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A FREE BOUNDARY PROBLEM ARISING IN PLASMA PHYSICS

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§4. A free boundary problem arising in Plasma Physics.

In this section we will study, with the aid of the abstract theorem 2.5, a free-boundary problem arising in Plasma physics. Our discussion follows a paper of Puel [Pu]. For other existence results concerning this problem see, for example, [Te 1,2], [Be-Bre], [Sc] and [Amb-Ma 1].

We study the equilibrium of a plasma confined in a toroidal cavity, whose meridian section is a bounded domain Ω. From the mathematical point of view, we have to study the following free-boundary problem:

Given $\Omega \subset \mathbb{R}^n$, and numbers $I, \lambda > 0$, to find $k \in \mathbb{R}$, an open subset $\Omega_p \subset \Omega$ and a C^1 function u such that

$$(4.1)\qquad \begin{cases} -\Delta u = \lambda u & \text{in } \Omega_p \\ \Delta u = 0 & \text{in } \Omega_v \equiv \Omega \setminus \Omega_p \\ u = 0 & \text{on } \partial\Omega_p \\ u = -k & \text{on } \partial\Omega \\ -\int_{\partial\Omega} \frac{\partial u}{\partial \nu}\, d\sigma = I & (\nu \text{ outer unit normal at } \partial\Omega) \end{cases}$$

The region Ω_p is the part of Ω filled by the plasma and Ω_v corresponds to the vacuum.

Let $u^+(x) = \max(u(x), 0)$. If ~~u is a solution of~~ the pair (k, u) satisfy

$$(4.2)\qquad \begin{cases} -\Delta u = \lambda u^+ & \text{in } \Omega \\ u = -k & \text{on } \partial\Omega \\ -\int_{\partial\Omega} \frac{\partial u}{\partial \nu}\, d\sigma = I \end{cases}$$

then u solves (4.1) with the same k, Ω_p being $\{x \in \Omega : u(x) > 0\}$. Of course, we will have free-boundary, i.e. $\Omega_p \subset\subset \Omega$ provided $k > 0$.

First of all, we remark that (4.2) has no solutions for $\lambda \le 0$. In fact if $\lambda \le 0$, every solution of

$$(4.3)\qquad \begin{cases} -\Delta u = \lambda u^+ \\ u = -k \end{cases}$$

is such that $\frac{\partial u}{\partial \nu} \ge 0$ on $\partial\Omega$.

From now on, we will take $\lambda > 0$.

The relation between (4.3) and (4.2) is given by:

4.1. Proposition. Given k, if u is a solution of (4.3) and $u \not\equiv -k$, then $\exists\, \alpha > 0$ such that $(\alpha k, \alpha u)$ is a solution of (4.2).

Proof. Remark that if u solves (4.3) and $u^+ \equiv 0$ then $u \equiv -k$. Hence $u \not\equiv -k$ implies $u^+ \not\equiv 0$, and it follows:

$$-\int_{\partial\Omega}\frac{\partial u}{\partial\nu}\,d\sigma = -\int_{\Omega}\Delta u = \lambda\int_{\Omega}u^{+} := \beta > 0$$

By a direct calculation, we have that $\tilde{u} = \alpha u$, $\alpha = I/\beta$ verifies

$$\begin{cases} -\Delta\tilde{u} = \lambda\tilde{u}^{+} & \text{in } \Omega \\ \tilde{u} = -\alpha k & \text{on } \partial\Omega \\ -\int_{\partial\Omega}\frac{\partial\tilde{u}}{\partial\nu}\,d\sigma = I & \end{cases}$$

This completes the proof. ∎

According to Proposition 1.1, we will study problem (4.3) for given λ, k.

Set

$$v = u + k$$

one has

$$(4.4)\qquad \begin{cases} -\Delta v = \lambda(v-k)^{+} & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega \end{cases}$$

Let λ_1 be the first eigenvalue of $-\Delta$ with zero Dirichlet boundary conditions and denote by φ_1 the corresponding eigenfunction, with $\varphi_1 > 0$ in Ω and $\|\varphi_1\|_2 = 1$.

We will discuss (4.4) distinguishing the cases: $\lambda < \lambda_1$, $\lambda = \lambda_1$ and $\lambda > \lambda_1$.

In the former, it is easy to verify the following:

For every k (4.4) has an unique solution, which, for $k \geq 0$ is $u \equiv -k$

If $\lambda = \lambda_1$, problem (4.4) has the only solution $u \equiv -k$ if $k > 0$, no solutions if $k < 0$ and for $k = 0$ the solution $\alpha\varphi_1$, $\alpha > 0$, besides the trivial one.

We will now study more in details (4.4) when $\lambda > \lambda_1$. We consider separately the cases: $k > 0$, $k = 0$, $k < 0$.

<u>Case $k>0$</u>: The solutions of (4.4) are the stationary points on $E := W_0^1(\Omega)$ of

$$f(v) = \frac{1}{2}\|v\|^2_{1,2} - \frac{1}{2}\lambda\int_{\Omega}[(v-k)^{+}]^2$$

We will show that f verifies the assumptions of Theorem 2.5. First of all, (f1) holds because the nonlinearity $g(s) := (s-k)^{+}$ is identically 0 for $s \leq k$ (recall that $k>0$!). To prove (f2), we take $v = \rho\varphi_1$:

$$f(\rho\varphi_1) = \frac{1}{2}\rho^2\|\varphi_1\|^2_{1,2} - \frac{1}{2}\lambda\int_{\Omega}[(\rho\varphi_1 - k)^{+}]^2$$

Since $\|\varphi_1\|^2_{1,2} = \lambda_1\|\varphi_1\|_2^2 = \lambda_1$, we get:

$$f(\rho\varphi_1) = \frac{1}{2}\rho^2\lambda_1\left(1 - \frac{\lambda}{\lambda_1}\int[(\varphi_1 - \frac{k}{\rho})^{+}]^2\right)$$

Passing to the limit for $\rho \to \infty$ and taking into account that $\lambda > \lambda_1$, we deduce that $f(\rho\varphi_1) \to -\infty$, as required.

It remains to show that (P-S) holds. First remark that the operator B defined by:

$$((Bv, w)) = \int v^{+} w \qquad w \in E$$

is compact. Moreover $f'(v) = v - \lambda B(v-k)$.
Let $v_n \in E$ be such that $f'(v_n) \to 0$. If $\|v_n\|_{1,2}$ is bounded we have done directly. Otherwise, let $z_n := v_n/\|v_n\|_{1,2}$. From $f'(v_n) \to 0$ it follows:

$$(4.5) \qquad z_n - \lambda B\left(z_n - \frac{k}{\|v_n\|_{1,2}}\right) \longrightarrow 0$$

Since $\|z_n\|_{1,2} = 1$, B is compact and $\|v_n\|_{1,2} \to \infty$, we deduce from (4.5) $z_n \to \bar z$ strongly in E (without relabeling) and one has

$$\bar z = \lambda B(\bar z)$$

In other words $\bar z \in E$ is solution of

$$(4.6) \qquad \begin{cases} -\Delta \bar z = \lambda \bar z^+ & \text{in } \Omega \\ \bar z = 0 & \text{on } \partial\Omega \end{cases}$$

The maximum principle implies in (4.6) $\bar z \geq 0$ and $\|z_n\|_{1,2} = 1$ gets $\bar z \not\equiv 0$. Then in (4.6) we have, in fact, $\bar z > 0$ in Ω and:

$$-\Delta \bar z = \lambda \bar z \quad \text{in } \Omega$$
$$\bar z = 0 \quad \text{on } \partial\Omega$$

which is in contradiction with the fact that $\lambda > \lambda_1$.
We are now in position to apply Theorem 2.5. It follows (4.4) has at least one non-trivial solution for $\lambda > \lambda_1$, $k > 0$, to which corresponds a non-constant solution u of (4.3).

Case $k < 0$: Let v be a solution of (4.4). Then $v \geq 0$ in Ω and hence $(v-k)^+ = v-k$

and

$$-\Delta v = \lambda(v-k) \quad \text{in } \Omega$$
$$v = 0 \quad \text{on } \partial\Omega$$

By easy computations it follows:

$$\lambda_1 \int v\varphi_1 = \lambda \int v\varphi_1 - \lambda \int k \int \varphi_1$$

or else

$$(\lambda - \lambda_1) \int v\varphi_1 = \lambda k \int \varphi_1$$

Since $\lambda > \lambda_1$, $\varphi_1 > 0$ in Ω, $v \geq 0$ in Ω and $k < 0$, we have a contradiction.
Lastly if $k = 0$, the same argument shows that $u \equiv 0$.

In conclusion we can state:

4.2 Theorem. For all $\lambda > 0$, $I > 0$, the free-boundary problem (4.1) has at least one solution. More precisely:
1) if $\lambda \leq \lambda_1$ such solution is positive in Ω and hence $\Omega_p = \Omega$ (no free boundary)
2) if $\lambda > \lambda_1$ then every solution of (4.1) takes negative values on $\partial\Omega$ and hence $\Omega_p \subset\subset \Omega$.

We end this section by showing a uniqueness result:

4.3. Proposition. If $0 < \lambda < \lambda_2$ then (4.1) has a unique solution.
Proof. Let u_1 and u_2 be two solutions of (4.1) in correspondence to a given $\lambda < \lambda_2$. By the preceding theorem $u_i|_{\partial\Omega} := -k_i$ have the same sign and

hence $\tilde{u}_i := \frac{u_i}{-k_i}$ satisfy

$$\begin{cases} -\Delta \tilde{u}_i = \lambda\, \tilde{u}_i^+ & \text{in } \Omega \\ \tilde{u}_i = 1 & \text{on } \partial\Omega \end{cases}$$

Therefore one has:

$$(4.7) \qquad \begin{cases} -\Delta (\tilde{u}_1 - \tilde{u}_2) = \lambda (\tilde{u}_1^+ - \tilde{u}_2^+) & \text{in } \Omega \\ \tilde{u}_1 - \tilde{u}_2 = 0 & \text{on } \partial\Omega \end{cases}$$

Set

$$h(x) = \begin{cases} 0 & \text{if } \tilde{u}_1(x) = \tilde{u}_2(x) \\ \dfrac{\tilde{u}_1^+(x) - \tilde{u}_2^+(x)}{u_1(x) - u_2(x)} & \text{otherwise} \end{cases}$$

From (4.7) it follows, setting $w = \tilde{u}_1 - \tilde{u}_2$:

$$(4.8) \qquad \begin{cases} -\Delta w = \lambda\, h(x)\, w & \text{in } \Omega \\ w = 0 & \text{on } \partial\Omega \end{cases}$$

Let us denote by $\mu_i(h)$ the eigenvalues of (4.8). If $w \not\equiv 0$ then $\lambda = \mu_i(h)$ for some $i = 1, 2, \dots$ Now, from $0 \le h \le 1$ it follows that $\mu_2(h) \ge \mu_2(1) = \lambda_2$, so that if $\lambda < \lambda_2$ we get $\lambda = \mu_1(h)$ and w has the same sign in Ω. For ex., let $\tilde{u}_1 \ge \tilde{u}_2$ and set

$$\Omega_i = \{ x \in \Omega : \tilde{u}_i(x) > 0 \}$$

From $\tilde{u}_1 \ge \tilde{u}_2$ in Ω, we infer $\Omega_1 \supset \Omega_2$. But λ is also the first eigenvalue of $-\Delta$ on Ω_1 and Ω_2 (with zero Dirichlet boundary data) and hence $\Omega_1 = \Omega_2$. ■

§ 5. Constrained extrema

Let M be a (non-empty) subset of the Banach space E and $f : E \to \mathbb{R}$. By a local minimum of f constrained on M we mean a point $u^* \in M$ for which the following holds: there exists a neighborhood U of u^* such that $f(u^*) \le f(u)$ for all $u \in U \cap M$. Similarly we define a local maximum, a minimum or a maximum of f constrained on M.

It is easy to see that, in analogy to Theorem 1.1, the following holds.

5.1. Proposition. Let $M \subset E$ be weakly closed and $f : E \to \mathbb{R}$ be w.l.s.c. If $f(u) \to +\infty$ as $\|u\| \to \infty$, $u \in M$, then f has a minimum constrained on M.

~~In view of later applications, we consider the following situation: Let $h : E \to \mathbb{R}$ be given~~

We consider here a specific situation, which will be interesting for later applications.

Let E be a Hilbert space with scalar product $(\cdot,\cdot)$ and norm $\|\cdot\|$. Let $h \in C^1(E;\mathbb{R})$ and suppose $M := \{ u \in E : h(u) = 0 \} \ne \emptyset$.

If $h'(u) \neq 0$ $\forall u \in M$, then M is a C^1-submanifold of E of codimension 1. Let $u \in M$ and set

$$TM_u = \{ v \in E \;:\; (h'(u), v) = 0 \}$$

and

$$\nu(u) = \frac{h'(u)}{\| h'(u) \|^2}$$

Define, for $u \in M$ fixed, the mapping $\psi = \psi_u : E \to E$ by setting:

$$\psi(x) = x - u - (h'(u), x-u) + h(x)\,\nu(u)$$

Such $\psi \in C^1(E,E)$ and ~~is a~~ satisfies

(5.1) $\qquad d\psi(u) = id_E$

Moreover, from $(h'(u), \psi(x)) = h(x)$ it follows that $\psi(x) \in TM_u$ if and only if $x \in M$. Using (5.1) we can deduce ψ is a local diffeomorphism at $x=u$, and hence there exist: a neighborhood V of 0, a neighborhood U of u (depending on u) and a mapping $\varphi = \varphi_u : V \to U$ such that $\varphi \circ \psi = \psi \circ \varphi = id$. Denoted again by φ the restriction to $TM_u \cap V$ of the preceding mapping, one has $\varphi(v) \in M$ $\forall v \in V \cap TM_u$. Moreover φ is C^1 and it results, by (5.1):

(5.2) $\qquad d\varphi(0) \cdot v = v \qquad \forall v \in TM_u$.

Now, let $f \in C^1(E, \mathbb{R})$ and $u^* \in M$ be such that u^* is a constrained extremum for f on M.

Then $f \circ \varphi_{u^*} : V \cap TM_{u^*} \to \mathbb{R}$ has a stationary point at $u = 0$. We write φ for φ_{u^*}. From the preceding remarks, it follows:

$$df(u^*) \cdot d\varphi(0) \cdot v = 0 \qquad \forall v \in TM_{u^*}$$

Taking into account (5.2) and the definition of TM_{u^*}, we infer:

$$(f'(u^*), v) = 0$$

for every v such that

$$(h'(u^*), v) = 0$$

Hence

(5.3) $\qquad f'(u^*) = \lambda\, h'(u^*)$, $\quad \lambda = \dfrac{(f'(u^*), h'(u^*))}{\| h'(u^*) \|^2}$

From now on, we will assume

(h1) $\qquad h \in C^{1,1}(E, \mathbb{R})$ and $h'(u) \neq 0$ for every $u \in h^{-1}(0)$.

For such a h we consider a map $F_{f,h}$ defined on the set $\{ u \in E : h'(u) \neq 0 \}$ by setting

$$(5.4)\qquad F(u) = F_{f,h}(u) = f'(u) - \frac{(f'(u),\, h'(u))}{\|h'(u)\|^2}\, h'(u)$$

By a critical point of f on M we mean a $u \in M$ such that

$$F_{f,h}(u) = 0$$

Sometimes we will also use the notation f'_M for $F_{f,h}$.
For example, a local extremum of f constrained on M is a critical point of f on M.
In the following the subscripts f,h in $F_{f,h}$ will be understood.

In analogy to what done in § 2, we will need some deformations. Let $x \in M$ be given. Define $\alpha_x(t) = \alpha(t,x)$ as the solution of the Cauchy problem

$$\begin{cases} \dfrac{d\alpha}{dt} = -F(\alpha) \\ \alpha(0) = x \end{cases}$$

Let us remark that from the definition (5.4) of F it follows:

$$(F(u),\, h'(u)) = 0$$

Then $\alpha(t,x) \in M$ (where it is defined). In fact, the mapping $h(\alpha(\cdot,x))$ satisfies:

$$\frac{d}{dt} h(\alpha(t,x)) = (h'(\alpha), \alpha') = -(h'(\alpha), F(\alpha)) = 0,$$

and hence $g(\alpha(t,x)) = \text{const.} = g(\alpha(0,x)) = g(x) = 0$.
Taking into account this fact, as well as that F is locally Lipschitz continuous, we can deduce that α is well defined in $(t^-(x), t^+(x))$ (cf. notations in § 2).
Moreover it results:

5.2 Lemma. The function $f(\alpha(t,x))$ is non-increasing as function of t.

Proof One has:

$$\frac{d}{dt} f(\alpha(t,x)) = -(f'(\alpha), F(\alpha))$$

Since $(F(\alpha), h'(\alpha)) = 0$, it follows:

$$\frac{d}{dt} f(\alpha(t,x)) = -(f'(\alpha) - \lambda h'(\alpha), F(\alpha))$$

Taking $\lambda = (f'(\alpha), h'(\alpha)) / \|h'(\alpha)\|^2$, we have

$$(5.4)\qquad \frac{d}{dt} f(\alpha(t,x)) = -\|F(\alpha(t,x))\|^2 \leq 0$$

as required. ∎

From (5.4) we also infer:

$$(5.5)\qquad f(\alpha(t_n,x)) - f(\alpha(t_m,x)) = -\int_{t_n}^{t_m} \frac{d}{dt} f(\alpha(t,x)) = \int_{t_n}^{t_m} \|F(\alpha(t,x))\|^2$$

5.3 Lemma. If f is bounded from below on M, then $t^+(x) = +\infty$ $\forall x \in M$.

Proof. We use (5,5) with $t_n = 0$ and $t_m = t$:

$$f(x) - f(\alpha(t,x)) = \int_0^t \|F(\alpha(t,x))\|^2$$

Since f is bounded from below on M and $\alpha(t,x) \in M$, it follows:

$$\int_0^t \|F(\alpha(t,x))\|^2 \le \text{const}$$

Let $t^+(x) < \infty$ for some $x \in M$; if $t_n \uparrow t^+(x)$, we have

$$\alpha(t_n,x) - \alpha(t_m,x) = -\int_{t_n}^{t_m} \frac{d}{dt}\alpha(t,x) = \int_{t_n}^{t_m} F(\alpha(t,x))$$

Hence:

$$\|\alpha(t_n,x) - \alpha(t_m,x)\| \le \int_{t_n}^{t_m} \|F(\alpha(t,x))\| \le$$

$$\le |t_n - t_m|^{1/2} \left\{ \int_{t_n}^{t_m} \|F(\alpha(t,x))\|^2 \right\}^{1/2} \le$$

$$\le \text{const.}\ |t_n - t_m|$$

Thus $\alpha(t_n,X)$ is a Cauchy sequence and $\alpha(t_n,x)$ converges to some $u \in M$. This leads to a contradiction, as in Lemma 2.2. ■

We now introduce the compactness condition (P-S).

Condition (P-S): we say that the pair (f,h) (or (f,M)) satisfies (P-S) if: every sequence $u_n \in M$ such that $f(u_n)$ is bounded and $F_{f,h}(u_n) \to 0$ has a converging subsequence.

We will use the following notations:

$$M_a = \{ x \in M : f(x) \le a \}$$

$$K = \{ x \in M : f'_{|M}(x) = 0 \}$$

$$K_c = \{ x \in K : f(x) = c \}$$

We are in position to state:

5.4. Theorem. Suppose $f, h \in C^{1,1}(E,\mathbb{R})$, satisfy (h1) and (P-S). If f is bounded from below on $M = h^{-1}(0)$, then f attains its minimum $\underline{u}$ constrained on M; such u is a critical point of f on M.

Proof. Let $\beta := \inf\{f(u) : u \in M\}$. Suppose, first, that $\exists\, \varepsilon > 0$ such that $M_{\beta+\varepsilon} \cap K = \emptyset$. Let $x \in M_{\beta+\varepsilon}$ be fixed. Since f is bounded from below on M, we know (Lemma 5.3) that $t^+(x)$. Moreover, as in such Lemma, we have :

$$\int_0^\infty \|F(\alpha(t,x))\| < \infty$$

Taking a sequence $t_n \uparrow \infty$, with $\|F(z_n)\| \to 0$, $z_n := \alpha(t_n, x)$, we have: $z_n \in M_{\beta+\varepsilon}$ (by Lemma 5.2). Using (P-S) we infer $z_n \to \bar z$ (along a subsequence) and $\bar z \in M_{\beta+\varepsilon} \cap K$, a contradiction. Therefore, we have that:

$$M_{\beta+\frac{1}{n}} \cap K \neq \emptyset$$

Again a direct application of (P-S) leads to $K_\beta \neq \emptyset$, as required. ■

~~[illegible]~~

5.5. Remark. From the proof above, it is clear that it is enough (P-S) to be satisfied in $M_{\beta+\varepsilon}$: namely, it is sufficient to consider sequences $u_n \in M_{\beta+\varepsilon}$.

As application of theorem 5.4 we have:

5.6 Theorem. Let $f \in C^{1,1}(E, \mathbb{R})$ be such that:

(f5.1) $f'(u)$ is compact and $f'(u) \neq 0$ $\forall u \neq 0$;

(f5.2) $f(0) < 0$ $\forall u \neq 0$

Then the eigenvalue problem

(5.6) $f'(u) = \lambda u$ $\|u\| = 1$

has at least one solution.

Proof. We will apply theorem 5.4, taking $M = S = \{x \in E : \frac{1}{2}\|x\|^2 = \frac{1}{2}\}$. Remark that in this case it results

$$f_S'(u) = F(u) = f'(u) - (f'(u), u)\, u$$

First of all, it is easy to see that f is bounded from below on S. In fact, if $u_n \in S$ and $f(u_n) \to -\infty$, we have: $u_n \rightharpoonup \bar u$ and $f(u_n) \to f(\bar u)$ (since f' is compact, then f turns out to be weakly-continuous, cf. [Kr] or [Va]), which is impossible.

~~To prove (P-S), we take, according to Remark 5.5, a sequence $u_n \in S$ such that, for some α~~

~~[illegible]~~

We prove now (P-S). By (f5.2) it follows that $\beta := \inf\{f(u) : u \in S\} < 0$. Hence, according to Remark 5.5, we fix $\alpha \in (\beta, 0)$ and take a sequence $u_n \in S$, with $f(u_n) \leq \alpha < 0$ and such that $f_S'(u_n) \to 0$. Passing eventually to a subsequence, we can assume $u_n \rightharpoonup \bar u$ and $f(u_n) \to f(\bar u)$; since $f(\bar u) \leq \alpha < 0$, (f5.1) implies $f'(\bar u) \neq 0$. Then from $f_S'(u_n) \to 0$ we deduce:

$$(f_S'(u_n), f'(u_n)) = \|f'(u_n)\|^2 - (f'(u_n), u_n)^2$$

Since f' is compact, the left hand-side tends to 0 and it follows:

$$\|f'(\bar u)\|^2 = (f'(\bar u), \bar u)^2$$

Using again (f 5.1) we infer $(f'(\bar u), \bar u) \neq 0$ and hence $a_n := (f'(u_n), u_n) \neq 0$ for n large, and $a_n \to \bar a$. Then one has:

$$u_n = \frac{1}{a_n}\left(f'(u_n) - f'_S(u_n)\right) \longrightarrow \frac{f'(\bar u)}{\bar a}$$

This completes the proof. ■

Theorem 5.6 can be applied to solve nonlinear eigenvalue problems of the type:

$$(5.7) \qquad \begin{cases} -\Delta u = \lambda\, g(x,u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

Precisely, let $\Omega \subset \mathbb{R}^n$ be a bounded domain and g satisfy:

(g 5.1) g is Hölder continuous and $|g(x,s)| \leq c_1 + c_2 |s|^p$ with $1 < p < \frac{n+2}{n-2}$ (if $n > 2$, if $n = 2$ any p is allowed).

(g 5.2) $s\, g(x,s) < 0 \quad \forall s \neq 0$

5.7 Theorem. If (g 5.1-2) hold, then (5.7) has at least one solution u, satisfying $\|u\|_{1,2} = 1$.

We leave to the reader the details of the proof, obtained as application of Theorem 5.6.