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DENSITY OF THE RANGE OF POTENTIAL OPERATORS

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ABSTRACT

Let L be a self-adjoint operator with a closed range in a Hilbert space H and let ψ be a differentiable convex function on H. Under a non resonance assumption, we prove that the range of $L+\partial\psi$ is dense in H.

INTRODUCTION

Let H be a real Hilbert space, let $L : D(L) \subset H \to H$ be a self-adjoint operator with a closed range and let $\psi : H \to \mathbb{R}$ be a twice Gâteaux-differentiable convex function. In [9] Mawhin showed that if there exists real numbers α, β and γ such that $0 < \beta \leq \gamma < \alpha$, $\sigma(L) \cap]-\alpha, 0[= \emptyset$ and, for every $u \in H$,

$$\beta I \leq \psi''(u) \leq \gamma I,$$

then $L+\partial\psi$ is one to one and onto. The following weaker condition were introduced by Dolph [5] in his study of Hammerstein equations

$$(o) \qquad 0 < \underline{\lim}_{|u|\to\infty} \frac{\psi(u)}{|u|^2} \leq \overline{\lim}_{|u|\to\infty} \frac{\psi(u)}{|u|^2} < \frac{\alpha}{2}.$$

If (o) is satisfied, under the supplementary assumption that the right inverse of L is compact, $L+\partial\psi$ is onto (see [4] wich extends some results of [2]). In the present paper we prove that (o) implies that the range of $L+\partial\psi$ is dense in H. We use the dual least action principle of Clarke and Ekeland [3] and the variational principle of Ekeland [6]. The abstract result is applied to periodic solutions of a nonlinear wave equation with a nonmonotone nonlinearity.

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1. A DENSITY THEOREM.

Let H be a real Hilbert space with inner product $(.,.)$ and corresponding norms $|\,.\,|$. Let $L : D(L) \subset H \to H$ be a self-adjoint operator with a closed range and let $\psi : H \to \mathbb{R}$ be a differentiable convex function.

Let α, β, γ and c be real numbers such that $0 < \beta < \gamma < \alpha$ and

(A_1) $\sigma(L) \cap \,]\alpha, 0[\, = \phi$, where $\sigma(L)$ denotes the spectrum of L,

(A_2) for every $u \in H$,

$$\beta \frac{|u|^2}{2} - c \leq \psi(u) \leq \gamma \frac{|u|^2}{2} + c.$$

Let us write

$$K = (L|\, D(L) \cap R(L))^{-1},$$

$$\psi^*(v) = \sup_{u \in H} [\,(v,u) - \psi(u)\,],\ v \in H,$$

and

$$\varphi(v) = \frac{1}{2}(Kv,v) + \psi^*(v),\ v \in R(L).$$

The function ψ^* is the Fenchel transform of ψ.

The present formulation of the "dual action" φ were introduced in [1] for hyperbolic problems and in [7] for hamiltonian systems. See [4] and [8] for other abstract formulations. The following lemma has been widely used in the study of hamiltonian systems (see [7]).

LEMMA. *Under assumptions* A_1 *and* A_2, φ *is coercive on* $R(L)$, *i.e.* $\varphi(v) \to \infty$, $|v| \to \infty$.

<u>Proof</u>. It suffices to observe that A_1 and A_2 imply that

$$\forall v \in R(L) \qquad \frac{1}{\alpha} |v|^2 \leq (Kv,v)$$

and

$$\forall v \in H \qquad \frac{1}{\gamma} \frac{|v|^2}{2} - c \leq \psi^*(v). \qquad \square$$

THEOREM 1. *Under assumptions* A_1 *and* A_2, *if* $\partial\psi$ *is uniformly continuous, the range of* $L + \partial\psi$ *is dense in* H.

<u>Proof</u>. Since, for every $f \in H$, the function $\psi(u) - (f,u)$ has the same properties as $\psi(u)$, it suffices to prove that $0 \in \overline{R(L + \partial\psi)}$.

Let $\varepsilon > 0$ be fixed. By assumption there exists $\delta > 0$ such that, for every $u, v \in H$,

$$|u - v| \leq \delta \Rightarrow |\partial\psi(u) - \partial\psi(v)| \leq \varepsilon.$$

Since φ is coercive by the lemma, it follows from a theorem by Ekeland [6, p. 444] that there exists $v \in R(L)$ such that, for every $h \in R(L)$ and for every $t > 0$,

$$\varphi(v) \leq \varphi(v + th) + \delta t \,|h|.$$

Thus

$$-(Kv,h) \leq \frac{\psi^*(v + th) - \psi^*(v)}{t} + \delta\,|h| + \frac{t}{2}(Kh,h).$$

If $t \downarrow 0$, we obtain

$$-(Kv,h) \leq \delta^+ \psi^*(v,h) + \delta\,|h|.$$

Since $\delta^+ \psi^*(v,.) + \delta\,|\,.\,|$ is positively homogeneous and subadditive, the Hahn-Banach theorem insures the existence of $w \in \operatorname{Ker} L$ such that, for every $h \in H$,

$$(w,h) - (Kv,h) \leq \delta^+\psi^*(v,h) + \delta\,|h|.$$

But then

(1) $\quad -\delta\,|h| \leq \psi^*(v + h) - \psi^*(v) - (u,h)$

where $u = w - Kv$. We shall now use a classical argument in convex analysis. It follows from (1) that the convex sets

$$C_1 = \{(h,s) \in H \times \mathbb{R} : s \geq \psi^*(v + h) - \psi^*(v) - (u,h)\},$$

$$C_2 = \{(h,s) \in H \times \mathbb{R} : s < -\delta\,|h|\}$$

are disjoint. Since C_2 is open there exists a (non vertical) closed hyperplane separating C_1 and C_2. It is then easy to verify that there exists $f \in H$ such that, for every $h \in H$,

$$-\delta\,|h| \leq (f,h) \leq \psi^*(v + h) - \psi^*(v) - (u,h).$$

The first inequality implies that $|f| \leq \delta$, the second that $(u + f) \in \partial\psi^*(v)$ or $v = \partial\psi(u + f)$. By the definition of u, $Lu + \partial\psi(u + f) = 0$. Since $|f| \leq \delta$, $|Lu + \partial\psi(u)| = |\partial\psi(u) - \partial\psi(u + f)| \leq \varepsilon$. □

Remark. Particular cases of theorem 1 were announced in [11] and [12]. The use of the "dual action" φ were suggested to us by J.L. Lions.

2. PERIODIC SOLUTION OF A NONLINEAR WAVE EQUATION.

This section is devoted to the existence of 2π-periodic solutions in t on x of the nonlinear wave equation

$$u_{tt} - u_{xx} - u + \partial j(u) = f(t,x)$$

where $j : \mathbb{R} \to \mathbb{R}$ is convex and differentiable and $f \in H = L^2(]0,2\pi[^2)$.

Let A be the linear operator defined by

$$D(A) = \{u \in C^2(]0,2\pi[^2) : u(0,.) = u(2\pi,.) = u(.,0) = u(.,2\pi) = u_t(0,.) - u_t(2\pi,.) = u_x(.,0) - u_x(.,2\pi) = 0\}$$

$$Au = u_{tt} - u_{xx}.$$

Let us write $A = A^*$. Then A is self-adjoint and $\sigma(A) = 2\mathbb{Z}+1 \cup 4\mathbb{Z}$ consists of eigenvalues which are of finite multiplicity except 0 (see [10]).

Let us define $\psi : H \to \overline{\mathbb{R}}$ by

$$\psi(u) = \int_0^{2\pi}\int_0^{2\pi} j(u(t,x))\,dt\,dx.$$

THEOREM 2. *Assume that there exists* $\beta, \gamma, c \in \mathbb{R}$ *such that* $0 < \beta \leq \gamma < 1$ *and, for every* $u \in \mathbb{R}$,

$$\beta\,\frac{u^2}{2} - c \leq j(u) \leq \gamma\,\frac{u^2}{2} + c,$$

assume further that ∂j *is Lipschitz*ian, then equation

(2) $$Au - u + \partial\psi(u) = f$$

is solvable for f *in a dense subset of* H.

Proof. It suffices to apply theorem 1 with $L = A - I$ and $\alpha = 1$. □

Remark . Theorem 2 applies for example to

$$u_{tt} - u_{xx} - \frac{1}{4} u + \frac{3}{4} \sin u = f(t,x).$$

In this case $\partial j(u) = \frac{3}{4} (u + \sin u)$.

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