\bigg|_{t_r} \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

and

$$\varphi(t) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(x,y,z)}{R}\bigg|_{t_r} \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z.$$

Lienard-Wiechert Potentials

For moving point charges we cannot obtain the potentials by integrating simply over the "volume" of the point charge. We must take the motion of the charge into account. The result of a proper integration are the Lienard-Wiechert potentials.

$$\mathbf{A}(P,t) = \frac{1}{4\pi c\epsilon_0}\frac{q}{R}\frac{\boldsymbol{\beta}}{1 + \mathbf{n}\cdot\boldsymbol{\beta}}\bigg|_{t_r}$$

and

$$\varphi(P,t) = \frac{1}{4\pi\epsilon_0}\frac{q}{R}\frac{1}{1 + \mathbf{n}\cdot\boldsymbol{\beta}}\bigg|_{t_r}.$$

These potentials describe the radiation fields of synchrotron radiation.