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REMARKS ON THE DUAL LEAST ACTION PRINCIPLE

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INTRODUCTION

Let H be a real Hilbert space, let $L : D(L) \subset H \to H$ be a
self-adjoint operator with a closed range and let $\psi : H \to \mathbb{R}$
be a continuous convex function. The surjectivity of $L + \partial\psi$ is
studied under a non resonance condition due to Dolph [8].
The basic tool is the dual least action principle of Clarke
and Ekeland. In contrast to the previous applications of this
principle, we consider the case when the right inverse of L is
not necessarily a compact operator.

1. THE DUAL LEAST ACTION PRINCIPLE.

Let H be a real Hilbert space with inner product $(.,.)$ and
corresponding norm $|.|$. Let $L : D(L) \subset H \to H$ be a self-
adjoint operator with closed range and let $\psi : H \to R$ be a
continuous convex function.

Let α, β, γ and c be real numbers such that $0 < \beta \leqslant \gamma < \alpha$
and

(A_1) $\sigma(L) \cap] -\alpha, 0 [= \phi$. where $\sigma(L)$ denotes the spectrum of L.,

(A_2) for every $u \in H$,

$$\beta \frac{|u|^2}{2} - c \leqslant \psi(u) \leqslant \gamma \frac{|u|^2}{2} + c.$$

Let us write

$$K = (L \mid D(L) \cap R(L))^{-1},$$

$$\psi^*(v) = \sup_{u \in H} [(v,u) - \psi(u)], \quad v \in H.$$

and

$$\varphi(v) = \frac{1}{2} (Kv,v) + \psi^*(v), \quad v \in R(L).$$

The function ψ^* is the Fenchel transform of ψ. The present
formulation of the "dual action" φ was introduced in [5] for
hyperbolic problems and in [9] for hamiltonian systems. See
[7] and [10] for other abstract formulations.

LEMMA 1. *Under assumption* A_2, *if* φ *has a local minimum on* $R(L)$,
then equation

(1) $-Lu \in \partial\psi(u)$

is solvable.

Proof. If φ has a local minimum at $v \in R(L)$, then for every $h \in R(L)$
sufficiently small and for every $t \in]0,1[$, we have

$$\psi^*(v) \leqslant \psi^*(v + th) + t(Kv,h) + \frac{t^2}{2} (Kh,h).$$

Thus

$$-(Kv,h) \leqslant \frac{\psi^*(v + th) - \psi^*(v)}{t} + \frac{t}{2} (Kh,h).$$

If $t \downarrow 0$, we obtain denoting by $\delta^+ \psi^*(v,.)$ the right Gateaux variation at v

$$-(Kv,h) \leqslant \delta^+ \psi^*(v,h).$$

Since $\delta^+ \psi^*(v,.)$ is positively homogeneous and subadditive,
the Hahn-Banach theorem insures the existence of $w \in \operatorname{Ker} L$
such that, for every $h \in H$,

$$(w,h) - (Kv,h) \leqslant \delta^+ \psi^*(v,h).$$

But then

$$(w - Kv,h) \leqslant \psi^*(v + h) - \psi^*(v),$$

i.e. $w - Kv \in \partial\psi^*(v)$. It follows that $v \in \partial\psi(w - Kv)$.
If $u = w - Kv$, $-Lu = v$ and u is a solution of (1). $\square$

The following lemma has been widely used in the study of hamiltonian systems (see [9]).

LEMMA 2. *Under assumptions* A_1 *and* A_2, φ *is coercive on* $R(L)$, *i.e.* $\varphi(v) \to \infty$, *as* $|v| \to \infty$.

Proof. It suffices to observe that A_1 and A_2 imply that

$$\forall \, v \in R(L) \quad -\frac{1}{\alpha}|v|^2 \leqslant (Kv,v)$$

and

$$\forall \, v \in H \quad \frac{1}{\gamma}\frac{|v|^2}{2} - c \leqslant \psi^*(v). \qquad \square$$

2. SURJECTIVITY THEOREMS.

The following result generalizes theorem 2.1 of [8] and theorem 3.2 of [8]. It extends theorem 1 of [7].

THEOREM 1. *Under assumptions* A_1 *and* A_2, *if* $\sigma(L) \cap]-\infty,0[$ *consists of isolated eigenvalues with finite multiplicity, then* $L + \partial\psi$ *is onto.*

Proof. Since, for any $f \in H$, the function $\psi(u) - (f,u)$ has the same properties as $\psi(u)$, it suffices to prove that (1) is solvable.

By assumption $\frac{1}{2}(Kv,v)$ is weakly lower semi-continuous (w.l.s.c.). Therefore φ itself is w.l.s.c. By lemma 2, φ has a minimum on $R(L)$ and, by lemma 1, (1) is solvable. $\square$

The following result extends theorem 3.7 of [1].

THEOREM 2. *Under assumptions* A_1 *and* A_2, *if*

(a) ψ *is differentiable and* $\partial\psi$ *is Lipschitzian with constant* k

(b) $\sigma(L) \cap]-k,0[$ *consists of isolated eigenvalues with finite multiplicity, then* $L + \partial\psi$ *is onto.*

Proof. As in theorem 1, it suffices to prove that (1) is solvable.

If $f_i \in \partial\psi^*(v_i)$, $i = 1,2$, we have $v_i = \partial\psi(f_i)$, $i = 1,2$. By Corollary 10 of [3], assumption (a) implies that

$$(v_1 - v_2, f_1 - f_2) \geqslant \frac{1}{k}|v_1 - v_2|^2.$$

Thus $\varphi_1(v) = \psi^*(v) - \frac{1}{k}\frac{|v|^2}{2}$ is convex.

Let $\{P_\lambda : \lambda \in \mathbb{R}\}$ be the spectral resolution of L and let us write

$$Q_1 = \int_{]-\infty,-k]} dP_\lambda, \, Q_2 = \int_{]-k,\infty[} dP_\lambda, u_i = Q_i u, i=1,2, u \in H.$$

It follows from assumption (b) that $\varphi_2(v) = \frac{1}{2}(Kv_2,v_2)$ is w.l.s.c. on $R(L)$. Moreover, for any $v \in R(L)$,

$$(Kv_1,v_1) \geqslant -\frac{1}{k}|v_1|^2 \geqslant -\frac{1}{k}|v|^2.$$

Thus $\varphi_3(v) = \frac{1}{2}(Kv_1,v_1) + \frac{1}{k}\frac{|v|^2}{2}$ is convex. Finally $\varphi = \varphi_1 + \varphi_2 + \varphi_3$ is w.l.s.c. and, by lemmas 1 and 2, (1) is solvable. $\square$

As an obvious consequence of theorem 2 we obtain :

COROLLARY 1. *Under assumptions* A_1 *and* A_2, *if* ψ *is differentiable*

and $\partial\psi$ *is Lipschitzian with constant* α, *then* $L + \partial\psi$ *is onto.*

Remark. Corollary 1 generalizes theorem 1.2 of [8], theorem 1 of [11] and extends theorem I.12 of [4].

3. PERIODIC SOLUTIONS OF A NONLINEAR HYPERBOLIC EQUATION.

This section is devoted to the existence of 2π-periodic solutions in t and x of the nonlinear hyperbolic equation

$$u_{tt} - u_{xx} + \lambda u + \partial j(u) = f(t,x)$$

where $j : \mathbf{R} \to \mathbf{R}$ is convex and $f \in H = L^2([0,2\pi])^2$. We shall only consider the case when $\lambda = 1$. The other cases are left to the reader. The case when $\lambda = 0$ is treated in [7].

Let A be the linear operator defined by

$$D(A) = \{u \in C^2([0,2\pi] : u(o,.)-u(2\pi,.) = u_t(o,.)-u_t(2\pi,.) = u(.,o)-u(.,2\pi) = u_x(.,o)-u_x(.,2\pi) = o\}$$

$$A u = u_{tt} - u_{xx}.$$

Let us write $A = A^*$. Then A is self-adjoint and $\sigma(A) = 2Z + 1 \cup 4Z^-$ consists of eigenvalues wich are of finite multiplicity except 0 (see [12]).

Let us define $\psi : H \to \overline{\mathbf{R}}$ by

$$\psi(u) = \int_0^{2\pi} \int_0^{2\pi} j(u(t,x))dt\, dx.$$

The following theorem extends the results of [13].

THEOREM 3. *Assume that there exists* β, γ *and* $c \in \mathbf{R}$ *such that* $0 < \beta \leqslant \gamma < 2$ *and, for every* $u \in \mathbf{R}$,

$$(2) \qquad \beta\frac{u^2}{2} - c \leqslant j(u) \leqslant \gamma\frac{u^2}{2} + c,$$

then equation

$$Au + 3u + \partial\psi(u) = f$$

is solvable for every $f \in H$.

Proof. It suffices to apply theorem 1 with $L = A + I$ and $\alpha = 2$.

□

4. THE GROWTH OF $\partial\psi$.

A sharp estimate of the growth of $\partial\psi$ under assumption A_2 is given.

PROPOSITION 1. *Under assumption* A_2, *thereexists* $c' \in \mathbf{R}$ *such that, for every* $f, u \in H$,

$$(3) \qquad f \in \partial\psi(u) \Rightarrow |f| \leqslant 2\gamma |u| + c'.$$

Proof. Let $u \in H$ and $f \in \partial\psi(u)$. Assuming $f \neq 0$, let us write $g = f/|f|$. We have, for every $t \in \mathbf{R}$,

$$(f, tg-u) + \psi(u) \leqslant \psi(tg)$$

or

$$0 \leqslant \psi(tg) - t|f| + (f,u) - \psi(u).$$

By assumption

$$0 \leqslant t^2 \frac{\gamma}{2} + c - t|f| + |u||f| - c''$$

where $c'' = \inf_{u \in H} \psi(u) > -\infty$. Then

$$|f|^2 \leqslant 4\frac{\gamma}{2}(c - c'' + |u||f|)$$

and (3) follows easily.

□

Remark. The argument of the proof is due to Brézis and Nirenberg
[4,p.312].

The following example shows that estimate (3) is sharp.

Let $\gamma \in \,]1/2,1\,[$, $\beta \in \,]0,2\gamma-1\,[$, $p = \dfrac{\gamma - \beta}{1 - \gamma}$ and

$q = \dfrac{p - \beta}{p - 1}$. Let us define inductively $t_0 = \gamma/\beta$, $u_n = pt_n$ and

$t_{n+1} = u_n/\beta$. Let us define $r : \mathbb{R} \to \mathbb{R}$ by

$$r(u) = \gamma u, \quad u \in \,]-\infty,1\,]$$
$$= \gamma , \quad u \in \,]1,t_0\,]$$
$$= qu + (1 - q)u_n, \quad u \in \,]t_n,u_n\,]$$
$$= u_n, \quad u \in \,]u_n,t_{n+1}\,] .$$

If $\psi(u) = \displaystyle\int_0^u r(s)ds$, for any $u \in \mathbb{R}$,

$$\beta \frac{u}{2}^2 < \psi(u) < \gamma \frac{u}{2}^2 .$$

But, for any $n \in \mathbb{N}$, $\partial\psi(u_n) = u_n$ and $u_n \to \infty$, $n \to \infty$.

Moreover, if $L : \mathbb{R} \to \mathbb{R}$ is defined by $Lu = - u$, assumptions

A_1 and A_2 are satisfied with $\alpha = 1$, and, for any $n \in \mathbb{N}$,

$Lu_n + \partial\psi(u_n) = 0$. Thus conditions A_1 and A_2 doesn't imply any

a priori bound for the solutions of (1) (see also [8]and [2]).

[1] H. AMMAN, Saddle points and multiple solutions of diffe-
rential equations, *Math. Z.* 169(1979)127-166.

[2] A. BAHRI and J.M. MOREL, Image de la somme de deux opéra-
teurs, *C.R. Acad. Sci. Paris* 287(1978)A719-722.

[3] J.B. BAILLON and G. HADDAD, Quelques propriétés des opéra-
teurs angle-bornés et n-cycliquement monotones, *Isr.
J. Math.* 26(1977)137-150.

[4] J.H. BREZIS and L. NIRENBERG, Characterizations of the
ranges of some nonlinear operators and applications
to boundary value problems, *Ann. Scuola Norm. Pisa*
5(1978)225-326.

[5] H. BREZIS, J.M. CORON and L. NIRENBERG, Free vibrations
of a nonlinear wave equation and a theorem of
P. Rabinowitz, preprint.

[6] A. CASTRO, Hammerstein equations with indefinite kernel,
Internat. J. Math. & Math. Sci. 1(1978)187-201.

[7] J.M. CORON, Résolution de l'équation $Au + Bu = f$ où A
est linéaire auto-adjoint et B est un opérateur
potentiel non linéaire, *C.R. Acad. Sci. Paris*
288(1979)A805-808.

[8] C.L. DOLPH, Nonlinear integral equations of the Hammerstein
type, *Trans. Amer. Math. Soc.* 66(1949)289-307.

[9] I. EKELAND, Oscillations de systèmes hamiltoniens non
linéaires III, preprint.

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[10] I. EKELAND and J.M. LASRY, to appear.

[11] J. MAWHIN, Contractive mappings and periodically perturbed conservative systems, *Arch. Math.* 12(1976)67-73.

[12] J. MAWHIN, Solutions périodiques d'équations aux dérivées partielles hyperboliques non linéaires, in "Mélanges Vogel", Rybak, Janssens et Jessel ed., Presses Univ. Bruxelles, Bruxelles, 1978, 301-319.

[13] J. MAWHIN, Periodic solutions of nonlinear dispersive wave equations, in "Constructive methods for nonlinear boundary value problems and nonlinear oscillations", Albrecht,Collatz and Kirch gässner ed., Birkhaüser Verlag, Basel, 1979, 102-109.